Fluid Mechanics

Prof. S. K. Som
Department of Mechanical Engineering
Indian Institute of Technology, Kharagpur

Lecture - 15
Conservation Equations in Fluid Flow Part – III

Good afternoon. I welcome you all in this session of fluid mechanics. Well, we are discussing the continuity equation and deduced the equation in different co-ordinate systems, Cartesian co-ordinate system and also in cylindrical polar co-ordinate system.

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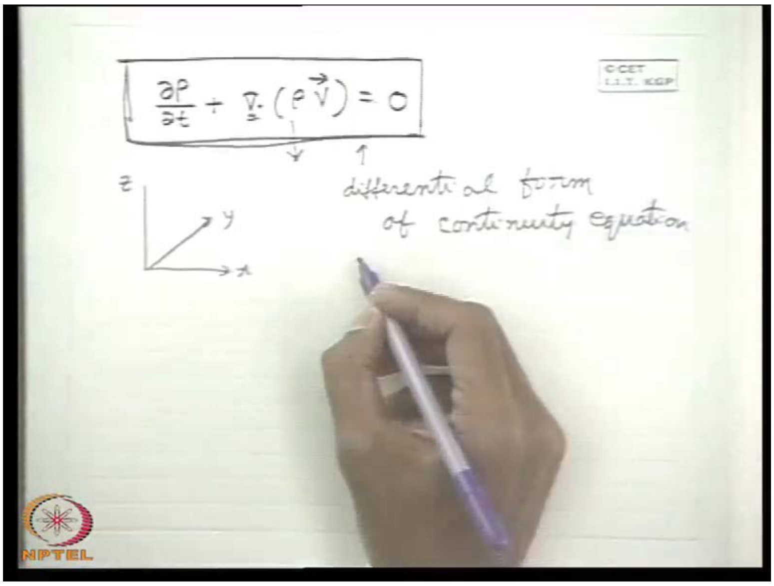

Well, let us see the equation that if we recollect the continuity equation is del rho del t plus divergence of rho into velocity vector, as we recall, this was the differential equation or this was the equation for continuity. That means, the statement of conservation of mass apply to a control volume what we have found out that by expanding these in different frame of reference, whether it is a rectangular Cartesian co-ordinate system in conventional x y z like this, x, y, z or in a cylindrical polar co-ordinate system or even spherical co-ordinate system, we can derive different form of the equations, that equations with respect to different co-ordinate system. Another way of deriving it is to use the fundamental principles of conservation of mass to a control volume, a fluid appropriate to the co-ordinate system.

So, in any case, this general equation in vector form is the differential form of the continuity equation. Since, this is a differential equation, you say this is a differential operator. So, this is known as differential form of continuity equation. So, this gives the differential equation. So, that is why it is known as differential form of continuity equation. Now today, we will find another or we will deduce another form of continuity equation, which is known as integral form of continuity equation That means continuity equation.

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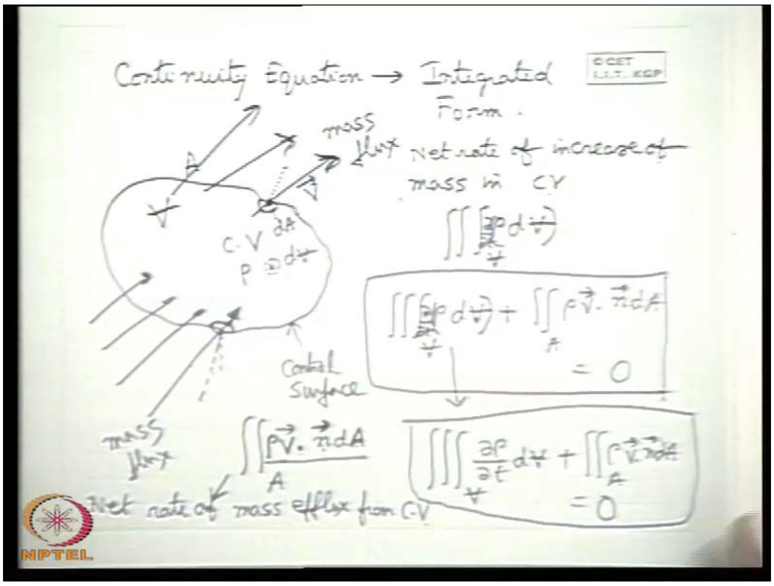

Continuity equation means the same, and that is equation derived from the same principle of conservation of mass applied to a control volume. So, that is the continuity equation by its physical significance and this continuity equation integrated form. So earlier, we discussed the differential form. Now, integrated form. Very simple. Let us consider a finite control volume like that. So, to derive the integrate form, we do not require any frame of difference. Just we consider a control volume, when arbitrary shape whose total volume is v and which is bounded by a surface area A.

