Basics of Noise and Its Measurements

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Lecture - 13
Transmission Line Equations

Hello, welcome to Basics of Noise and Its Measurements. This is the third week of this particular MOOC course. And I once again welcome all of you to this course. I hope that, the going in this course has been relatively smooth till so far, and you are benefiting from whatever we have been discussing and elaborating upon in this particular area of topic in this particular area. So, what we are going to do this week is essentially solve those pressure and velocity wave equations, and solve them with a purpose because at the end of the week, we will introduce a term called impedance, and the mathematical process underlying this whole solution methodology is important because that will help you understand this whole notion of impedance. And by the time, we close this week, we would have introduced this notion of impedance, and we would have also discussed as to how you can actually physically measure impedance of a material, so that is what we will do and that is how we are going to do this week.

So in this particular lecture, what I am going to talk to you is about something known as transmission line equations. So what do these equations mean, what they mean is that they represent the solution of the velocity and pressure wave equations specifically 1-D pressure wave, and 1-D velocity wave equations. And they are valid for any one-dimensional flow of noise or sound, for instance, if noise or sound is travelling in a duct or a tube of uniform cross section or in a wave guide then these equations are valid and so that is how we are going to start this.

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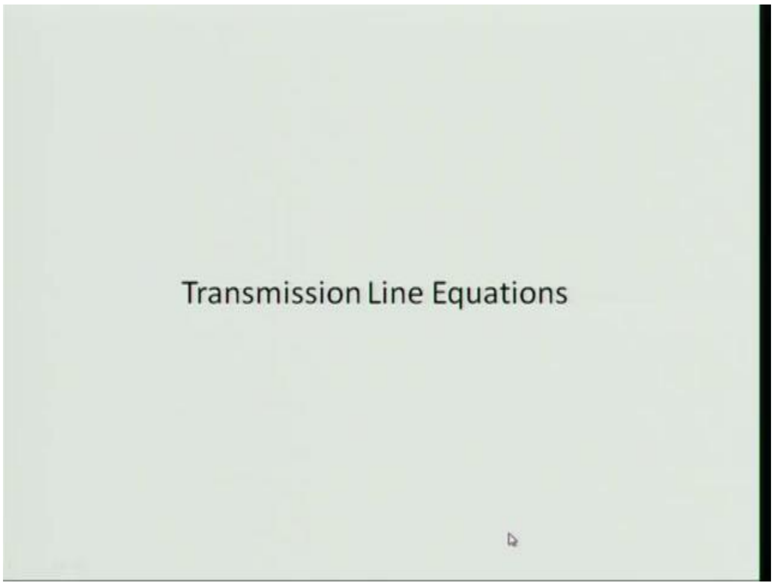

So, in this particular lecture, as I mentioned, we will be discussing the transmission line equations which represent the solution of velocity and pressure wave equations. They are valid for at least in the context of what we will be discussing this week; they will be valid for one-dimensional flow of acoustical energy.

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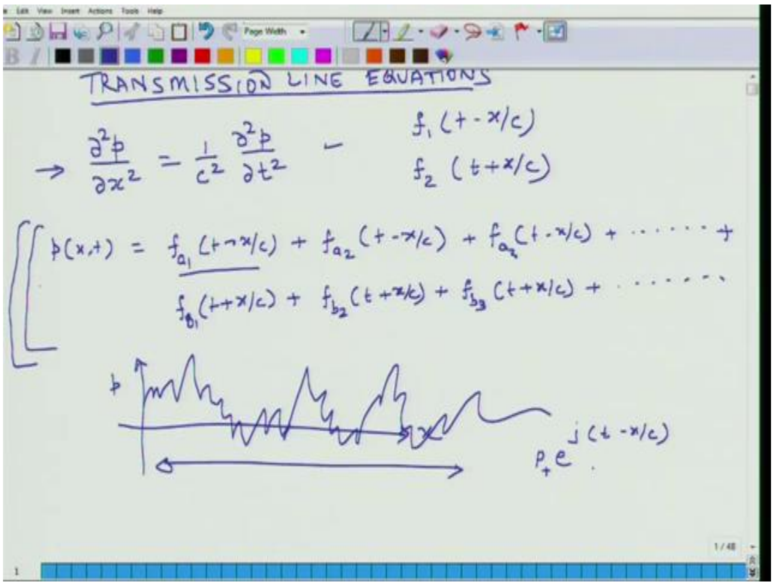

So, we are going to discuss transmission line equations. And once again, they represent the solution of 1-D pressure and velocity wave equations. So for start us, we will write the pressure wave equation partial derivative, second derivative of pressure, which respect to x equals 1 over c square second derivative partial derivative of pressure with respect to time. And in the last week, we had discussed that one possible solution of this wave equation would be f 1 of t minus x over c. Then another class of this solution would be f 2 of t plus x over c. So, if both of these represent the solution then if I add both of these functions also, then the sum of these two functions also would represent a solution for this pressure wave equation.

