Introduction to Composites
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Lecture – 64
Relations Between Force and Moment Resultants and Mid-plane Strains and Curvatures

Hello, welcome to Introduction to Composites. Today is the 4th day of this particular week. What we have developed in the first 3 days of this week are strain displacement relations, relations between stresses and strains for individual layers, and also yesterday we introduced or defined terms known as force and moment resultants.

And essentially what they mean physically are that suppose you have a plate and it is subjected to external forces and external stresses, then the relationship between these stresses and the forces and moment it experiences on a per length basis. So, that is what is known as force and moment relations.

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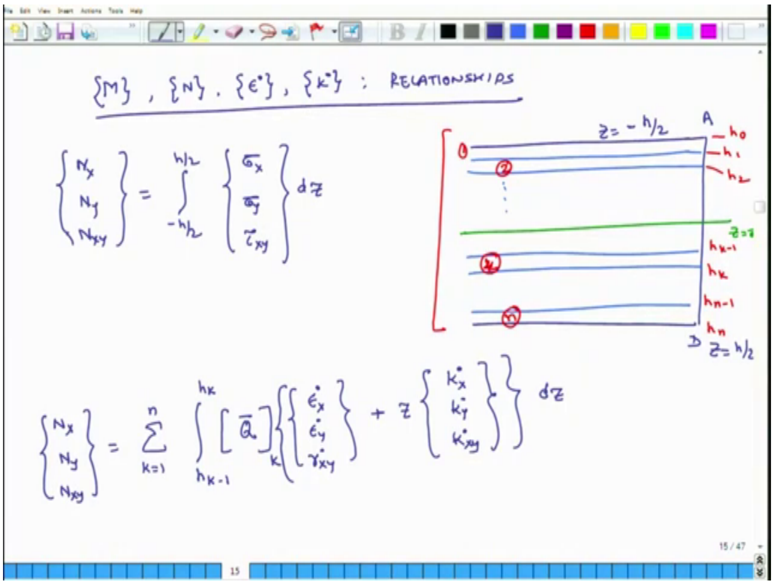

So, the next thing we want to do is we want to expand this topic further by relating this force resultants force and moment resultants and strains, midplane streams and midplane curvatures. So, this is the relationship we want to develop. So, essentially what we will do today is we will develop the relationship between force and moment resultant and strain midplane and midplane strains and midplane curvatures.

So, first we will start with force resultants. So, we have shown that N x, N y, N xy this is equal to integral it is integral of from minus h by 2 to h by 2 integral of sigma x sigma by tau xy times d z this is what we have developed. So, we have developed 3 different equations, but here what we have done is we have just combined them into a single vector equation.

Now, before we work on this further we will develop a convention. So, suppose I have a laminate ok, and this is my point A, this is my point D, the midplane is located here. So, this is z equals 0, and let us say it has n different layers. So, this thing is having several layers. So, layer 1, layer 2 and so on and so forth and let us say this is kth layer and this is the nth layer. So, this is layer 1, layer 2 layer k and layer n. The overall thickness is h. So, from the mid plane the top point is here h by 2 this is the coordinate of point A and the bottom one because z is going down positive, a positive. So, this is z equals h by 2 and here z equals minus h by 2 on the top layer, ok.

And then we write down the coordinates of the beginning and end of each layer. So, I say that this is h 0, this is h 1, this is h 2 ok. So, what is the thickness of the top layer? It will be h 1 minus h naught. What is the thickness of second layer it will be h 2 minus h 1. What is the layer thickness of kth layer? It will be h k minus 1 and the other 1 is h k, so it will be h k minus h k minus 1. And what is the thickness of the last layer? It will be h n n h n minus 1. So, this is the convention we will follow throughout this class, ok.

So, with this convention we developed this relation further. So, we say that N x, N y, N xy equals. Now, what we have to do is we have to integrate sigma x across the thickness of the specimen ok, and we know that from when we go from 1 layer to other layer sigma x can when sigma y or tau xy it can arbitrarily jump it jump discontinuously because the Q matrix changes right because the Q matrix changes. So, and we also know that for a layer what is stress stress is nothing, but Q times the strain, ok.

