Introduction to Composites
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Lecture - 52
Generalized Hooke's law for Anisotropic Materials

Hello, welcome to Introduction to Composites course. Today is the 4th day of the on-going week. Yesterday we had introduced the in tensor form the relation between stress and strain for anisotropic materials, and then we had shown that we start with 81 different elastic constants. And because the stress tensor is symmetric especially when there are no point moments in the body then the number of elastic constants it drops down from 81 to 54.

Today we will extend this discussion and we will continue to explore if we can reduce the number of these constants to as low as possible because at the end of the day if we have a lesser number of constants which are required to characterize a material, then it makes our life much more easy in terms of doing experiments and understanding these materials with less effort and time. So, what we have shown is that from 81 we are down to 54 constants because of stress symmetry.

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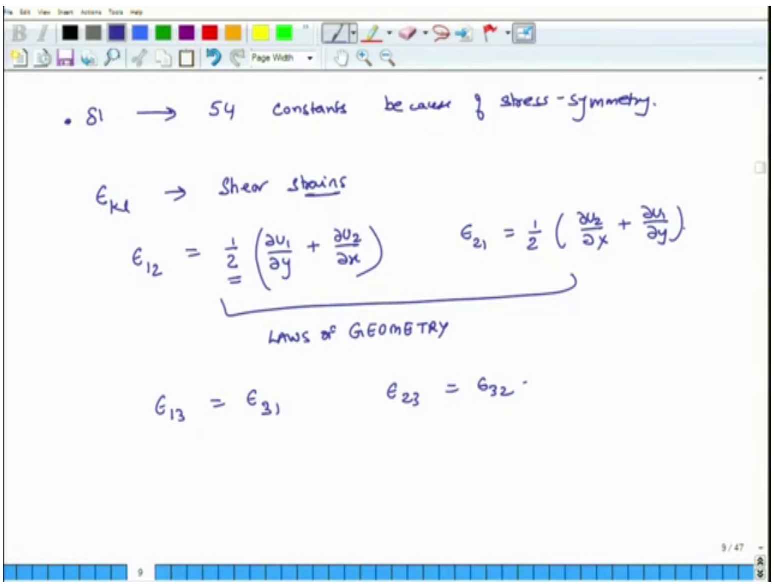

Next, look at the strain tensor. So, epsilon kl ok, for shear strains let us we will see the description of shear strain. So, epsilon 12 is defined if all these are linear strains and these definitions you have already seen in your earlier textbooks is half of del U 1 over del y plus del U 2 over del x ok. In your engineering books this half may not be there because we there we are talking about in earlier books because there we may be talking about engineering strains, but shear strain is half of that and epsilon 21 is half of del U 2 over del x plus del U 1 over del y.

So, the mathematics, so this comes from geometry, these conclusions are from geometry laws of geometry and like Newton's laws, laws of geometry are always true they cannot be violated. So, if these strains are linear in nature then epsilon 12 is always equal to identically equal to epsilon 21, in all cases similarly epsilon 13 equals epsilon 31 and epsilon 23 equals epsilon 32 which means that we do not have 9 independent strains, but we have only 6 independent strains.

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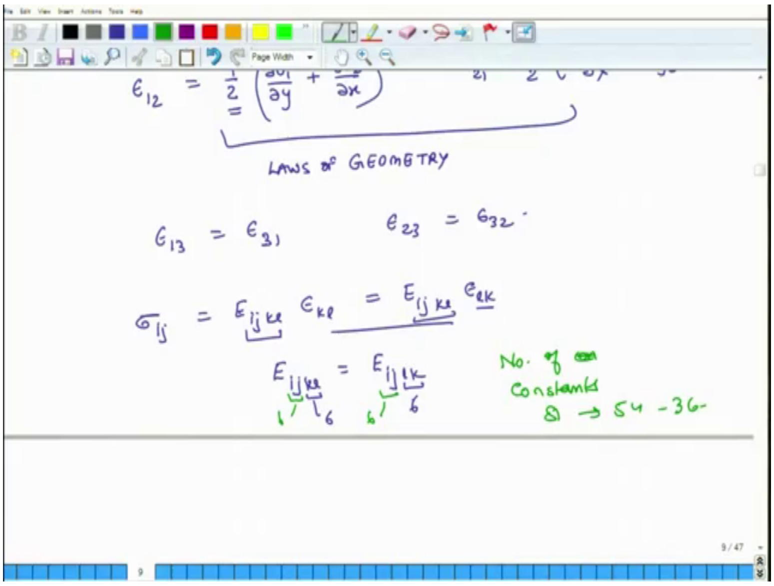