So, in general, the most general solution for this pressure wave equation, we could write it as p of x and t equals f a 1 t minus x over c plus f a 2 t plus, excuse me minus x over c plus f a 3 t minus x over c, and then I can go on for an extremely larger infinite number of terms. And each of these functions f a 1, f a 2, f a 3, f a 4 and so on and so forth, they represent the sound wave travelling in the positive x-direction. So this is and then I can add some more terms to this solution, so all these terms represent the solution or represent sound waves travelling in positive x-direction, but as we had discussed waves could also be travelling in the negative direction. So then those backward travelling waves could be represented by another series of function f b 1, and which is a function of t plus x over C c f b 2 t plus x over c plus f b 3 t plus x over c and so on and so forth. So, this is an extremely general perhaps, the most general form of the solution which is valid for the pressure wave equation.

Now, if I have sound and let us say this x-axis represents time and y-axis represents pressure then sound pressure could be fluctuating something like this. And what we have learnt in our earlier classes, not in this particular course, but earlier in while we were doing a first or second year of physics or engineering that some of these functions, especially if they are repetitive in nature they can be represented as a sum of sine and cosines. So one possible solution for, or one possible form of f a 1 could be in sinusoidal or a cosine format and then I can add up a series of sine and cosines, and also those also represent the solution for this. Making this more general, I can represent it in exponential format; so in format something like exponential to the power e to the power of j t minus x over c, and some complex function p plus and this plus could be p plus 1, p plus 2, p plus 3 and so on and so forth.

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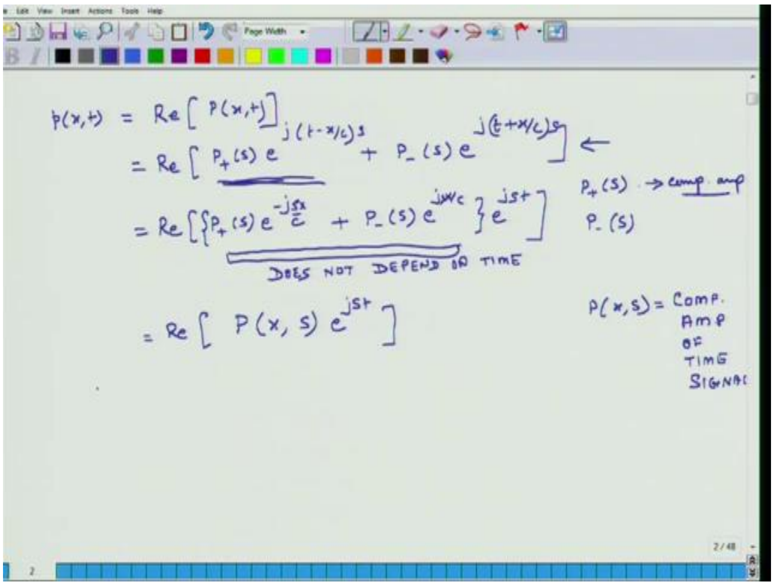

So one possible solution especially if the wave function which we are trying to analyze is repetitive in nature, it could be of this form that pressure is the real component of some complex pressure p of x and t. And where p of x and t equals real of this number p plus which could be a function of s e to the power of j t minus x over c plus p minus s e to the power of j t plus x over c. So, in this solution, there is just one term for forward travelling wave, and one term for wave travelling backwards, but I can make this far more general by having a series of p pluses and p minuses, so I could have p plus 1, p plus 2, p plus 3 and so on and so forth. But for the sake of simplicity, we are just considering one term for forward travelling wave and one term for a wave which is travelling in the backwards direction.

Couple of other things this p plus term, it depends on complex frequency. So it is, in earlier class, we had talked about complex time function right, so this whole thing you know this is something like that, so p plus is the complex amplitude of this whole term and it does not vary with time or x, but it can potentially vary with s, which is the complex frequency. So, this is complex amplitude. Similarly, p minus, it can also potentially depend on complex frequency s. It can also potentially depend on complex frequency s.

