Fluid Mechanics
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Lecture - 32
Incompressible Viscous Flows Part-IV

Good afternoon, welcome you all to this session fluid mechanics we are discussing about the different types of parallel flows. Now today, we will be discussing another type known as Hagen Poiseuille flow.

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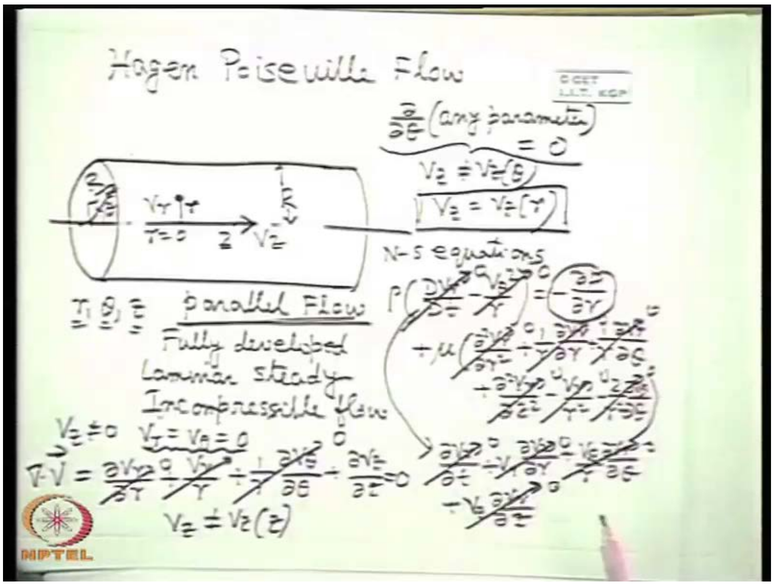

Let us concentrate here Hagen, so now we will discuss Hagen Poiseuille flow; we have discussed the plane Poiseuille flow in the last class; names of the two people, who developed the equations for this flow, after they are named this Hagen Poiseuille flow. This is basically flow through a circular tube, so if we consider like this is a circular tube like this or rather yes. So, this is Hagen Poiseuille flow the flow through a, I am sorry. So, you can just change this, so this is not good. So, we can change it, we can make it again Hagen Poiseuille flow. I am sorry we can make it again Hagen Poiseuille flow, this is, this designates the flow through a tube, circular tube let this is the flow through a tube of circular cross section. So, this is the flow, so this is the axis.

So, here similarly if we first we fix the coordinate axis for all types of fluid, we fix the cylindrical polar coordinate; that means, z from here you measure z, any point we measure at r and azimuthal location. That means, if this with the r this location with the r here this is the surface not here, so this is the azimuthal location. That means, r theta z at the coordinate, r is the radial coordinate at the surface, it is R; that means, this one is R, R, the radius of this circle. Any point has got a radial coordinate r from the axis, this is the z in this direction and theta is the azimuthal direction. So, these three coordinates as you know; defines the cylindrical polar coordinates, compatible with the geometric flow, through a cylinder tube of circular cross section.

Now, again if we consider a parallel flow, parallel flow fully developed incompressible laminar flow; which is already told fully developed, I also write. The meaning of this is not totally appreciated at this moment, fully developed laminar study all these things are there, study incompressible flow, incompressible flow, All right. Now, when it is a parallel flow definition wise here, we define V z naught is equal to 0. The flow is always in the axial direction, there is no component of velocity in the radial or in the azimuthal; that means, tangential direction; that means, V r is 0 V r and V theta all 0.

So, as per as our routine application, if we write the incompressible flow continuity equation in cylindrical coordinate system if you recall del V r del r plus V r by r plus 1 by r del V theta del theta plus del V z del z 0, V r V theta V z at the V z z direction velocity, V r is r direction velocity and V theta is theta direction velocity, tangential velocity azimuthal direction, but incidentally they are 0 for parallel flow.