So, this integral instead of having integral if once we are going from 1 layer to other layer we have to add and within the layer we can integrate because within the layer the stress is linearly varying across layers stress jumps. So, once I move from 1 layer to other layer I have to add, but within layer I have to integrate. So, using this thinking I can say that I have to integrate this thing add this thing 1 k times k is equal to 1 to n, if there are n layers then I have to do n additions and then within each layer I integrate it. So, let us say I am integrating it for the kth layer. So, it will be h k minus 1 to h k. And what is the for the kth layer, what is the stress it is Q bar times epsilon excuse me epsilon x epsilon y gamma xy plus z times K x, K y, K xy d z, ok. So, this is what I get.

So, essentially what I have done is first what I have done is I have developed the stress strain relations for the kth layer and I am integrating this on the kth layer and that range of z is h k minus 1 to h k because those are the coordinates for the top and bottom surfaces of kth layer. And then for I do similar operations for each layer and I add all of them up and this subscript k indicates that this is for the kth layer Q bar is for the kth layer. So, this is what I get, ok.

And this entire integration is happening for over ind over this variable said ok. Now, we know that epsilon x naught, epsilon y naught, gamma x y naught they do not change with z K x, K y, K xy they do not change with z. Also within a layer Q bar is constant because the same thing and because I am integrating this layer by layer, so within a layer the only thing which gets involved integration is this thing z right. So, when I integrate it this z becomes z square by 2 and here I get n, when I integrate the midplane strain vector I get another z term here, ok.

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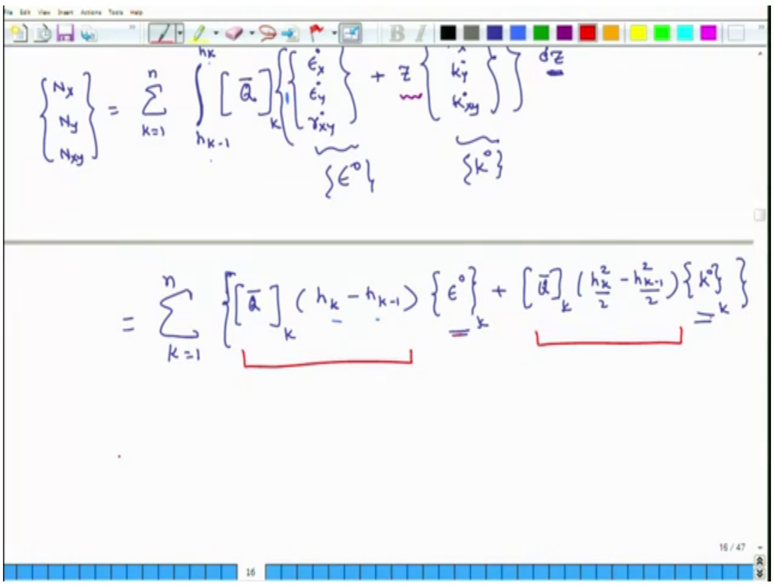

So, when I integrate this what I get is, so this is equal to k is equal to 1 to n and Q bar times h k minus h k minus 1 times epsilon naught bar. So, this is epsilon naught bar and this is k naught bar k. So, I am just for purposes of brevity I am just doing this.

How did I get h k minus h k minus 1? See there is a 1 here you can consider when I integrate 1 with respect to z I get z, and when I take the limit z from h k to h k minus 1 we get h k minus h k minus 1 ok. And this is for the kth layer. We are still doing this for the kth layer and then I do the same thing with the other thing, other one.

So, I get Q bar for the kth layer and when I integrate z it becomes z square by 2 and when I take the limits h k to h k minus 1, I get h k square minus h k minus 1 square by 2 times the midplane curvature for the kth layer, and this entire thing I do for every single layer and I add it up this entire thing on operation for every single layer and I add it up.

Now, across all the layers the mid plane strain does not change, midplane strain is because it is at mid plane strain. So, essentially what I have to do when I am adding this up is I do not have to worry adding up epsilon, epsilon naught is common same thing in this term k naught is common. So, I have to add up products of these for individual layers ok. So, so essentially what I get is now this is a 3 by 3 matrix and this is just a number. So, when I add multiply this and I add add this for all the layers I get still get a 3 by 3 matrix and I call that 3 by 3 matrix A matrix, ok.