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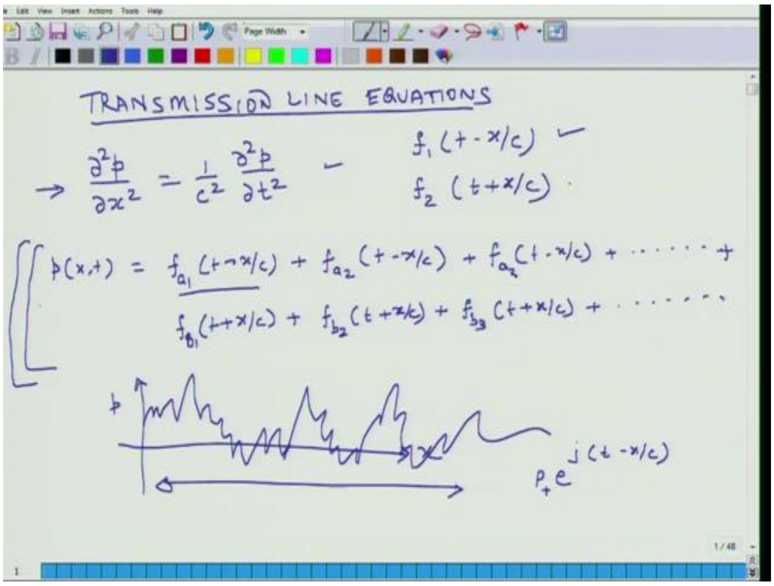

Now, I can rearrange this entire thing as and this function is a valid function for the pressure wave equation, because wherever t and x are coming, they are coming in either t minus x plus c format or t plus x over c format. So this function, we have seen like that it belongs to the same class of functions which we had discussed as f 1 t minus x over c or f 2 t plus x over c. So, this function satisfies the wave equation. So, now what I can do is I can rearrange the terms inside the parentheses, so what I get is e minus j x over c plus p minus s e j x over c, excuse me, e j, I missed something, I should have term's' also in the exponent for e, so it is s t, excuse me, s x over c, and I should have an s here also and e j s t.

So this entire term, it does not depend on time; it is only dependent on frequency and the position of the wave. So this entire term could be rewritten as another complex number, which is a function of x and s. And this p of x and s, it is we can call it now complex amplitude of the time signal. So p plus s was complex amplitude of the forward travelling wave, but p of x and s is the complex amplitude of the time signal, and the value can change from one point to other point, so this complex amplitude to time signal and then e to the power of j s t.

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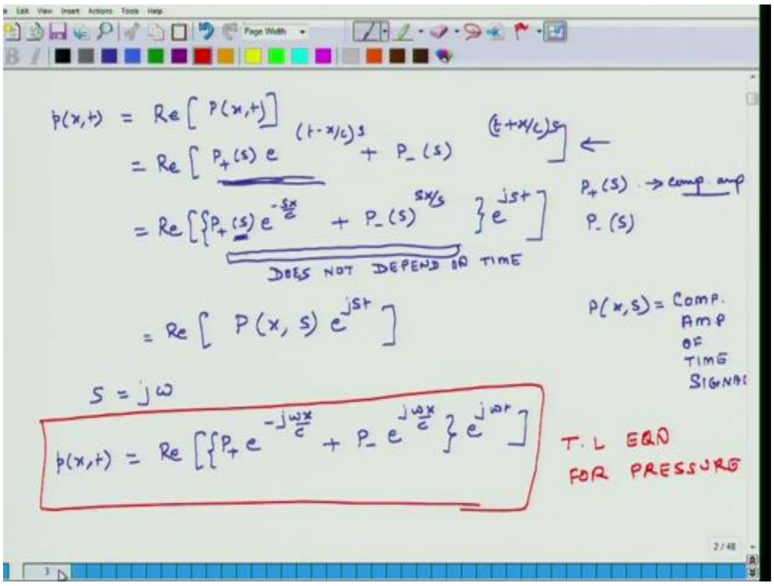

Now, in most of the cases, s is equal to j times. So, I think I made one more error here, s there should not be any j here, so e to the power of s x over because j is embedded in x. So, in most of the regular cases, s equals j omega, s equals j omega. So then I can write it as p of x over t equals real of p plus; and for the sake of brevity, I am going to not write this s term, but we should always realize that p could vary with frequency. But anyway, so we are just going to omit this s term wherever it comes along with p plus just for sake of brevity, but we should always realize once again that p plus can vary with respect to frequency. So then it is real of p plus e to the power of minus j omega x over c plus p minus e j omega x over c. I can put these in brackets, and e j omega t. This equation, which I have bracketed in red, is called the transmission line equation for pressure.