So, continuity equation that is divergence of the velocity vector in cylindrical coordinates stands like that, as I told you earlier while discussing continuity equation. Now V r 0 V theta 0 only one component of velocity exist and as is consequence is well consequence; that means, V z is not a function of z because, del V z del z 0 another thing we have to consider here that the flow is symmetrical about these axis symmetrical flow about the axis r is equal to 0 which means that the flow is azimuthally, azimuthally symmetrical. That means, del l theta; that means, there is no variation of any parameter with respect to theta all parameters are functions of r and z we consider. That means, with theta there is no variation del l theta of any parameter is 0 which means, that any r z plane at any theta is represents the similar flow situations. That means, it is symmetrical therefore, V z cannot be according to this definition, cannot be function of theta the flow is symmetrical about the axis.

So, therefore, only go is that V z is a function of r; that means, V z is a function of the perpendicular coordinates, coordinates in the perpendicular direction to the velocity. This is the usual consequences; which we found in case of Cartesian coordinate system, we use a function of y where, V is the velocity component in x direction and only existent velocity component. So, here also like that V r V theta 0, V z is a function of r.

Now, I write Navier-Stokes equations, Navier-Stokes equations. That is equation of motions, in art directions. So, what is the equation rho D V r D t that is, the acceleration minus V theta square by r in art direction is equal to minus del p del r. If you write this equation you have to just make practice, that now for incompressible flow, so, only these Laplacian term will be there and that in cylindrical coordinate takes this form, del square V r 1 has to develop this habit to write the Laplacian del V r del r plus 1 by r del V r del theta plus del square V r del z square minus V r by r square due to the curvature this two terms comes 2 by r square del V theta del theta. So these are the terms for the Laplacian and for incompressible flow, this is the art direction Navier-Stokes equation, the equation of motion in art direction.

Let us see what does it give, D V r D t is 0 there is no V r even, if we split it if one wants to see del V r del t the temporal term plus V r del V r del r convective term plus that you as you know V theta by r del V r del theta well, plus V z del V r del theta. So, each and every term is 0, temporal derivative is 0. So, V r 0 del l theta is 0, more over V theta is 0. Though V z is not 0, but del V r del z 0 there is no V r.

So, therefore, as a whole this substantial derivative is 0 there is no V theta 0 all these terms are 0 because, V r is non existence. So, there is no V r and its gradient V theta V r, so, therefore, the only consequence is that del p del r is 0.

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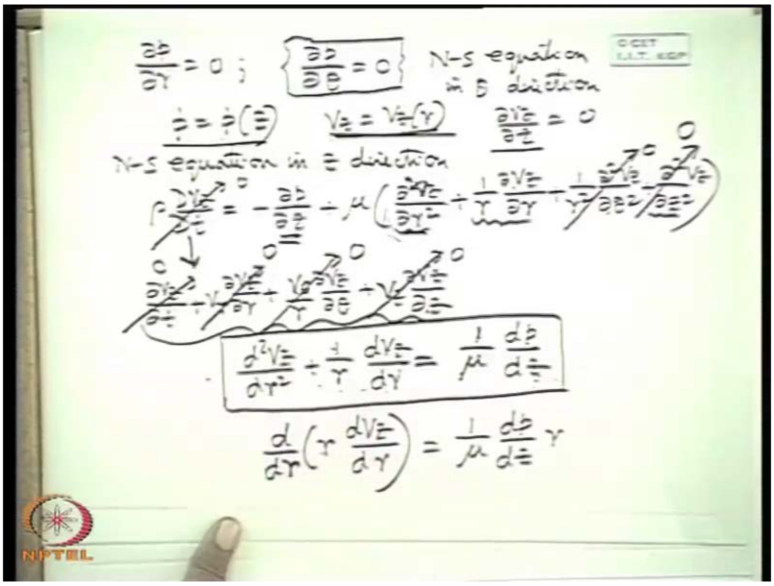

So, therefore, we get del p del r again we get from azimuthal symmetry del p del theta is 0, but this del p del theta 0, we can obtained also from Navier-Stokes equations in theta direction. But it is not required now because, unnecessarily we may not write this equation to get del p del theta 0 because, we have considered then azimuthally symmetric flow that is, flow is symmetrical about r is equal to 0 axis, del l theta of any parameter is 0; that means, del p del theta is 0.