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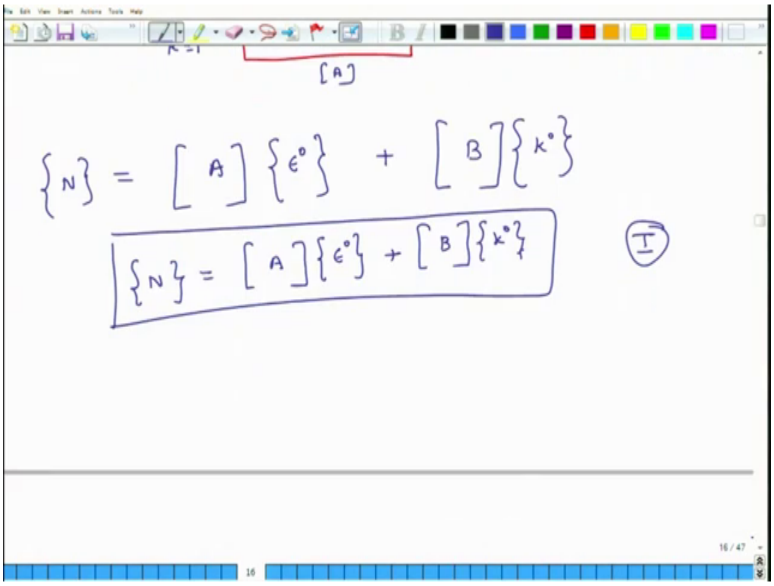

So, that is equal to a 3 by 3 matrix A times the midplane strain. So, from here I get A ok. And same thing I do on the second one. So, here I this is a 3 by 3 matrix and h k square by 2 minus h k minus 1 square by 2 is a number. So, when I multiply that and I add up all the consecutive terms I still get a 3 by 3 matrix.

So, I the second term I get is another 3 by 3 matrix which I call B times the midplane curvature k and this is my stress no force resultants. So, I can say that force resultants which are N x, N y, N xy are equal to A matrix which is a 3 by 3 matrix times mid plane strain plus B matrix which is a 3 by 3 matrix times midplane curvatures this is one set of equations and these equations connect force resultants to midplane strains and midplane curvatures.

So, if for a composite laminate I know force resultants on the outside edges if I know what is the value of force per unit length which is being applied then I can calculate midplane strain and midplane curvatures using these relations. So, this is set I. Similarly, these are the relations for Ns right, these are the relations for N's. Similarly let us see what we can do with the relations for M's. So, these are the relation for forces resultants.

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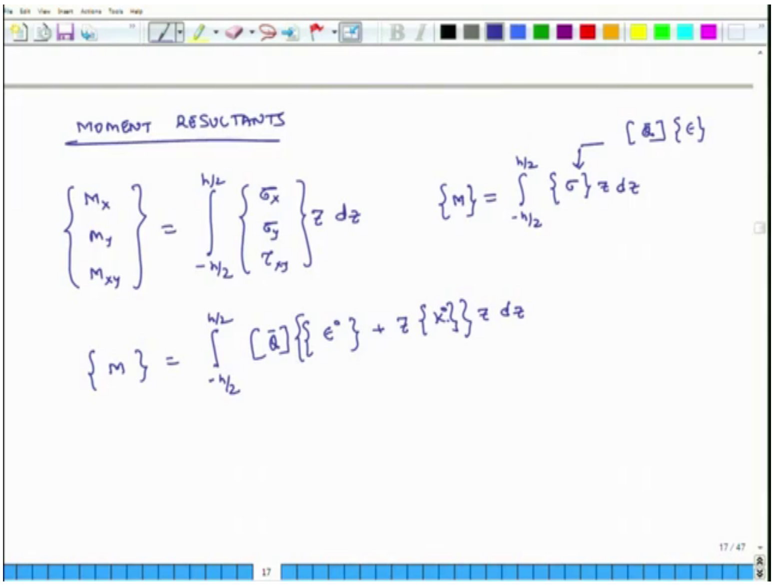

Now, we do for moment resultants. So, how do they look like? We know that moment result M x, M y, M xy are what they are nothing, but integral minus h by 2 to h by 2 and you integrate sigma x sigma y tau xy times z times d z ok, times it is z times d z or in short we can say that moment resultants are this vector is equal to integral minus h by 2 to h by 2 of stress vector times z d z, ok. So, this is just a short form the second one. Just takes lesser space that is all.

But sigma which is the stress vector is what it is Q bar matrix for each layer if I have to compute for at any layer. Then what do I do? It is Q times strain. And what is a strain? It is midplane strain plus z times curvature ok. So, I can say that M equals minus h by 2 to h by 2 Q bar matrix times epsilon naught plus z times k naught this entire thing multiplied by z d z, ok.

