Introduction to Composites
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Lecture - 51
Generalized Hooke's law for Anisotropic Materials

Welcome to Introduction to Composites. Today is the third day of the on-going week which is the 9th week of this course. And in the last class we just started our journey into the area of generalized Hooke's law for anisotropic materials, and what we had done was we had just defined the strain tensor and the stress tensor. Both of these tensors we had shown that they are of second order and thus they can be represented by 9 elements which can be organized in a 3 by 3 matrix form.

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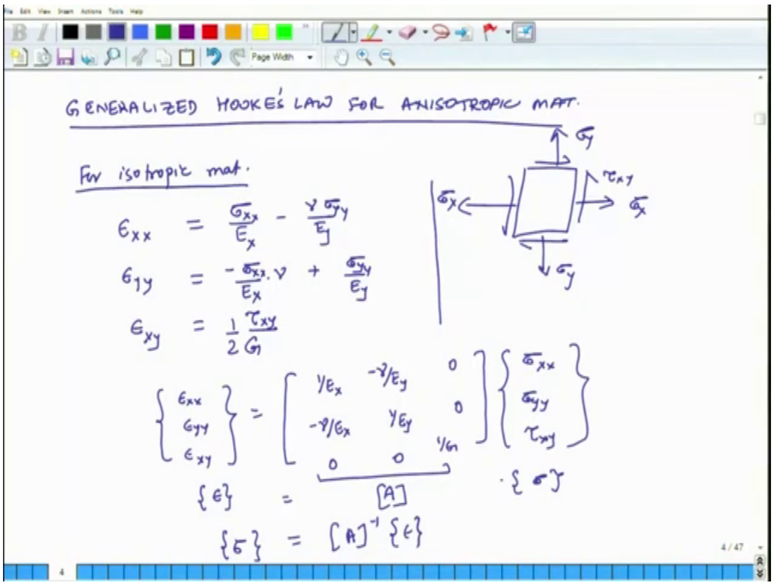

So, now what we will do is we will develop relationship for the generalized Hooke's law for anisotropic materials. That is what we are going to develop. And what we will do is we will first see what kind of stress strain relationships exist for isotropic materials. So, for isotropic materials for isotropic materials what is the let say epsilon x, epsilon y, gamma xy. So, epsilon x, so suppose there is a material and I am pulling it and there is also a shear stress ok.

So, what is the relation for the strain in x direction it is sigma x by E x minus nu sigma y by E y and there is no dependence on the shear stress for isotropic materials because extensional stress cause only extensional strains, so no, no term for tau xy in this relation. For strain in the y direction it is minus sigma x by E x times Poisson's ratio plus sigma y by E y and the shear strain is tau xy by G. And I know I can also if I want to use engineering strain, if I want to use engineering strain then it is gamma xy, but if I want to use tensor strain then I have to use the symbol epsilon xy and the tensor strain is half of the shear strain, so this is 2, 1 over 2 ok. And when I use tensors I always have 2 indices. So, then I have to write epsilon xx epsilon yy ok. So, this is for isotropic material for in if it is loaded in 2 dimensions. If it is loaded in 3 dimensions then I add another term here minus nu times sigma z by E y and so on and so forth ok. So, that is the thing.

So, essentially what I can. So, just for this 2 dimensional stress strain state I can express this in a matrix form epsilon xx, epsilon yy, epsilon xy is equal to 1 over E x minus nu over E y minus nu over E y, E x 1 over E y and 1 over G like this. And we have and then is because I am using tensor notation, so I have to put 2 indices. So, sigma xx sigma yy and tau xy ok

So, this is a 3 by 3 matrix because here I am having only 2 stress, I mean you know only stresses and strains only in 2 directions. But essentially let us call this matrix as A, sum matrixes A. So, and let us call this strain vector as epsilon, and let us call this stress vector as tau. Then I can rewrite this whole set of equations as sigma equals inverse of a times epsilon. So, this is the strain tensor, this is the stress tensor, this is the strain tensor and this is a matrix which connects stress and strain tensors ok. And let us call this A inverse as B then this is equal to B times epsilon ok.