Now, let us consider the mass flux like this. Let us consider the mass flux. This is the mass flux, which crosses the control volume. Let us consider these are the mass fluxes crosses the control volume. This is the mass flux. So, we define a control volume in a flow field, where the mass flux crosses the control surface. You know, the boundary of the control volume is known as control surface. This is the control surface.

Now, first of all we have to find out the total mass efflux and the rate of mass increase in the control volume. Now, if we specify a small elemental area d a on the control surface, where the velocity vector is free. This is the velocity vector at this point, enclosing this small elemental area d a. Then we can tell that the mass efflux from this elemental surface d a is what? Which is v dot, that is, v is the velocity vector dot n d a. What is this? This is the mass efflux. I am not writing everything. I am telling that mass, if this quantity represent the mass efflux from this area d a.

What is n? This n is a unit vector in the direction perpendicular to this area element d a and taken positive when directed outwards from this area. So that means, d a is the scalar magnitude of the area and this is multiplied with a unit vector to give a vector sense of it, where unit vector is along the perpendicular direction and taken positive when directed outwards. So, physical significance of this means, that the cross sectional area and the velocity of flow normal to this cross sectional area. As you know, to find out the flow across any section, we multiply the cross sectional area with the velocity of flow perpendicular to this area. This is expressed by this dot product in vector form.

So now, if we integrate this over the entire area of the control volume; that means, the entire area bounded by the control surface, then we get this as the net rate on mass efflux from the control volume. So, I can write this as net rate of mass efflux from the c v control volume, the total area. Now here, if we consider an area, there is a mass influx. How it is indicated? This is because in that case, v the velocity vector and this, for example, this is the normal direction outwards which is taken positive outwards. Very important, so that, this will be in the opposite direction. So, there will be a negative sign; that means, if we integrate this over the entire surface area depending upon the mutual direction of velocity vector and this vector in the net efflux will be determined like that. That means, efflux minus influx which will be taken care of by the entire surface in this way.

So mathematically, one can tell that v dot n bar d a n d a, where this n is an unit vector as I have defined is the net rate of mass efflux from control volume. rho, very good. rho should be there. This could have been the volume efflux. Very good. I am sorry. Very good. It is rho v n d a. So, rho should come because otherwise this is the volume flow rate. So, net rate of mass. Now, what is net rate of increase in of mass in the control volume? This is very simple. If we take for example, at a point where the density is rho and a small elemental volume d v enclosing that point, then rho d v is the mass at that point. So, simply we tell that it is the integral of rho because rho is varying. rho can vary within the control volume. So, simply rho d v over the control volume.

All right. So therefore, we can write according to our statement of conservation of mass applied control volume, if you recall and it is very simple. Very simple physical intuition that the net rate of mass efflux from control volume plus the net rate of increase of mass in control volume is 0. That means, we can write that this as the first term d v plus the net rate of mass efflux rho v. This is one vector and this is another vector.

So, this is the general from of the continuity equation in integral form. So, this is the continuity equation integrated form or integral form. This is the general expression. Now, you tell me, I am sorry. This is the rate of change del del t del del t of rho. So, there will be a del del theta del del t of rho d v. I am sorry. del del t of rho d v. So, this can be written of course, with another because since the volume d v s del del t of rho d v, let us write this way that this should be the total volume del rho del t. So, del rho del t I to d v plus it is this rho v dot as it is. So, this is the volume integral and this is the surface integral. Triple integral. This is all right. So, this is the triple integral. This is the integral form of the continuity equation.

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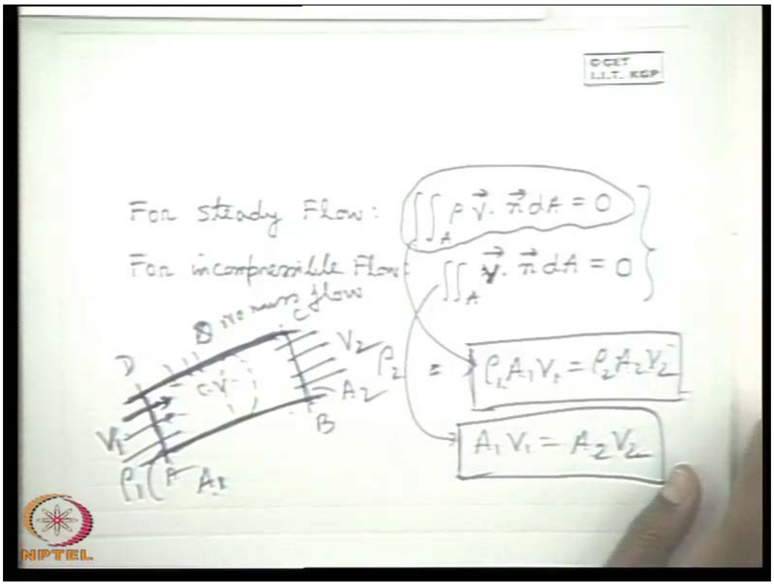