So, only go is this, p is a function of z, its complete, so, V z is a function of r and p is a function of r this you keep in mind. Now write the z deduction Navier-Stokes equation, N-S equation in z direction, N-S equation in z; that means, along the axis of the pi z direction. Write rho D V z D t so, D V z D t is the only acceleration in z direction unlike r and theta direction no cross component come. So, this must be equal to del p del z plus mu Laplacian of V z; that means, del square V z del r square. So, Laplacian in the cylindrical polar coordinate you know the operator is like that, del l del square del r square plus 1 upon r del del r, 1 upon r del del r. That means, this is operating on V z plus 1 upon r square del square del theta square operating on V z, plus del square del z square operating on V z. So, this is the Laplacian.

Now, let us see this is 0, why? If you just now it difficult to accept this is 0, why? Because V z is not 0 why D V z D t 0 let us split it, then only you will accept plus so, this becomes V r del V z del r the convective part plus V theta by r del V z del theta plus

V z del V z del z All right. Now it is clear that each and every term 0 for steady flow, this is 0 del V z del t now V r is non-existent 0. So, this term is 0 V theta is 0 and more over del del theta of any parameter is 0, it is 0. Now though V z is non 0, but del V z del z is 0 from continuity. So, that we write V z is a function of r because, continuity has already told us del V z del z 0 for these, this is 0.

So, therefore, we see all the terms as 0, so, D V z D t c. Here you see this is not 0, now where we find that this is 0, this is also not 0, but this is 0 because, no parameter is a function of theta. So, second derivative of theta is also 0. Now this can be neglected with respect to del square V z del r square, why? Now del V z del z may be 0, but del square V z del z square may not be 0, but if we consider del V z del z 0 del square V z del z square may be 0, but we can argue in a different way that the length of the tube is much more compared to the radius. So, that variation in z direction is neglected compared to that in the r. Mathematically also one can tell if, del V z del z 0, del del z of del V z del z is also 0. So, both ways we can tell this is 0.

So, therefore, we can we are having with this these and these, but it has been proved that p is a function of z only. So, we get rid of del, so, we write only d ordinary differential similarly, it has been proved that V z is a function of r only. So, therefore, this is also 0, so, therefore, V z del, del we get rid of d. So, therefore, I write this first that d square V z d r square plus 1 upon r d V z d r is equal to 1 by mu d p dz. It is a similar equation that we arrived in Cartesian coordinate there we arrived d square (( )) d y square is 1 by mu d p d x, here x coordinate is the z direction. That means z is the direction of flow r is this direction the radial direction, the governing equation is d square V z d r square plus 1 upon a d V z. V z is a function of r that is why it is an ordinary differential equation and p is a function of z. With this same argument we can tell that if d p d z is constant and left hand side is constant; that means, if V z is a function of r only. So, d square V z d r square can either be constant or a function of r 1, 1 by r d V z d r is either a function of r or constant. That means, as a whole left hand side option is there either a function of r or constant, and since p is a function of z the option for right hand side is either is a function of z or constant, depending upon either p is a linear function of z or non-linear function of z. But a function of r cannot be made equal to a function of z. So, therefore, the equality between the LHS and RHS the left hand side and right hand side tells that this is constant and this is they can only match at a constant value which means that V z is a quadratic function of r. So, that this gives constant value independent of r and p is a linear function of z. So, that this gives a constant value; that means, d p d z is constant.

Now, you just integrate it how to integrate you write in this fashion by multiplying r this is a simple calculus differential equation. So, there is nothing fluid mechanics any if you just multiple r. So, you get rd square V z d r square plus d V z d r which can be written as d V z d r; that means, if you multiple r on both the sides I get this multiply at r d square V z d r square plus d V z d r; that means, r d square V z d r square plus differentiation of r is one; that means, I can write them by multiplying with r left hand side like this.