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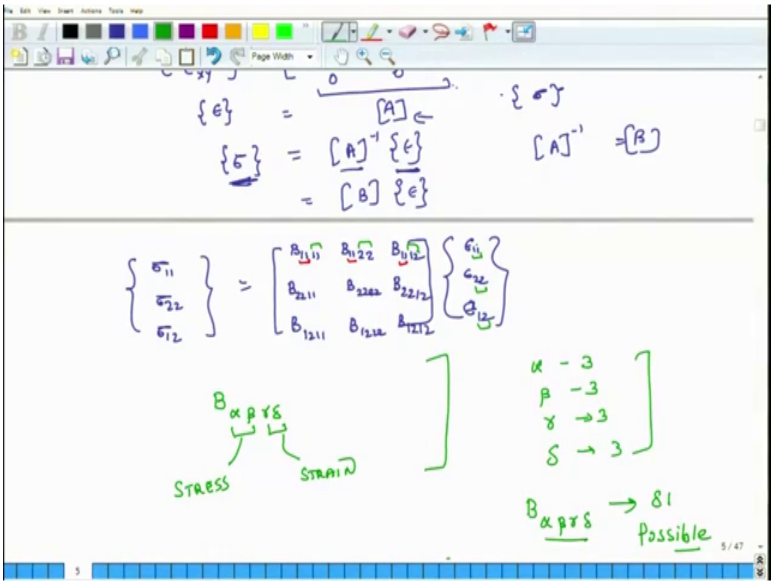

So, in general I can write it as now x, if x corresponds to the first direction then sigma 11, sigma 22, sigma 12 right is equal to this B matrix times epsilon 11, epsilon 22, epsilon 12. And let us number these, so B we will, we can calculate all these elements we can calculate all these elements from this matrix I can find what is the value of a from this and if I take its inverse I know what are the values in B matrix.

But anyway I will just use general symbols. So, let us say this is B matrix is B 11 and I will explain this for a moment. So, then it is B 11 22 and B 11 12 and then here it is B 22 11, B 22 22, B 22 12 and this is B 12 11, B 12 22 and B 12 12 these are the 9 elements of the B matrix. And we know how to calculate these different values using the process above and let us just understand this what all these terminologies mean.

The first 2 indices B 11 tell us that they are related to the first equation sigma 11 ok. So, B 11, B 11 11, B 11 22, B 11 12 they are being used to compute the value of sigma 11 in terms of epsilon 11 ok. And the second 2 indices relate to the indices for these guys strains ok. So, what that means, is that B in general if we call it alpha beta gamma delta this is in general 4 terms. The first 2 terms can tell us connect this B matrix to the stress and the second 2 terms indices connect to the strain tensor, and alpha, beta, gamma delta for this they can assume 2 values 1 and 2. So, in that case they have 2 sets of values.

But for a 3 dimensional system for a 3 dimensional system if the material is not isotropic if the material is not isotropic alpha can have 3 values, beta can have 3 values, gamma can have 3 values, and delta can have 3 values, right. So, there will be B alpha beta gamma delta will have can have 81 possible values. For what? For a 3-dimensionally fully loaded anisotropic material, for isotropic material some of those values will be same for instance B 11 11 and B 22 22 they may be same they will be same. But for anisotropic generally anisotropic material this B because it has 4 indices alpha beta gamma delta and these indices are related to each index can assume 3 different values, so it can have 81 possible different values that is one thing.