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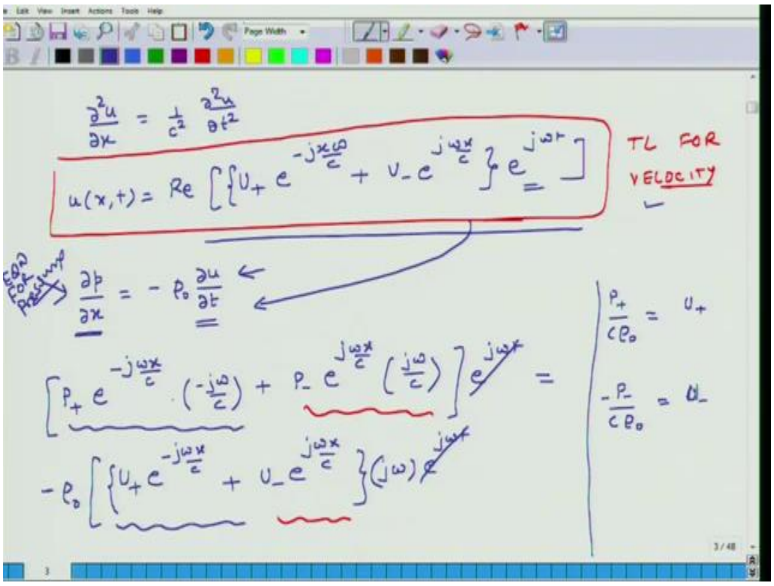

Similarly, I can also have a transmission line equation for velocity. Now, we know that the wave equation for velocity was nothing but second derivative of u with respect to x is equal to 1 over c square second derivative of u with respect to time. And if I do the same math, which I had done for pressure, I will get a transmission line equation for velocity as well and that equation will be u of x and t is equal to real of u plus e minus j x omega over c plus u minus, and this equation is the transmission line equation for velocity. So we have two transmission line equation; one equation it represents a solution for velocity; another equation represents the solution for pressure.

Now, at this stage, they appear independent of each other, but in reality the pressure value of pressure depends on velocity and the value of velocity depends on pressure back and forth. And the relation between these two is governed by the momentum equation. So, what we will do is that we will connect these two equations, transmission line equation for velocity with that of pressure using Newton's second law that is the momentum equation. So, what we will do is that the momentum equation, if you remember states that partial of pressure with respect to x equals minus rho naught partial of u with respect to time. So, what I am going to do is I am going to plug in the left hand side, the solution for pressure using the transmission line equation for pressure, and here I am going to plug in this equation, and here equation for pressure. And using that what I will get is set of relationships between p plus p minus u plus and u minus, and then we will see what those relations tell us, so that is what we are going to do.

So, on the left hand side, we get so I am going to differentiate the transmission line equation for pressure with respect to x, so I get p plus e minus j omega x over c times minus j omega over C plus p minus e j omega x over c times j omega over c. And I am going to omit the notation real from both sides, because we have real on both sides, so whatever is inside the brackets they should be equal. So, this should equal the partial derivative of this entire thing with respect to time multiplied by negative of rho naught. So, this should be equal to minus rho naught and then u plus e minus j omega x over c plus u minus e j omega x over c. And now I am going to differentiate e j omega, this e j omega with respect to t times j omega e j omega t, so e j omega t e j omega t cancel out.

This equation whatever we have left of this equation because Newton's law is valid for all values of time and it is also valid for all values of x. It is a universal law, so it does not matter which position we are. So that can happen only if this term equals this term, and this term equals this term of course, multiplied by j omega and rho naught. So, I will establish these two equivalences so what that means, is p plus by c rho naught should be equal to u plus. So, if I establish this equivalence then this is what I get and if I equalize the things which I have been underlined and red together then I get p minus by c rho naught negative equal u minus.