Now, you tell me under steady condition, what will happen for steady flow? For steady flow, what is this steady flow? This will be 0. So, for steady flow, the equation is here. I write surface integral rho v dot d a is 0. For incompressible flow, whether steady or unsteady, incompressible flow means density is constant. Incompressible flow, what will be this term? That is over the area. Incompressible flow also this will be 0 and more over, rho will come out because rho is invariant in this entire control volume.

So, rho will come out. So, simply for incompressible flow, surface integral, sorry, v dot 0; that means, it is the volume flow rate which is conserved. Simply, physically in a compressible flow, mass flow rate is conserve and for incompressible flow, it is the volume flow rate which is conserve. Now, for a steady flow, this is the equation for incompressible flow. This is the equation. So, this is a vector form. Sometimes, it is difficult to recognize physically.

So, from engineering point of view, simply you can guess this way. Let us consider a channel formed by two stream lines. These are the two stream lines. Sometimes, we describe a stream tube. It is a tube formed by different stream lines that the surface of the tube is the stream surface, whose cross sectional area is a circle. But we can think of a channel also, which has got some depth in this direction and which is bounded by two stream lines. So, in that case, if you think from two dimensional point of view, you see that if we define a control volume like this for example, A B C D. Then in this case, the flow will, let us consider the velocity vector is such that it is like this.

So, mass flux only crosses the control surface or comes into the control volume. This is formed by these two stream lines across this surface A D and across the surface B C because this surface or this line across which there will be 0. That means 0. No mass flow; by definition of no mass flow; by definition of the stream line. So now, if we tell that this cross sectional area is A 2 and this cross sectional area is A 2 and this is A 1, and if we tell the velocity vector normal or velocity normal to this area is v 1 and velocity normal to this area is v 2. Then this equation simply tells if rho 1 is the density or rho 2 is the density, the integration over the entire control volume will simply yield rho 1 A1 V 1 is equal to rho 2 A 2 V 2.

So, this is a very simple case that when in case of flow through a pipe, if we consider the inlet section of area A 1, where the velocity is V 1 perpendicular to the cross section, then rho 1 A 1 V 1 is the mass flow coming in. So, we do not go through all these detailed equation from simple physics or physical concept. We tell this is the mass flow coming into the control volume. For example, in a pipe there is no flow across the rigid boundary. That is why a rigid boundary always behaves like a stream surface or stream line because across which there is no mass flow and that mass be equal to that going out, rho 2 A 2 V 2. In case of incompressible flow, it is A 1 V 1. That is very simple consequence of continuity; that means, the volume flow rate which is coming in to the control volume, and mass going out the control volume. In case of incompressible flow, it is the volume flow rate which is conserved. While in case of compressible flow, it is the mass flow rate which is conserved. Well, I think this is clear.

Now, we will switch on to the next section, momentum equation. Now, what is momentum equation? So far, we have derived the continuity equation. That is the conservation of mass applied to a fluid flow system. We have considered the control volume approach, that we have considered a control volume where the conservation of mass we have applied. That is the very preliminary and fundamental conservation statement and we have derived the continuity equation.

Now, one thing we have seen that in deriving the continuity equation, rather while applying the conservation of mass to a fluid flow, the concept of viscosity or question of viscosity does not come into picture. Why? This is because no force is coming into picture. It is only the flow of fluid. So, when the flow field is described, whether it is a viscous flow or it is an ideal flow or inviscid flow does not matter. If flow field is defined, it has to satisfy continuity equation along with the density field to make a possible flow because the conservation of mass has to be satisfied.

Now, we come to equation of motion. Now, equation or statement of conservation of momentum to derive the equation of motion for a fluid flow. Conservation of momentum. What is the conservation of momentum? The conservation of momentum, if you recall, the principle of conservation of momentum applied to a system. First, consider the conservation of linear momentum. That is rate of change of linear momentum of a system is equal to the force acting on the system in the same direction. As you know, for a system mass is constant. So, rate of change of momentum; that means, mass into velocity is a vector can be written as mass coming out.