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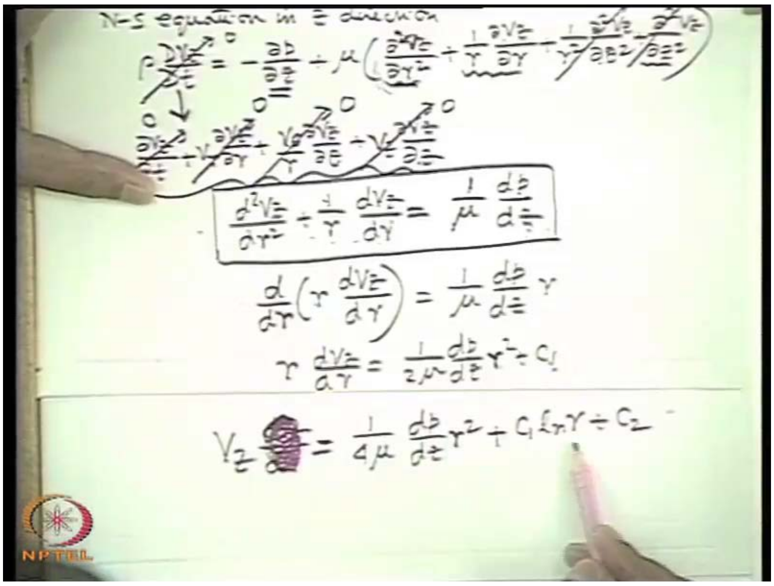

Then what I can do I can integrate this. So, if you integrate this first line will be r d V z d r is equal to 1 by 2 mu d p d z r square very simple plus c 1. So, second integration give d V z d r now this will be c 1 by r and this will be again r; that means, 1 by 4 mu d p d z into again r square because, this will be divided. So, r square by r r; that means, r d r again r square by 2 4 mu plus c 1 by r will give you c 1 l n r, c 1 by r d V z d r second integration c 1 another integration constant c 2.

So, therefore, I m also playing like (( )) V z is 1 by 4 mu, then integration gives V z is 1 by 4 mu d p d z r square plus c 1 l n r plus c 2 very good. So, this is the differential equation. Now you tell me what are the boundary conditions to find out c 1 c 2, you must have a look to this that, this is the picture what are the boundary conditions you tell me, tell me the boundary conditions; that means, if I keep it like this what are the boundary conditions this is the flow.

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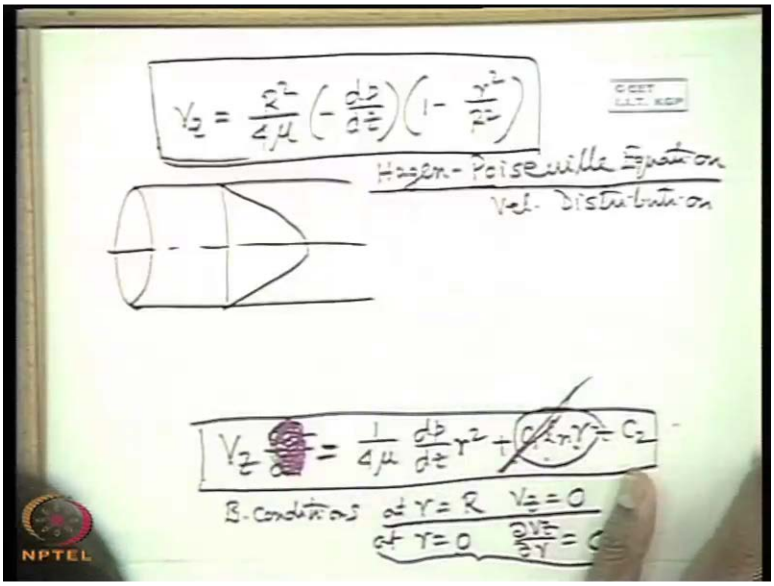

So, what are the boundary conditions, so, boundary conditions, boundary conditions at r is equal to R what V z is equal to 0, but one boundary condition then what to do about c 1 c 2 at r is equal to 0 very good tell me at r is equal to 0 del V z del r is 0. Because, this is symmetry about this symmetrical about this axis, but this is not required vigorously, mathematically, one can simply argue that at r is equal to 0, the flow flied is defined. So, this term cannot exist because, this term cannot define the equation physically that r is equal to 0, it is undefined; with an l n r terms. So, c 1 is 0 or if you tell that r is equal to 0 both way we can tell this is physically, we can eliminate this term or we can eliminate this term mathematically also by telling that since it is symmetric about r is equal to 0. So, del V z del; that means, the derivative has to be 0 with a maximum or minimum at r is equal to 0 and mathematically if we look that, d V z d r r d V z d r just the previous step, before integration is this. So, at r is equal to 0 d V z d r 0 means c 1 0.