So, we can also say that sigma ij is equal to E ij kl epsilon kl and this is also same as E ij kl epsilon lk because epsilon kl and epsilon lk are same ok. And if that is the case then from this I can I have to say that is equal to E ij kl I am sorry is equal to E ij lk otherwise this relation would not hold, ok. So, these which means that these now get 6 different combinations not 9, ok. And in our original discussion because of stress symmetry these have 6 different possible configurations because of strain symmetry these guys have 6 combinations.

So, number of constants it goes down from 81 to 54 to 36 ok. So, instead of having now 9 equations, we have 6 equations involving 6 stresses, 6 strains and strain tensor which is of dimensions 6 by 6. So, this is our sigma 11, sigma 22, sigma 33, sigma 23, sigma 31, sigma 12 these are the 6 stresses and they relate to they are connected to the stress tensor matrix.

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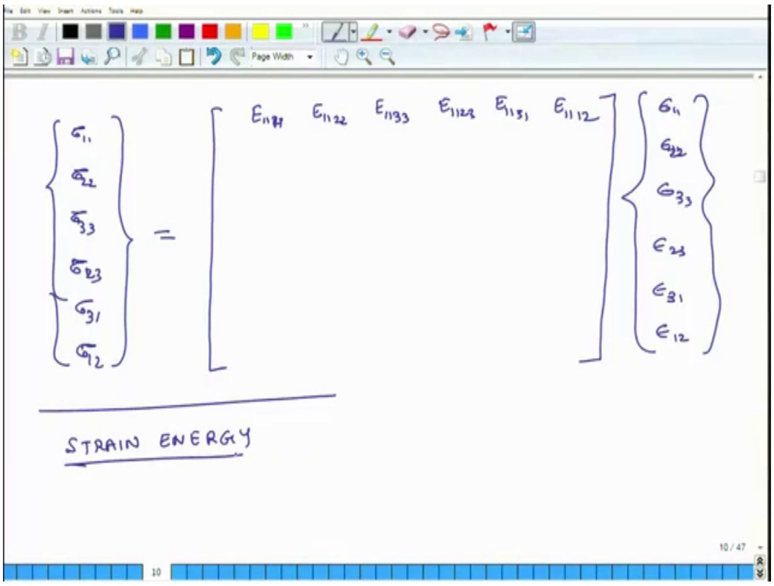

Epsilon 11 and the strains epsilon 22, epsilon 33, epsilon 23, epsilon 31 and epsilon 1 to 6 strains and the terms are something like this E 11 11, E 11 22, E 11 33, E 11 23, E 11 31, E 11 12 and likewise we populate this entire matrix ok, and these are 36 constants.

Next what we will do is we will see what other simplifications we can make. So, here we talk about strain energy considerations. We look at strain energy strain energy, and I will not go into very rigorous mathematical proof, but I will give you an idea what we are trying to do. And we will again start by understanding the basic thought process using the example of a isotropic material.

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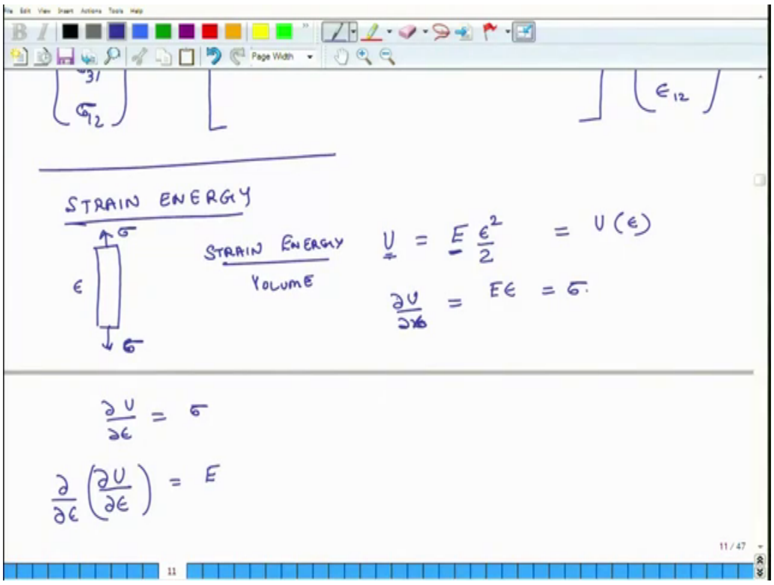