So, mass into rate of change of velocity; that means, mass into acceleration in a particular direction is equal to the net force acting on that system in that direction. The same equation of motion, that Newton's second law, which we have read in the preliminary physics, and that is applied to a fluid system also. That means, if you consider a system in a fluid flow of mass m, then this mass times into acceleration, if it is accelerated in a particular direction must equal to the net force acting on that direction. So, this is simply the momentum theorem or conservation of momentum. Or you can tell the equation of motion applied to system of solid or fluid as well.

Now here, one question comes when you talk about the total force acting on a fluid body in a particular direction. You know, there are two types of forces that act on a fluid system. As we have read earlier, we have discussed earlier that one force is the external force acting on the fluid body. For example, the gravitational force. There may be other external forces, some magnetic field in which the fluid flow is exposed or electrostatic field. These are the external forces acting throughout the mass. Another force which is acting, if we take a fluid system in isolation, these are the surface forces which appear in the surface contact forces. As you know, when it is taken as a free body in isolation from its surrounding mass, it is the action or reaction as the way you can think, that it is because of this mutual interaction between this fluid system with the surroundings. When it has been kept in isolation, this appears as the force at its surface.

So, these forces are of two types. One is the normal force and another is the tangential force. So, if the fluid viscosity is taken into account, all fluids have viscosity. So, fluid with viscosity, this surface force yields to tangential forces also, apart from the normal compressive forces. As you know, in a fluid flow, the normal forces are only compressive forces, which is the pressure. But if we consider that the fluid to be ideal, this is a hypothetical situation. Theoretically, if we consider that fluid is frictionless, the only way we consider a frictionless mechanical system and when we analyze it, though no mechanical system can be divided of friction. Similarly, if we consider the fluid to be ideal fluid; that means, 0 viscosity fluid, then the only surface force is the pressure forces. That is normal compressive forces on the surface.

So therefore, the shape of the equation or the form of the equation will change, if we take viscosity or if we neglect viscosity of the fluid. So, depending on that, the final expression which we derived from the application of one of the basic laws of conservation of momentum will differ.

So firstly, in your syllabus, we will discuss the conservation of momentum or the principle of conservation of momentum theorem applied to an ideal fluid, and that is fluid without viscosity. Then finally, the equations will be derived by the application of this principle. We will see the velocity field that is relationship between the velocity field and the pressure field and the resulting equation is what is known as Euler's equation, because Euler was the scientist who first deduced this equation. That means, they applied the equation of motions Newton's second law to the flow of an ideal fluid. Let us see the Euler's equation.

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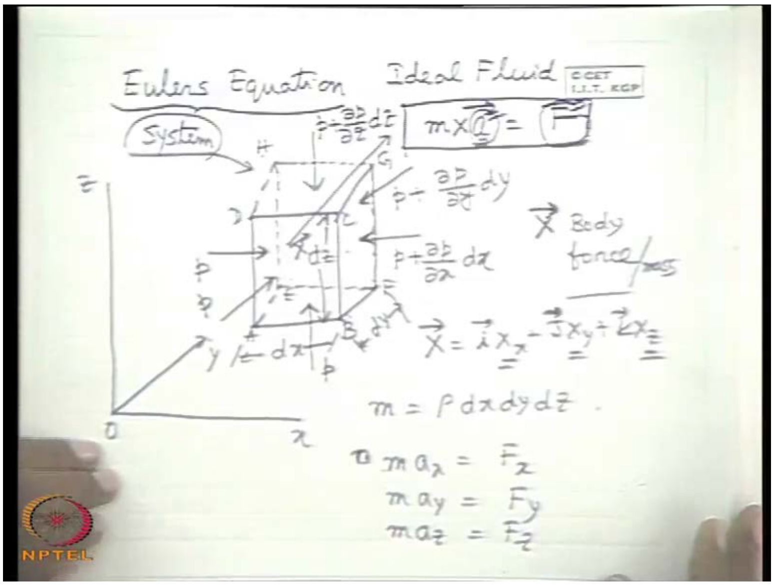

So, if anybody asks what is Euler's equation, do not try to; Euler's equations, d u d t d u d t, like that you must say that Euler's equation is finally, an equation relating the velocity and pressure field, for an ideal fluid which is deduced from the application of the conservation of momentum. So now, let us see that conservation of momentum, as you know that mass into acceleration is equal to the force impressed on the system. Let us first consider the Euler's equation in a x y Cartesian co-ordinate system z and let us define the fluid system compatible to this. Let us define a fluid system like this, A B; it is a parallel (( )) as you know, A B C D compatible to the appropriate or compatible to the co-ordinate system.

Now, this is a fluid system now. Be careful, this is a system and not a control volume. There is the difference. You will see afterwards how the statement relates for both system and control volume. First of all, we are taking a fluid system because we are utilizing this expression, mass into acceleration A in a particular direction is force in that direction. Simply as you have read earlier in school that is the Newton's second law of motion.