So, either way we can tell c 1 0. So, from this equation precisely we get c 2 and finally, what we get is V z is equal to after having this R square by 4 mu because, this is the value of c 2 d p d z. So, minus d p d z I always write like this, into 1 minus r square by R square. This is precisely the velocity distribution equations and this is a parabola; that means, if you draw it again I am drawing this; that means, if you draw it. So, it is a again sort of parabola, it has to be a parabola we have seen the quadratic equation, this is known as Hagen Poiseuille equation or simply Poiseuille equation if the what plane they are adjective Hagen Poiseuille equation.

Plane Poiseuille equation is the equation velocity distribution between two fixed plate. So, either Hagen Poiseuille or Poiseuille equation, this is the velocity Poiseuille equation or Poiseuille velocity distribution or Hagen Poiseuille velocity distribution. The word Hagen Poiseuille or only Poiseuille without any adjective means, that it is the equation for a parallel flow through a pipe or through a duct of circular cross section through a cylinder.

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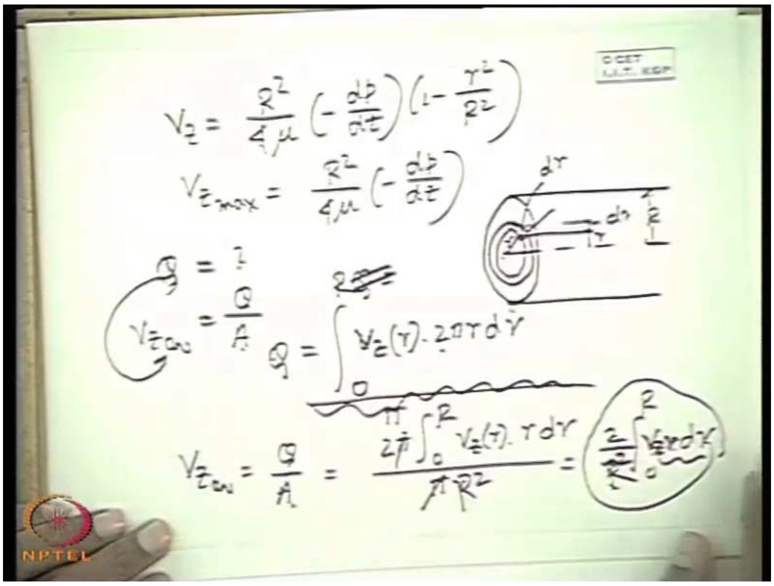

So, the velocity distribution is this V z R square by 4 mu minus d p d z 1 minus, so again I write V z R square by 4 mu minus d p d z into 1 minus r square by R square; where is V z maximum at r is equal to 0. So, maximum V z is R square by 4 mu minus d p d z, another thing is clear that, the sign of V z represented by sign of d p d z when d p d z is 0 V z is 0; that means, flow is entirely governed by the pressure gradient. A negative pressure gradient causes a velocity in the positive z directions and vise verse; obviously, it has to be that this is the z direction a positive V z means the negative pressure gradient. That means, pressure here is higher, pressure here is lower and the opposite is also true and if d p d z 0 V z is 0.

Now, question comes how to find out average velocity; that means, first of all we will have to find out Q because, we know average flow velocity is Q by area, what is the definition of average velocity V z average is Q by area. So, what is Q? How to define Q? So, to define Q, what we have to do, we have to consider an elemental ring at a distance r, at a distance r and elemental ring you understand? Of thickness the thickness d r here better if I write like if, I draw like this. You consider an elemental ring this is the circular cross section here at any radius r, that at any radius r you consider an elemental ring of thickness d r. This is the radius r we consider an elemental ring thickness here d r, and find out what is the flow rate through this annulus. Flow rate through this annulus is, flow rate through this annulus is the u or V z here V z at that r which is a function of r given by this, into what is the area of this annular ring twice pi r d r. So, simply integrate it 0 to r, this is capital R that is the radius.