So, if I have an isotropic material, if I have an isotropic material and I apply a stress sigma because of this it experiences some strain and the strain energy. So, when I stretch it expands and some energy gets stored into the system. So, strain energy is what you please go back and again check your books. So, let us call that strain energy as U and this is strain energy per unit volume. So, we are talking about strain energy per unit volume this is equal to the Young's modulus times epsilon square by 2 this is the strain energy in the body per unit volume for a body which is pulled in one direction only, ok.

So, what happens? So, what you do is del U over del x, I am sorry del U over del E if you differentiate this strain energy per unit volume then you get E epsilon and this is nothing, but sigma ok, because E times epsilon for this situation for a unidirectional E pulled specimen which is isotropic is sigma and then. So, here you can also express it as U times. So, U, so as strain energy is what it is it is a function of strain only because this is a constant E is a constant. So, it is a function of strain. So, what you can say is that if you differentiate U with respect to the strain you get sigma, right. So, I am just giving you a rough idea what is have, so partial derivative with respect to partial of the strain, and if you differentiate it once again then you get E.

Now, this is for a very simple test specimen, but in general we can mathematically show without making any approximations that for a linear material for an isotropic material, we can show that partial of U with respect of partial of epsilon ij this gives us sigma ij.

So, for an isotropic material we can show with a little bit of mathematics not a whole lot that if we differentiate the strain energy with respect to a particular strain.

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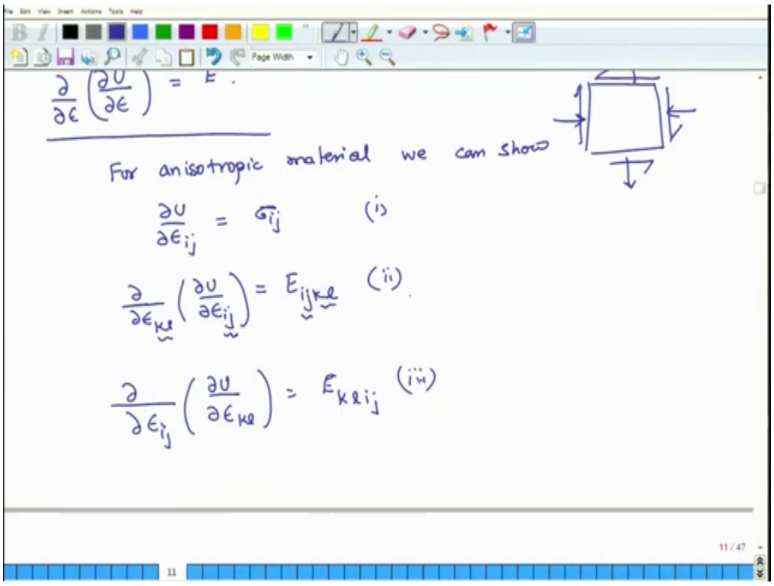

Now, here for an isotropic material we do not have to necessarily load the material only in 1 direction it can be loaded in all the directions. It can have all sorts of stresses in all the 3 directions ok. So, it will be experiencing sigma 11, sigma 22, sigma 12, sigma 13 all those stress and it will also experience all those strains and the total energy in the body is U. And if I differentiate that total strain energy with one particular strain then I get stress in that direction ok. So, this is there. So, this is one thing we can show and just as for a simple uni-axially loaded isotropic specimen the second derivative is nothing, but the elastic module elastic constant we can also say that partial with respect to E ij kl. So, this is the second thing ok. So, this is the second equation.

Now, what I can do is in the second equation I have first differentiated with respect to E ij epsilon ij and then I have again differentiated it with respect to a some different strain epsilon kl, but I can reverse the order of this differentiation. So, I can also say that partial of. So, here ij is in the bracket. So, that is why ij comes first and then second differentiation is epsilon kl. So, say epsilon kl becomes first like this ok. So, this is the third equation.