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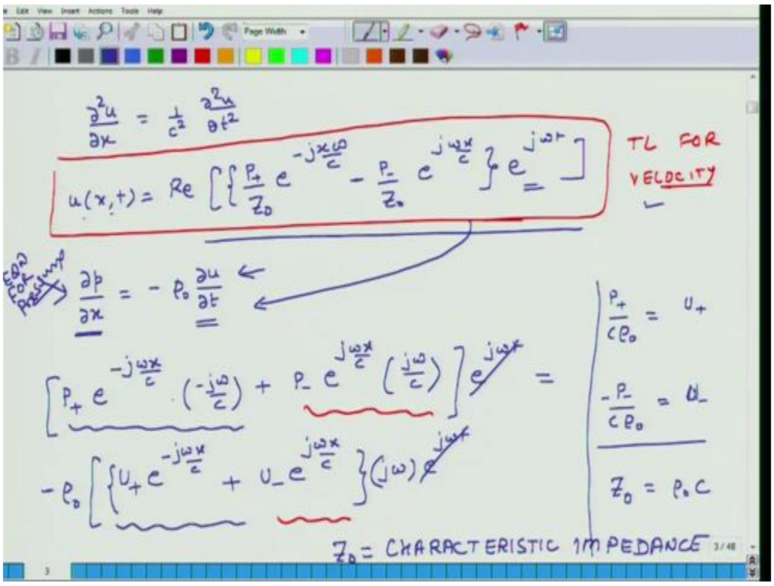

So, once I have this then I can modify this equation which is in red and I can modify it by replacing u plus by p plus by c rho naught, and also by replacing u minus by modifying these two minus of p minus by c rho naught. And then I define an entity called Z naught and I call this Z naught as rho naught c. So I can replace this c rho naught in both the equations by Z naught or Z 0 and the Z naught is called characteristic impedance, it is known as characteristic impedance.

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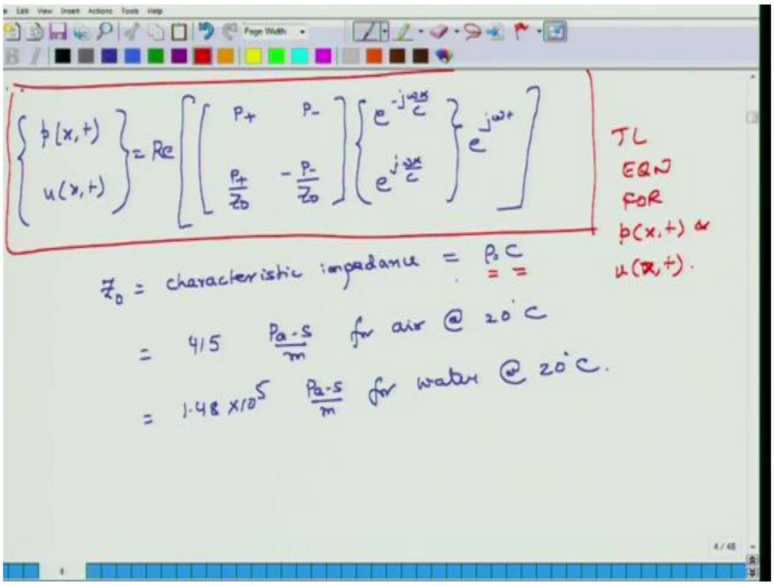

So, finally I get two sets of equations p of x and t, and u of x and t and I am going to write this in matrix form is nothing but real of p plus p minus p plus by Z naught p minus by Z naught e minus j omega x over c e j omega x over c e j omega t. Where Z naught is characteristic impedance and that equals rho naught c. The value of this is 415 Pascal second per meter, for air at 20 degree C and it is equal to 1.48 times 10 to the power of 5 Pascal second per meter for water at 20 degree C. So these are my final, forms of characteristic equation, final form of transmission line equation for p of x and t, and u of x and t.

So, I think this is what we planned to cover in this particular module and next, we will look at several applications of this transmission line equations and learn a little bit more about them. And slowly, we will develop this concept of impedance. So, this is the first time you are hearing this term impedance in context of this course; and this particular impedance, which is rho naught c is known as characteristic impedance. It only depends on the density of the medium and the velocity of sound in the medium. So, it does not depend on the location or time, so that is why it is known as characteristic of the medium and its units are Pascal second per meter. So that is all I wanted to cover today, and have a great day, I look forward to seeing you tomorrow.

Thank you.