So therefore, acceleration only a system can have. Control volume may be fixed. Control volume may move with fluid which will come afterwards. But control volume may be fixed. But it is system; that means, the Newton's second law is basically defined for a system that will be modified for a control volume afterwards with the help of transport. It denotes transport theory which I will tell afterwards.

So, now we consider a system. System means which is moving in the flow or moving with the flow field. System has got a definite mass. So, volume may not be constant, but mass is constant with fixed identity. You can think your system is composed of different Lagrangian particles of fixed identity. That means, system is a fixed mass with fixed identity, which is moving in the flow. Just like the Lagrangian particles move because it is composed by Lagrangian particles of fixed identity. So, they are called a system master acceleration. So, please think. This is not a control volume as we did in case of continuity system.

Now, in this system, let us consider these are the phases. What is the body? What are the pressure forces? First, surface forces acting on this. Then what are the forces on the surface? If we consider the fluid to be an ideal fluid or ideal flow, whatever you call, flow of ideal fluid is the ideal flow. Then pressure forces, which is the only force acting here as P, on this surface A E H D. You know how to define or specify this surface, the x surface. So, the pressure force in another x surface parallel to it. B F G C will be simply p plus del p del x into d x. Before that, I must define that this is d x. So, this is d y; that means, in y direction, this is the dimension of the system and this is d z; that means, this is d z dimension. Any dimension within the system. Neglecting the higher order term, we can tell the pressure is changed over a distance d x for these two planes p plus del p del x d. Extremely simple.

Now, if we define the pressure here, on this A B C D plane, the y plane, then the pressure on the another y plane which is separated by d y distance in the positive y direction; that means, E F G H will be p plus del p del y d y. My first job is to fix the force because acceleration, I know now from the kinematics. Similarly, if the pressure is p, if we specify at this plane A B E F, then the pressure at another z plane, that is D C G H will be p plus del p del z into d z d x d y d z at the dimensions of the system. So therefore, after recognizing these pressure forces, let us consider a body force acting on this system. Let us consider a body force like this, which is denoted by x bar at any direction. Let x bar is the body force per unit mass. As I have told, the body force is the external force which is acting throughout the mass. So, it can be expressed in terms of the mass per unit mass. So, body force per unit mass. x bar is the body force per unit mass, which can be written in terms of its component x y and x z, in terms of its component in x direction, y direction and z direction with i j k as the unit vectors in x y z direction. Now, next task is what? What is m? m of the body is the system is the rho d x d y d z. Now, if we write separately the equation of motion, it is equal to m into a x is net force in the x direction.

So, m into a y is the net force in the y direction. Let us define in terms of vector. So, m a z is the net force in the z direction. So now, you see what are the net forces in x y and z direction. So, this if you recall, m a, now let us see that what are the forces. Just a minute. Just this well, I think you can better see this.

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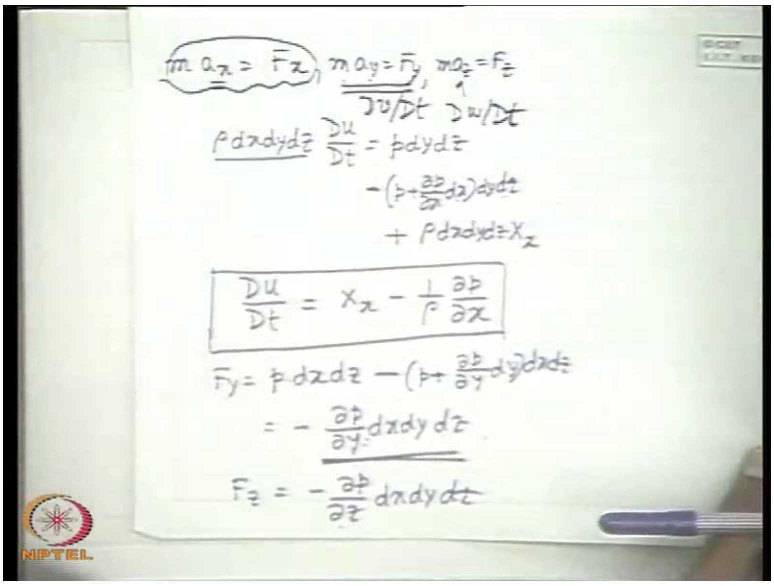

So now, so if we now, you see that what are the forces now. If we recall that this m in x direction a x is F x, m a y is F y and m a z is F z. Now, what is a x? a x is d u d t. So, when if I write this for x direction, then I can write rho d x d y d z into d u d t. What is the net force in the x direction? That is p plus del p del x into this one, first p into d y d z minus p plus del p del x d x into d y d z. So, net force in the x direction is p d y d z minus p plus del p del x d x into d y d z. Well, d y d z. Well, this is the net force. p into d y d z, you can see p plus del p del x into d y d z d x into d y d z, this is the area plus the x component of the body force.