Now, using the same logic which we explained earlier, within a layer strains are constant midplane strains are constant and curvatures are constant and also Q bar is constant. So, I can integrate within a layer, but as I move from one layer to other layer I have to add up because the function jumps because Q bar jumps from one layer to other, ok.

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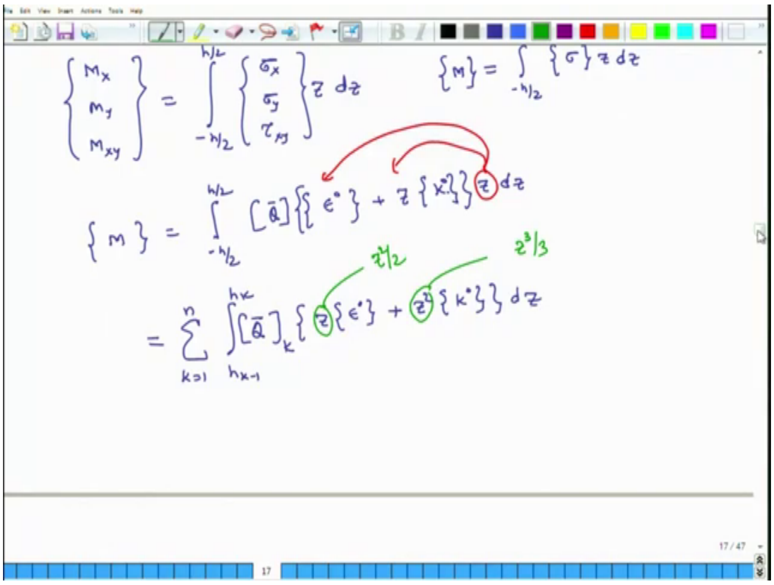

So, for a k layer laminate this thing becomes summation of k is equal to 1 to n and then if I am just doing my job for kth layer then it will be Q bar for the kth layer times epsilon naught plus z times. So, actually what I will do is I will move this z in inside the bracket, ok. So, then this becomes z times epsilon naught plus z square times the curvature vector the entire thing we close the brackets times d z, ok. So, this is my moment resultant and this has to be integrated from h k minus 1 to h k, ok.

Now, within a layer when I integrate then what happens to this z square when I integrate it becomes z cube by 3 and this z becomes z square by 2 ok. So, what I end up getting is Q bar. So, once I have integrated I get Q bar for the kth layer times and when I apply limits on z square by 2 I get h k square minus h k minus 1 square by 2 times epsilon naught plus if I integrate this. So, I get h k cube by 3 minus h k cube by 3 ok, minus 1 times Q bar matrix for kth layer times the mid plane curvature vector ok, and this is in parentheses.

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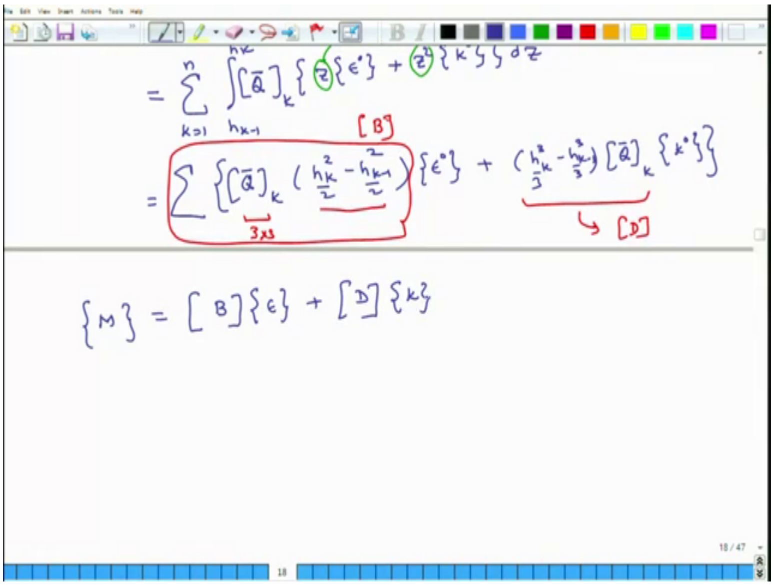

So, once again this is a 3 by 3 matrix this is just a number. So, and this entire thing we have defined it as the B matrix earlier, this is the B matrix, ok. So, this is nothing, but B matrix similarly if you sum up this thing it gives you another matrix we will call a D matrix, ok. So, I can say that the moment result ends M x, M y, M xy nothing, but a 3 by 3 matrix times midplane strains plus a 3 by 3 matrix called D times midplane curvatures.