So, therefore, if I write pi so, V Q is this sorry Q is this. So, what is V z average? So, V z average is I like to show you what I told you last class that V z average is Q by A; that means, 0 to r now twice pi I can take. So, V z as a function of r into r d r divided by pi r square. So, is equal to 2 by R square 0 to r V z r d r. So, you see it is not a simple arithmetic mean it is an weighted average V z into r d r not V z d r by into 2 by r. This could have been the arithmetic average in this case, it is not an arithmetic average another typical V z average 2 by r square V z r d r. So, how do you know that this type of average will be there, this depends upon the geometry because, we know the basic definition of V z average is Q by A. So, simply we define Q and then divided by area, we can define the expression for the average velocity. Here it is not like that u d y divided by 1 by h, so, it is not V z d r divided by 1 by r either it is V z r d r 2 by R square; that means, it is a typical average y z average V z into r d r which has come from the definition of the flow rate. I am sorry, I am sorry this is R; this has to be R square very good.

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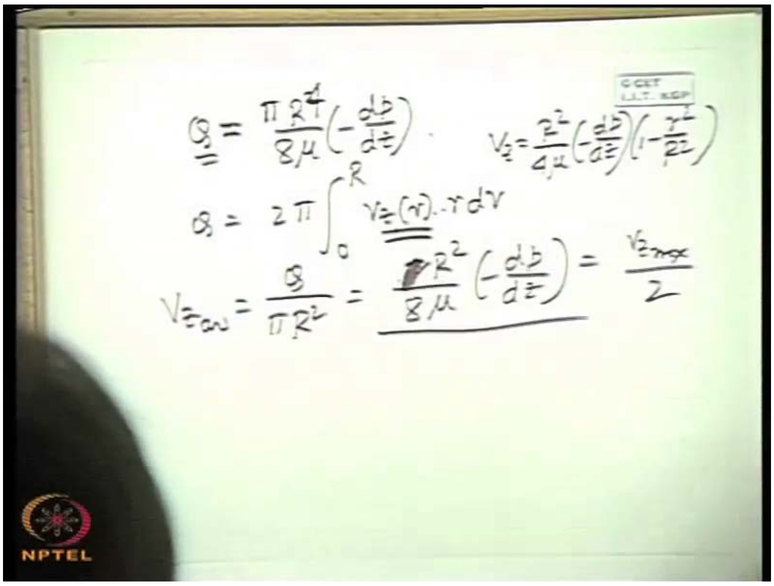

Now we find out the value of Q, if you substitute this V z r there is a function of r; that means, this function if we substitute with 2 pi at here 2 pi, you take out and you find out the integration draw do the integration, you will find that Q becomes equal to R square by 8 mu into minus d p d z into pi R square rather I write pi R 4 this is the value of Q. If you solve this integration; that means, it comes Q is equal to twice pi integration of 0 to r V z as a function of r into r d r; that means, that function of r you substitute.

What is that V z is equal to R square which we have got as Hagen Poiseuille velocity distribution minus d p d z into 1 minus r square by R; that means, if you substitute this integrate you get this. So, therefore, V z average as Q by pi R, R square that is the cross sectional area is simply, pi R square by 8 mu minus d p d z the relationship with the V z maximum is V z max by 2 maximum is pi R square by 4 mu. If you recall the maximum velocity, the maximum velocity we got as, pi R square by 4 mu V x maximum is R square by 4 mu minus d p d z. So, pi is equal to R square by 8 mu d p d z, V z maximum is R square by 4 mu minus d p d z.

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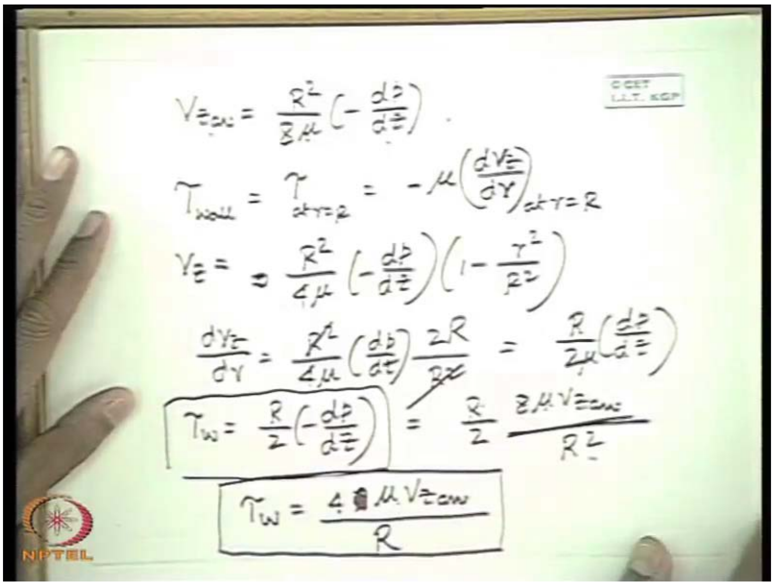