So therefore, this is rho per unit mass. This is the part. x bar nomenclature is the body force per unit mass. So, mass times, its x component. All right. So, if you make this cancel d x d y d z from both these sides, you get an expression like this, x sides. Take first minus 1 upon rho del p del x. Any problem? This is the x direction. Final equation, we get by equating the forces. Similarly, if we do for this, a y is d v d t and mass remains the same. What is F y? F y is, let us write F y. What is F y? F y is d x d z p into d x d z minus p plus this force del p del y d y times d x d z p plus del p del y. This is the pressure there times d x d z.

So, this becomes equal to minus del p del y d x d y. That means in any direction, the net pressure forces either in x y z will come out to be the corresponding gradient. It was del p del x d x d y d z and it is del p del y d x d y d z. Similarly, if you consider the net forces in z direction, it will be minus del t del z d x d y d z. This is because in z direction p into d x d y minus p plus del p del z d z into d x d y and it will ultimately yield to that. If we use this as F y and a y is D v D t and a z is D w D t.

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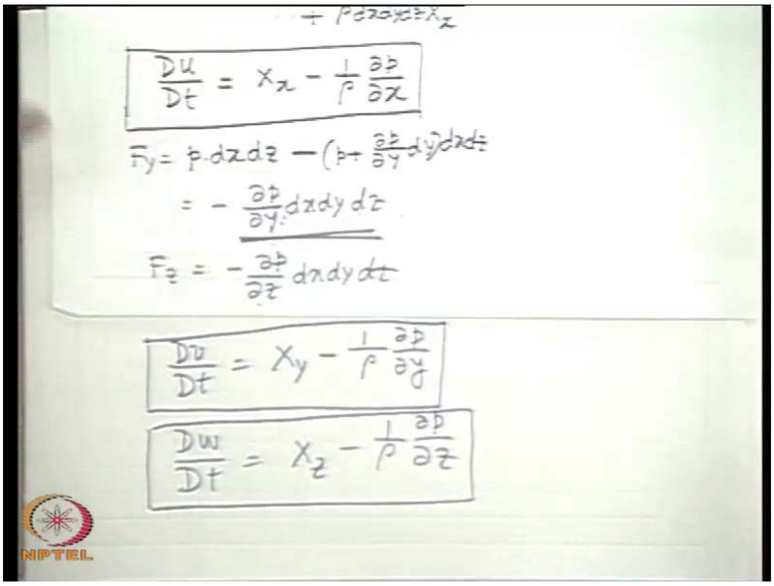

So finally, we get the same type of thing that we get for y component D v D t is equal to x y, in consideration of the body force, per unit mass, y component of the body force per unit mass and D w D t is equal to x z minus 1 upon rho del p del z .So, this is the x component and this is the y component. All right. Now again, I write this thing.

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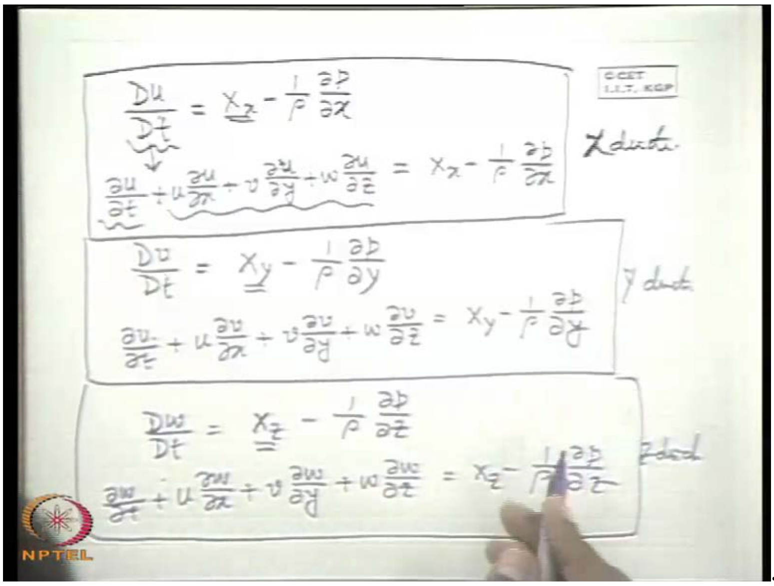

So therefore, what we get in x direction, we get D u D t equation of motion in x direction is x minus 1 upon rho del p del x. Similarly, for y direction I write, what is y direction? D v D t is equal to x y minus 1 upon rho del p del y and for z direction, D w D t is equal to x z minus 1, where p is the pressure in the flow, u is the x component of velocity, v is the y component of velocity and w is the z component of velocity in a typical x y z co-ordinate system and, xx x y x z at the x component body force per unit mass, y component of the body force per unit mass, and z component of the body force per unit. About this, we do not have any control from fluid mechanics point of view. This is described by the physics of the body force field.