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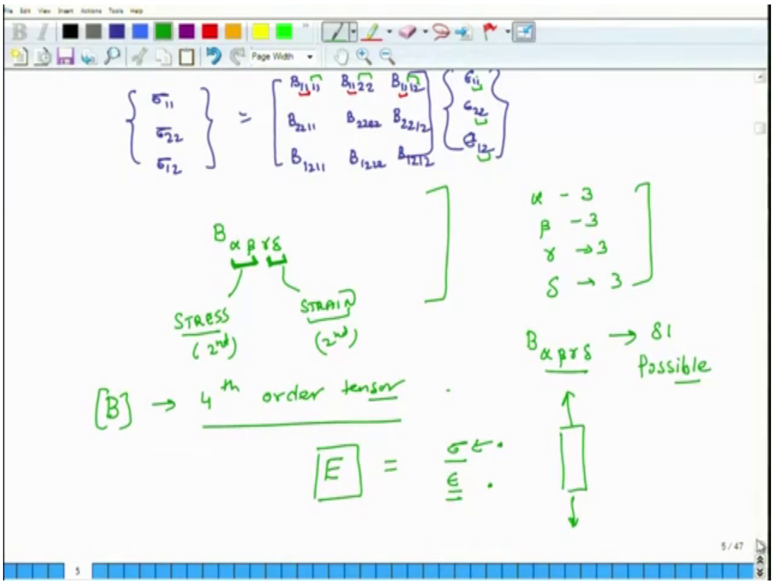

The second thing is that B is a 4th order tensor. Why is B a 4th order tensor? Because it is a constant which connects stress and strain, and the first two index indices relate to the stress portion and the second two relate to the strain thing and it connects stress and strain. The stress tensor itself is second order right, the strain tensor itself is second order. So, if I have to calculate B, if I have to find B it depends on 4 different independent set of directions, two directions are associated with the stress tensor.

What are those two directions? They are the direction of the force and they are the direction of the plane on which it is acting. And similarly the strain tensor is relates to the length, the direction of the length, and the direction of elongation, the direction of length and the direction of elongation. So, 4 different stress directions are associated with this B term. So, that is why it is a 4th order tensor.

Another way to look at it is that for instance E for isotropic material for isotropic material. E is what? Sigma by strain for a uniaxial loaded specimen. But then epsilon is a if it is second order tensor and sigma is a second order tensor then E becomes a 4th order tensor because this is associated two directions, this is associated with two directions. So, E will depend on 4 independent set of directions ok, 4th order tensor.

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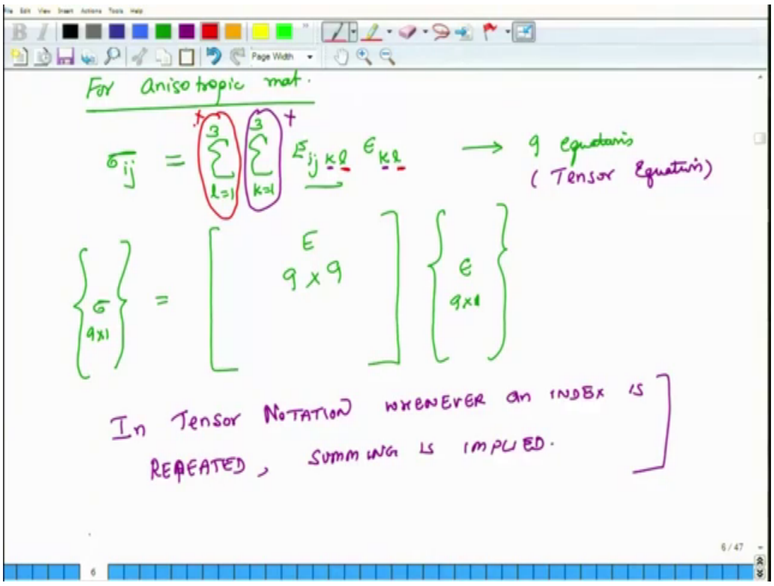

So, in general for an isotropic material can say that sigma ij equals E ij kl is the 4th order tensor times epsilon kl. And then I have to add them up like here when I do this matrix operation what do I do sigma 11 equals B 11 11 times epsilon 11 plus B 11 22 times epsilon 22 plus B 11 12 times epsilon 12. So, I am adding on these second two indices, right.

So, similarly I will add them on the second two indices. So, just to make it clearer E ij kl epsilon kl and I am adding on index k and k can have values from 1 to 3. First axis is x axis y axis z axis, so it can have 3 different values and l can also have values from 1 to 3 ok. So, these are 81. So, this is what how many equations? 9 equations, 9 equations. Why are the 9 equations? Because I can have 9 different values of sigma, sigma 11, sigma 22, sigma 33, sigma 13, sigma 23, sigma 33 and so on and so forth.