V z average R square by 8 mu minus d p d z. Tau we are interested now tau wall shear stress at the wall; that means, is equal to tau at r is equal to R it is minus mu d V z d r, d V z d r is the velocity gradient and that is equal to the shear rate. Why minus sign I am writing here, can you tell? Why minus d V z d r at r is equal to R; this is because, here the coordinate axis has been chosen in such a way that, the r is measured from this axis and the velocity, if you see here the velocity. I think it will be a better figure here if you see here, that the velocity profile is such that the velocity is decreasing with the increasing r. That means, r is not measured from the solid surface, solid surface always velocity is 0 when the solid surface is at the rest, that is the no sleep condition. So, velocity must increase from the solid surface and always the normal coordinate should be measured from the solid surface, we will see afterwards we define some wall coordinate or normal coordinate from the wall; which is perpendicular to the wall along outwardly, but here in this conventional cylindrical coordinate the r is measured from the centre. So, an increasing r shows a decreasing velocity to make account for this, we make conventionally a minus sign in defining tau.

So, now rest part is the simple mathematics. So, d V z d r what is V z better I write V z also. So, all at a time space is so small here 4 mu minus d p d z into 1 minus r square by R square. So, what is d V z d r? d V z d r is now it is twice r and R is equal to r means it is twice R, R square let us write. So, R square 4 mu minus minus cancels d p d z into 2 R R square and 2 R means at R is equal to R 2 R; that means, R square R square cancels.

So, it is R by 2 into d p d z. So, therefore, R by 2 I am sorry 2 mu, so, therefore, tau wall is minus mu, so, mu mu will cancel; so R by 2 minus d p d z. So, this is the value of wall shear stress are there we should make it in terms of the V z average; that means, R by 2 into 8 mu V z average understand 8 mu average V z average by R square; that means, I just substitute d p d z in terms of the V z average. That means tau w becomes is equal to 4 R mu V z average by, I am sorry 4 mu V z average by this cancels R. So, this is the expression of tau w in terms of V z average; that means, minus d p d z is V z average 8 mu by R square. So, this is V z average 8 mu by R square, so, V z average is 8 mu by R square. So, R R cancels, so, 4 mu V z average by R.

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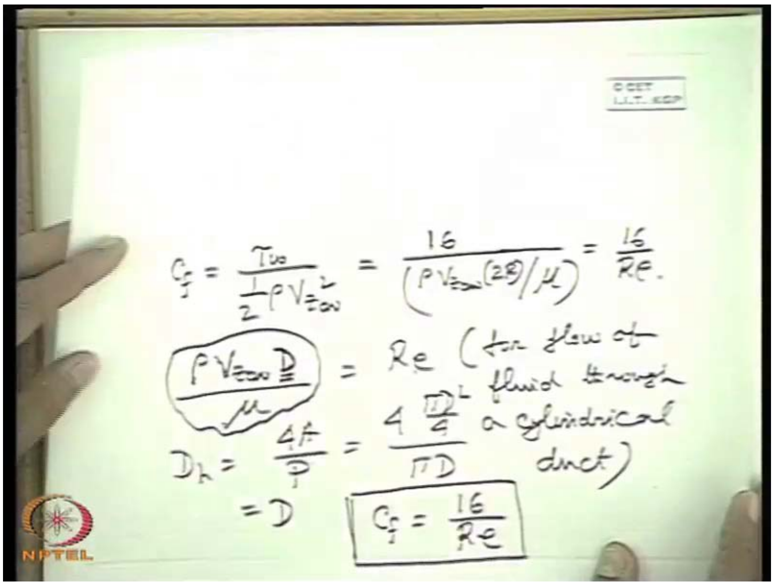

Now, what is the next step? Next step is skin friction coefficient C f. So, C f is defined as tau w by half rho V z average square, if you do that, you get an expression you can see it 16 by rho V z average into 2 R actually it is 8 I just multiplied divided by mu. So, this rho V z average 2 R is diameter of the D by mu is equal to Reynolds number for flow, flow of fluid through circular duct, through a cylindrical duct, circular cross section, through a cylindrical duct.