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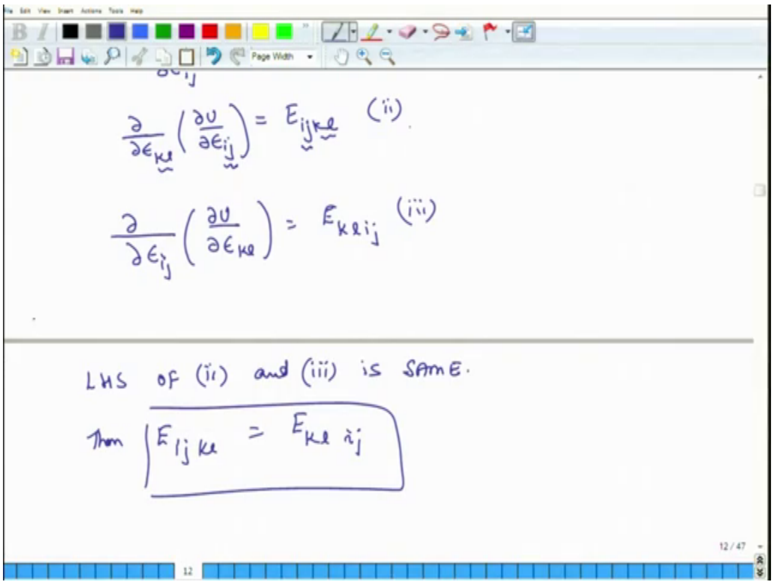

When we look at these 2 equations we see that LHS of 2 and 3 is same it is same in most of the situations it is same because U is from a mathematical standpoint we assume that math it is a continuous differentiable function of strain ok, it is a continuous function. So, if I differentiate it first with ij, epsilon ij, and then with respect to epsilon kl or vice-versa; I will get the same answer.

So, if the LHS of both and 3 is same then E ij kl is same as E kl ij, E ij kl is same as E kl ij. So, what does that mean? So, just give me a moment I will open up a slide and explain to you what; that means, specifically.

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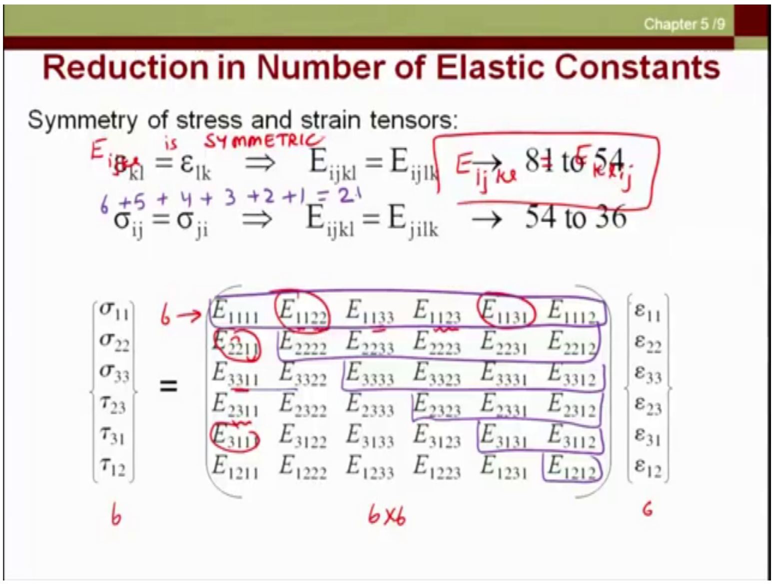

So, what we have shown mathematically is that E ij kl equals E kl ij. So, this is the general equation stress-strain equations this is the 6 by 6 matrix 6 strains 6 stresses and these are 36 independent constants.

But if ij kl is equal to kl ij what does that mean? Here kl is 22 and ij is 11 right, ij is 11 here ij is 22 and kl is 11 which means and if this equality holds then this term is same as this term similarly this term is same as this term, similarly this term is same as this term, similarly this term is same as this term and so on and so forth. So, basically what; that means, is that E matrix the matrix for E is symmetric, it is symmetric. So, what does that mean? So, the total number of constants which are unique now earlier it was 36, but now it will be 6 from here, right. So, what will be the unique terms? These are the 6 terms from here the first term is repeated. So, 5 terms from here, then these 2 terms are repeated. So, 4 terms from here then 3 terms from here then 2 terms from here and 1 term from here, ok. So, all these things if. So, it if it is symmetric then total number of constants is equal to 6 plus 5 plus 4 plus 3 plus 2 plus 1. So, that comes to 21, 21.