So, in other words I can have 9 different equations sigma 11, sigma 22, sigma 33, sigma 12, sigma 23, sigma 31, sigma 21, sigma 32 and so on and so forth. So, we will have 9 different equations and. So, essentially I will have a matrix for sigma a vector for sigma and there will be 9 rows and this will be the E vector E matrix will be a 9 by 9 matrix represented by E ij kl and this will be again the epsilon vector which is the strain tensor vector for strain tensor. So, this is E. So, this is 9 by 1, this is 9 by 1. So, this is the generalized tensor relationship for a fully anisotropic material.

Next we know, so what we do is that we make the writing of this equation a little simpler. And we do that, so in tensors in tensor notation in tensor notation whenever an index is repeated summing is implied this is a convention we follow 4 tensors not for other situations, but wherever we write tensor equations. So, this is a, so this is a tensor equation ok. So, in tensor equation whenever an index is repeated it implies by default summation. So, what do we see? That k is repeated k is in ij kl and k is in epsilon kl ok. What that means, is that we are going to add whether we have this symbol or not because it is a tensor equation it is a on the right side all the entities are tensors on the left side all the entities a tensor it is a tensor equation. So, this symbol is not needed because it is implied by default ok. If it was not repeated then we cannot say that some addition is happening, but if k is repeated then the addition is implied.

Similarly, we see that the index l is also repeated. So, we do not need this explicit symbol for addition also. So, for this reason we can simplify because that makes things simpler and easier to manage that is all for for management purposes only.

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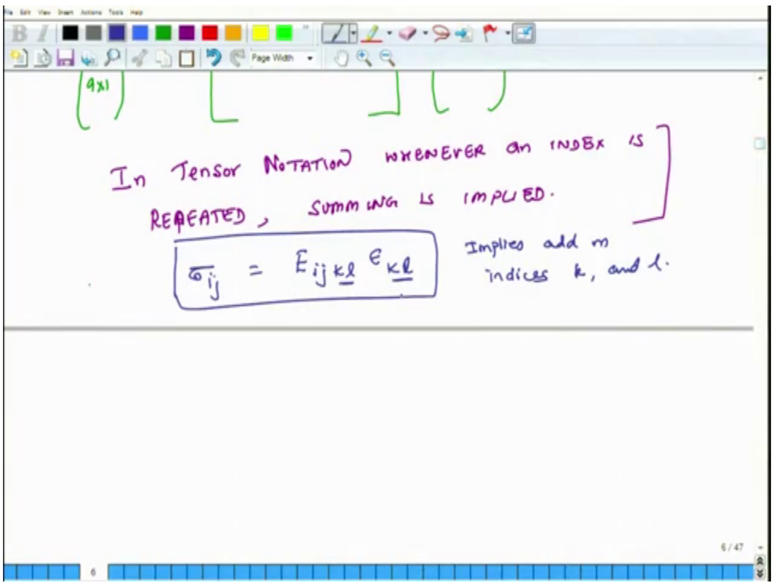

So, sigma ij is equal to E ij kl, epsilon kl and these indices are implied are repeated. So, implies add on indices k and l ok. We do not have to explicitly put these symbols for summation.

So, in books where they use tensors you will always find this kind of a notation and you will not find summation symbol when it comes to tensors because if an index is implied it by default implies if it is repeated then it by default implies that the thing is added up.