The Reynolds number is defined rho the characteristic velocity is V z average, that is flow rate by cross sectional area times the diameter of the tube; which is the hydraulic diameter, D h 4 A by p cross sectional area by weighted perimeter 4 pi D square by 4 divided by 5. So, it is simply D, so this is the definition of Reynolds number we will come again. So, therefore, C f is 16 by Reynolds number; which is very important that in a pi flow that Hagen Poiseuille flow, C f is 16 by Reynolds number. It is inversely proportional, in case of plane Poiseuille flow it was 12 by Reynolds number, C f is 16 by Reynolds number. So, Reynolds number definition is, rho V z average D by mu. So, this is about the Hagen Poiseuille flow, all aspects of Hagen Poiseuille flow. So, any queries?

So, today well I will leave here and thing is that, next class we will start a new chapter that is, the viscous flow through pipes the application of viscous flow. So, about the exact solutions I just like to make a brief closer before ending this section that, today we started this class we started with the derivation of viscous momentum equation or equation of motions for a viscous flow; and we first recognized that for a fluid at rest the only stresses at the pressure. The normal stresses, which or compulsive in nature and same from all direction the viscous flow. For an ideal fluid flow we have considered that, if we take a fluid element the only surface forces at the normal forces which or thermodynamic pressure or static pressures, but for a frictional fluid or viscous fluid along with the normal forces, there are shear stresses. So, therefore, if we recognize the external forces apart from the body forces the surface forces consist of both the normal stresses and these shear stresses for a viscous fluid.

So, then we developed the equation of motion; that means, we equated the mass time acceleration with the external forces, we wrote the external forces in terms of the stress components. The most vital part is the relationship between stress and the strain at components, if we what to express the equation of motion in terms of the velocity components and the pressure; which we did not do in this class, but we recognize the different assumption based on which stress and strainless relationship is developed and finally, accepted that relationship. The assumptions well like that the fluids behave in such a way that, the relationship between stress and rate of strain are linear, and there is a class of fluids; which behaves so and known as Newtonian fluid, which stress and rate of strength behavior or relationship is linear. This has been found experimentally, and this relationship is invariant with coordinate transformation. Another assumption was that the fluids for which the equation of motions will be developed they are viscous fluids, but the second coefficient of viscous it is 0. And another assumption last assumption was that the equation of motion will be derived in such a way the continuity to hydro static should be there. That means, you see the equations of motion; that is the Navier-Stokes equation, if you put all u v w 0 you get the equations of hydro static, if you put mu is equal to 0, you will get the Euler's equations. So, all continuity are there to known in viscid flow, friction less flow, and even for the hydro static cases; so, therefore, we will see that the continuity is there. So, based on this the stress and strain less relationships are developed and finally, this were substituted you got the equations of motions in terms of the velocity and pressure, and this equations are known as Navier-Stokes equation, named it after the two person Navier and Stoke.

Then we recognized the nature of the Navier-Stokes equation that, it is a practical differential equation definitely u v w being the dependent variables are functions of x y z and t. There are 4 independent parameters x y z and t, and there are four dependent variables u v w and pressure. We have 4 equations 3 equations of motion in three coordinate directions and one is the equation of quantity, so, they can be solved.

So, now the nature of the equation is they are practical differential equation, there order is 2, second order is governed by the viscous duct, Laplacian duct. Then the linear about the linearity and non-linearity; this is a non-linear equation because, if you see the acceleration part it is the convective acceleration which is non-linear in nature. u del u del x v del v del y, this gives a power of 2 for the dependent variable velocity component.

So, as a whole the Navier-Stokes equation is a non-linear second order of practical differential equation, which do not have a close form solutions, but there are certain cases simple cases of flow; where we have exact solutions close form solutions or Navier-Stokes equations. In those simplified cases by the physics of the problem, the basic Navier-Stokes equation; which is non-linear second order equations are been transformed to a very simple ordinary differential equations, even in second order, but linear ordinary differential equations. Those cases are parallel flows and we discussed the solutions of some parallel flows, again Poiseuille flow, plane Poiseuille flow, and quiet flow; that was the coverage of this section.

Thank you.