So, number of elastic constants because of strain energy considerations it drops down from 36 to 21, 36 to 21.

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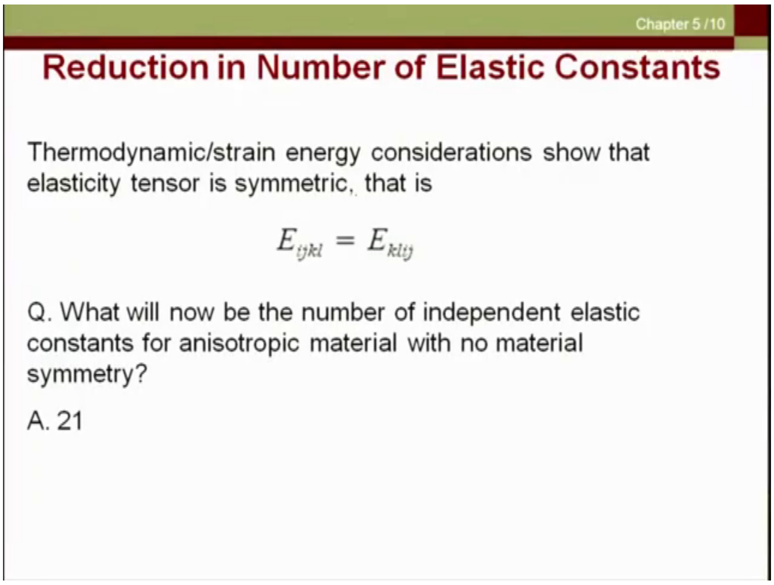

So, this is what we have said because of strain energy considerations we have we have shown that the elasticity tensor is symmetric that is E ij kl is equal to E kl ij and because of this the number of independent elastic constants for an isotropic material with no material symmetry it comes down to 21.

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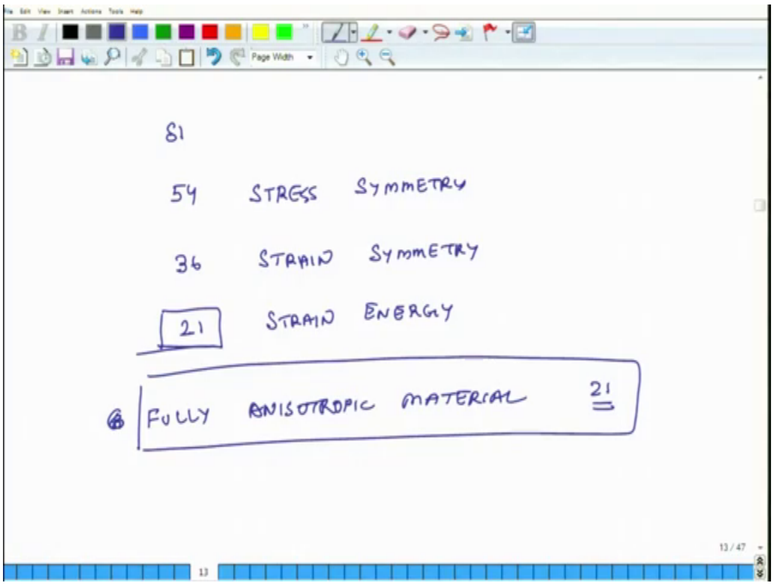

So, what have we accomplished? So, initially we started with 81 constants then because of stress symmetry the number of constants required is only 54, then because of strain symmetry the number of constants required is only 36. And then because of strain energy it is 21.

Going forward we will continue this discussion even further, but at the end of the day 21 is the total number of constants which are required for we cannot reduce this number any further, ok. So, for generally, for fully anisotropic material, for fully anisotropic material ok, for fully anisotropic material we can say that we need 21 material propertiesm independent material properties 21 independent properties. But luckily most the times the type composites which we use they are not fully anisotropic, they have some further you know planes of symmetry. So, this number for special ah cases this number will go down further. So, start tomorrow we will discuss those special cases.

So, thank you and we will continue this discussion tomorrow as well.

Thank you.