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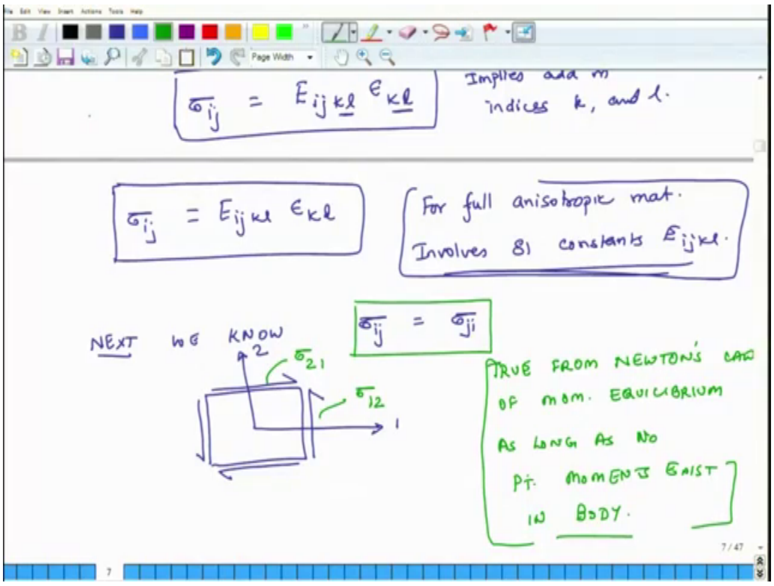

So, so what we have got is the tensor equation for I will once again write at E ij kl, epsilon kl this is the tensor equation for a fully anisotropic material. It is the tensor equation for fully anisotropic material and it involves 81 constants, E ij kl. These constants may be independent or not independent we will talk about that later but we are starting with 81 constants.

So, what that means, is that if we do not know anything else then if I have to characterize an anisotropic material. I have to find its 81 different elastic constants that is what it means. If I have an engine if I have an isotropic material I have to only find 2 1 is E and the other 1 is Poisson's ratio. Because the shear modulus what is it? It is E over 2 times no yeah, E over 1 plus 2 nu, right and so on and so forth.

So, I can express the shear modulus and the bulk modulus of a solid isotropic material in terms of E and nu. It has only two elastic constants independent elastic constants. For anisotropic its more than two much more than two, much more than two and right now we see that it requires 81 constants now all of these are not independent, but it is certainly more than two and we will try to explore how many actually we really need. So, this is there.

Next we know what do we know that sigma ij equals sigma ji ok, what does that mean? What that means, is that if there is a body on which I am applying some shear stress and let us say this is my direction 1, let us say this is direction 2. Then what is this stress this stresses sigma the first index is the direction of the plane direction the plane is 1, second index is direction of the force, force is in 2 direction. So, this is sigma 12 ok.

Now, let us look at this shear stress it is the direction of the plane is in the 2 direction right. So, sigma 2 and the second index is the direction of the force, force is acting in 1 direction, so sigma 21 ok. So, we know that sigma ij equals sigma ji. And why do we know? This is true from Newton's laws of Newton's law of moment equilibrium.

So, if you go back to your solid mechanics books you will see that there is a proof that if we apply Newton's laws of moment equilibrium on this body if the body is in equilibrium then sigma 21 and sigma 12 they have to be equal to each other they have to be identically equal as long as no point moments exist in body, no point exist in body. And in most of the cases this condition is always true. So, if that is true then sigma 12 is always equal to sigma 21, similarly sigma 13 is always equal to sigma 31 and sigma 2 3 is always equal to sigma 32.

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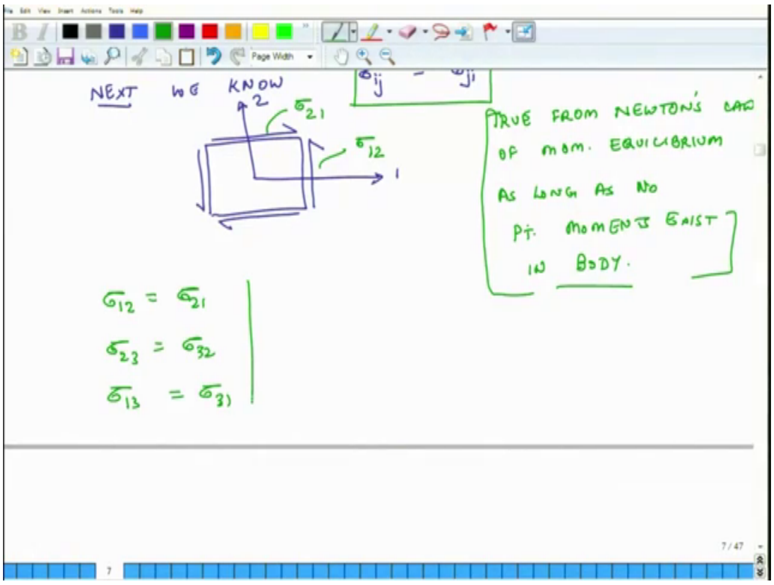