Now, we can write this in a different form by splitting of this total acceleration. That means, this can be written as del u del t plus u del u del x. This is another form. We can write explicitly u del u del x v del u del y plus w. That means, splitting of the total derivative or substantial derivative in terms of temporal and convective; that means, explicitly showing the acceleration in its components temporal and convective is equal to x minus 1 upon rho del p del x.

So, any one of this is the x component equation of motion. Similarly, this is del v del t, the temporal acceleration plus the convective acceleration, del v del x associated with u, and del v del y associated with v. This you know earlier, w del v del z from the kinematics is equal to x y minus 1 upon rho del p del y. So, this is the y direction motion; equation of motion in y direction and this will be del w del t plus u del w del x plus v del w del y well plus w del w del z is equal to x z minus 1 upon rho del p del z. So therefore, we see this is the z direction. So, this is x direction equation of motion rather than y direction and z direction. These three sets of equation, this, this and this or this, this and this, they constitute the Euler’s equation. This is known as Euler’s equations. That is the Euler’s equation, E u l e r s, the scientist, big scientist, Euler’s equation. Euler’s equation or equations of motions for the flow of an ideal fluid. That means, it is the conservation of momentum for the flow of an ideal fluid.

Now, you see that by the method of induction, as we did in case of continuity equation, these three equations can be put in a vector form. Can anybody tell what is the vector form? These three equations D u D t x minus 1 upon rho del p del x, D v D t is x y minus corresponding gradient it is the y component del p del y, it is the z component del p del z and the body force z component, this is the y component of the body force per unit mass and this is the x component of the body and gradient with respect to x direction.

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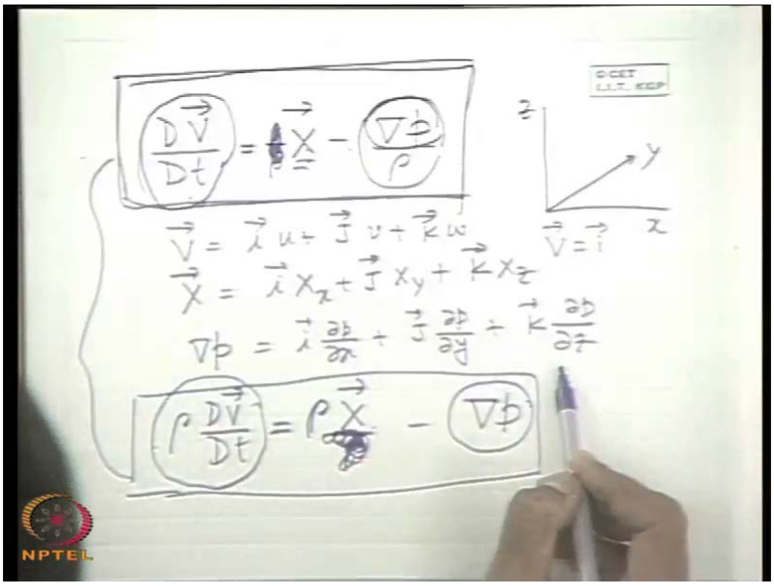

So, can you tell this can be written like that, d of v D v D t, where v is the vector is equal to x minus gradient of v gratitude. This is the vector form. Why v is given by three scalar components in three co-ordinate directions? Well, any problem?

Student: 1 by rho.

1 by rho, very good. Always I am missing something.

Student: (( )).

Which side.

Student: (( )).

Very good. grad p by rho, very good. Very good. So, I am happy that today you are very careful. k x z and grad p. Just by the simple preliminary vector that gradient of scalar field is just their gradient in the respective co-ordinate direction. So, it is the definition of, please any other thing. This is all right? So therefore, this is the definition or this is the vector form of the Euler's; D v D t is equal to x bar minus grad p by rho or somebody or some book or somewhere there, I draw here D v D t is x bar by rho minus x bar into rho minus rho x bar into grad. This is another rho. Does not matter here. This is the acceleration. Physical significance, you see. As you define the equation of motion, each and every term is the force. This is the acceleration; that means, force per unit mass. This is the negative of the inertia force per unit mass and this is the body force unit mass and this is the pressure force per unit mass. If you write in this way, rho D v D t is the acceleration time the density. That means, inertia force per unit volume. Negative of that because inertia force defined with a negative sign and this is the body force per unit volume and this is the pressure force per unit volume. Only the units are different. Any one of the two ways we can express.