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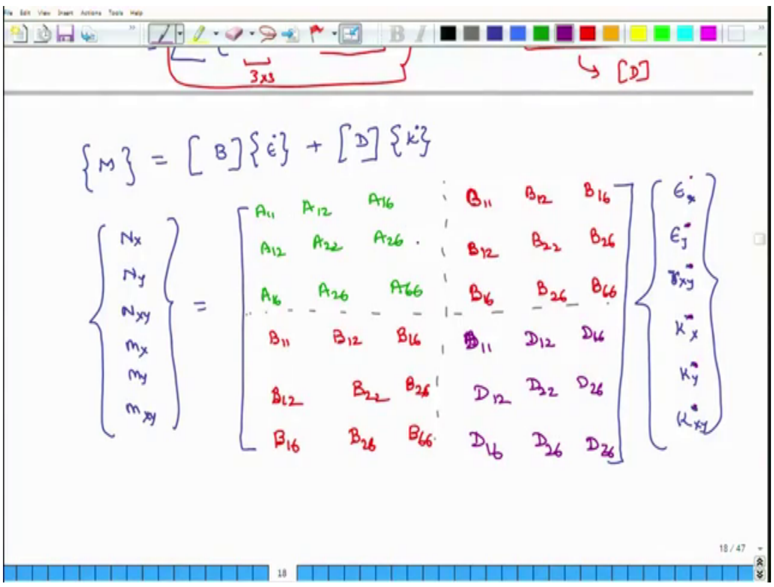

So, we will write this more explicitly. So, this is N x, N y, N xy, M x, M y, M xy. So, if I have all the 6 relations 3 force resultants and 3 moment results, then they are related to mid plane strains stress midplane curvatures using these relations.

And here this is a partition thing. So, in the first quadrant we have A matrix terms ok, in the second we have, in the third quadrant we again have B matrix and in the fourth quadrant we have a D matrix. So, if you know the moments force and moment result rains on the edge of a plate then you can calculate all the midplane strains and midplane curvatures using the help of A B and D matrix terms and the definitions of A B D we have already described ok. We have already described using these relationships. So, what you have to do is you have to layer by layer compute these parameters and add them up for all the layers and you will get individual A B and D matrices.

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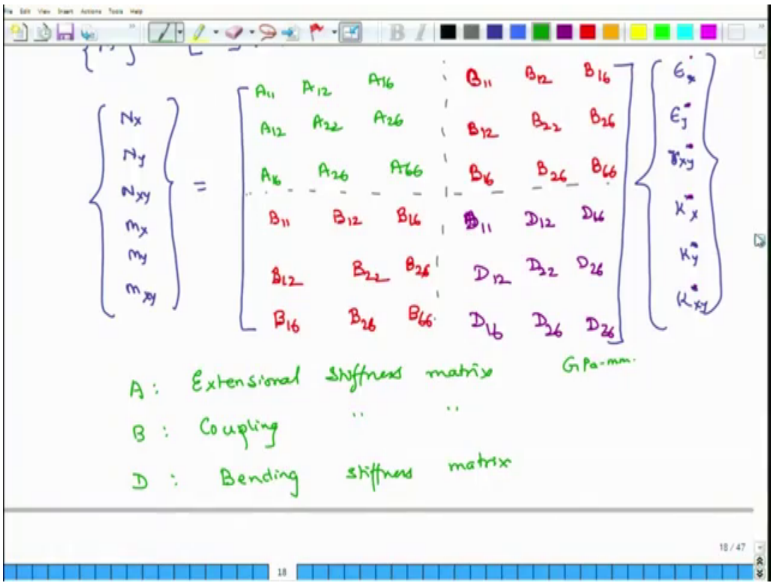

So, the A matrix, A matrix is known as extensional stiffness matrix extensional stiffness matrix, and its terms are you can say GPa millimeters because it is sigma x times now it is Q times z Q times z ok. The B matrix is called coupling stiffness matrix coupling stiffness matrix. And the D matrix is called bending stiffness matrix, so bending stiffness matrix ok. So, these are the definitions of A B and D matrices.

Tomorrow we will try to understand what is the importance of these matrices, and we will also maybe discuss a little bit some related things which help us standardize if we want to specify the lamination sequence of a composite, so we will also talk about it. But that concludes our discussion for today and we look forward to seeing you tomorrow.

Thank you.