So, what that means, is that sigma 12 equals sigma 21, sigma 23 equals sigma 32, and sigma 13 equals sigma 31, ok. So, with this understanding let us look at. So, what does that mean, ok?

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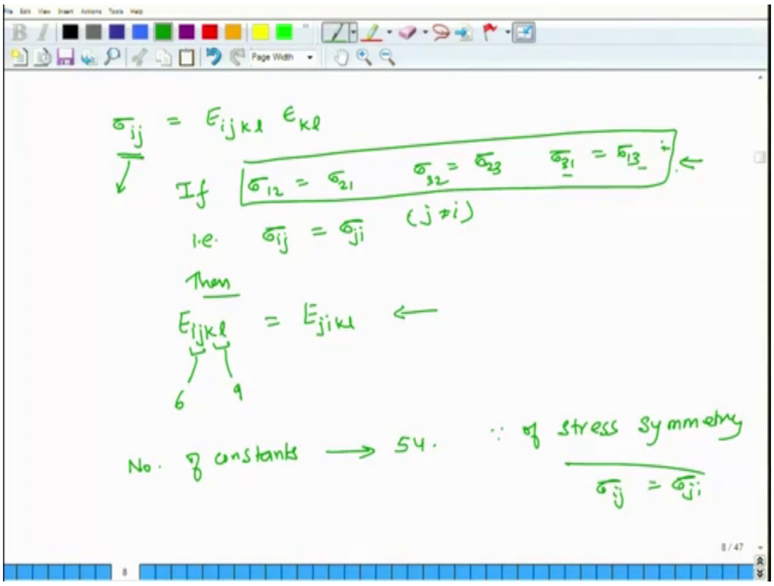

So, this is there. So, our original equation is sigma ij equals E ij kl, epsilon kl and we are summing on indices k and l ok.

Now, if sigma 12 equals sigma 21, sigma 13 equals sigma 23, and sigma 31 equals sigma 13 that is sigma ij is equal to sigma ji, when j is not equal to I when j is not equal to i because sigma 22 31 is equal to 13 12 is equal to ok. So, this is sigma 32 ok. So, what that means, is that if these things are true if these things are true then at the end of the day how many equations we will have. This will be 6 parameters only because now we have only 6 different stresses sigma 11 sigma 22 sigma 33 sigma 12 sigma 23 sigma 31 all the other stresses are just repeated right if we calculate sigma 12 we have already calculated sigma 21 and so on and so forth.

So, in this term E ij kl the total number of independent combinations for, so so what that means, is E ij kl now that can we will be true only if E ij kl is equal to otherwise E ji kl right. So, if this is the case then E ij kl is equal to E ji kl, otherwise these equivalences will not exist, but they have to exist because of Newton's law. So, if they are they are true because of Newton's law then this has to be true because this cannot violate the Newton's law. So, and that if that is the case then what that means, is, so initially indices I and j were collectively responsible for 9 different combinations right, i was responsible for 3 combinations, j was for 3. So, 3 times 3 is 9.

Now, these things become 6 right and this is still 9. So, because of stress so the total number of so because E ij kl is equal to E ji kl number of constants it goes down to 54 it goes down to 54 and this is because of stress symmetry that is sigma ij is equal to sigma ji, ok.

So, this is where we will conclude today we will continue this discussion on elastic constants for an isotropic materials and we will continue this discussion and we will see if we can reduce this number even further down. So, that is all for today and we will meet once again tomorrow.

Thank you.