So, this is precisely the Euler's equation of motion in a vector form. So, why I am writing in the vector form? Again, similar to continuity equation, if I have to derive the Euler's equation of motion in a different frame of reference, not in a Cartesian frame of reference like x y z, rather if we have to define it in a cylindrical polar co-ordinate system of spherical polar co-ordinate system, one way is to follow the mathematical approach. That means, you have to write this thing in the corresponding, so view will be all right and v bar, it will be defined with its scalar components in the respective co-ordinate access. So, only thing is that the grad p has to be expressed in terms of the respective co-ordinates.

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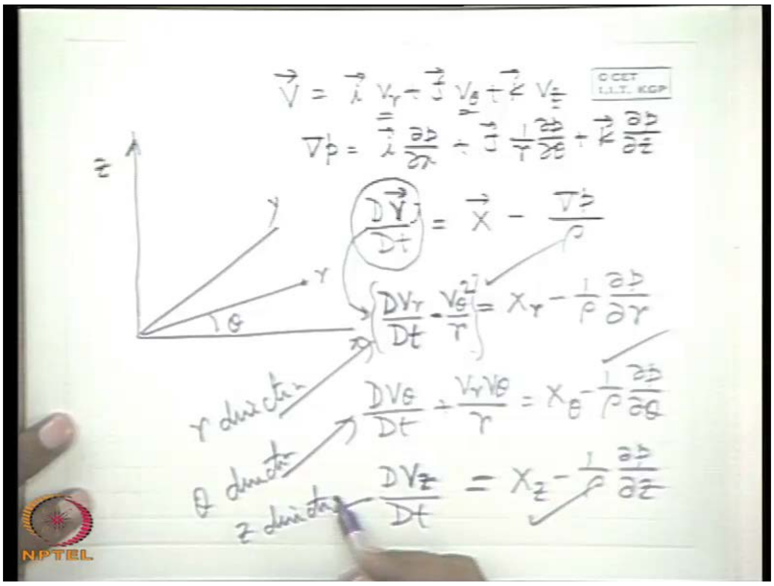

For example, if we think of a cylindrical co-ordinate system for example, this is x and this is y. That means, r theta and this z co-ordinate, then we define v as I v r plus j v theta plus k v z with the v r v theta v z, with the r theta z component, a velocity and I j k are the unit vectors along that. So, what is the grad p in that case? I del t del r plus j 1 upon r del p del theta plus k, this probably you know, del p del z.

So therefore, one can write this D v D t is equal to what? x minus p by rho. Now, if I write the r component, what will be this? D v r D t is equal to x r. If I define x r is the body force per unit mass in r direction minus 1 upon rho del p del r, is it all right? Can anybody tell is it all right? I have written the equation with a gap. Is it all right? What is that? There will be another component, that is the beauty that v theta square by r. Because of the v theta, I told you earlier there is an inward radial acceleration; that means, in the negative at this minus v theta square by r is the acceleration because of v theta. That component comes.

So, even if you follow this D v D t, so when you convert the total differential with the curvature of the, in consideration of the curvature of the co-ordinate system, mathematically also you arrive to this. So, you better write D v or D t as the total change in the, when you make a vector transformation. Some of you may be very strong in mathematics. You can do it, but you can follow this way also without going for the mathematical complication. When you make the D v D t scalar component in r direction, it will be the total differential of v r with respect to t with an additional term. So, it comes from the mathematics considering in the curvature of the co-ordinate system. But one can think in a physical way also, that it is the rate of change of radial velocity with time. It is the radial acceleration and acceleration at the radial direction along with another radial acceleration which arises wholly from the tangential velocity. That means, in a flow field where the radial velocity is 0, having only tangential velocity, there also a radial acceleration is there. This is known as centripetal acceleration as you have already seen.

So similarly, in theta direction, it will be D v theta D t plus v r v theta by r is equal to x theta minus 1 upon rho del p del theta. Similarly, D v z D t, there only D v z D t; z direction acceleration is responsible for the change of v z z component of velocity with time; x z minus, please.

Now, these three equations, this one, this one and this one, this is r direction, this is theta direction and this is z direction can also be derived by a similar fashion. That is by taking a control volume appropriate to r theta z co-ordinate and making its force balance and equating between the mass time acceleration, according to Newton’s second law of motion.

Thank you.