Viscous Fluid Flow
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Module - 02
Steady One-dimensional Rectilinear Flows
Lecture - 05
Example Problems

Hello everyone. So, in the last few lectures of this module, we have solved some exact solutions of Navier-Stokes equations. You know the assumptions and how to write the ordinary differential equation from the partial differential equations and invoking the boundary conditions, how you can derive the velocity distribution and the volumetric flow rate.

So, today we will solve some example problems from the knowledge of whatever you have already carried out in this lecture.

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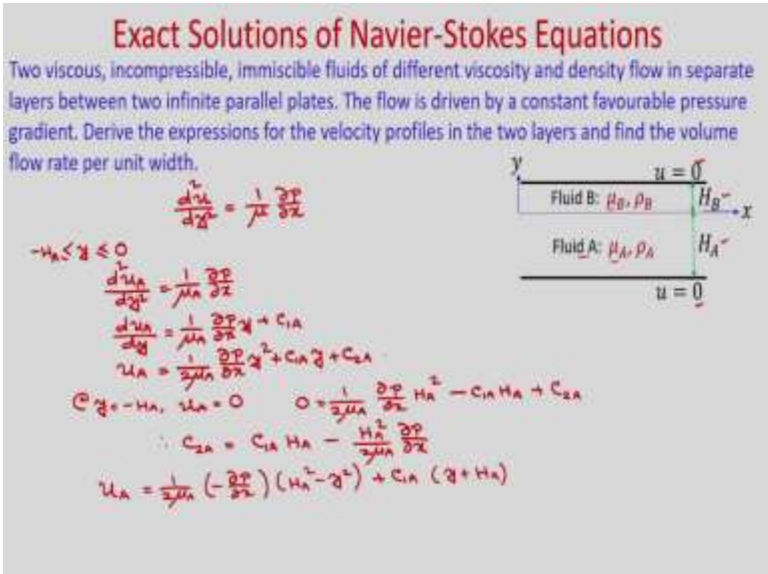

So, let us take the first problem, two viscous incompressible, immiscible fluids of different viscosity and density flow in separate layers between two infinite parallel plates. The flow is driven by a constant favorable pressure gradient. Derive the expressions for the velocity profiles in the two layers and find the volume flow rate per unit width.

This is the problem; so we have two parallel plates; stationary parallel plates, where you have velocity is 0. And two layers; one is fluid A, whose viscosity is μ_A, and fluid B, whose viscosity is μ_B. For convenience, we have taken the axis from this interface of these two layers. So, y is measured from this interface and x is the axial direction and these are the height of these layers H_A and H_B.

So, you can see this is a similar problem which we have already solved in two layers quite flow; but in this particular case, a top boundary is stationary, velocity is 0. And we have taken the axis in the interface of these two layers. So, we will start from the ordinary differential equation.

For this particular case you know, what is the ordinary differential equation, obviously it is a pressure-driven flow. So, you can write the ordinary differential equation neglecting the gravity as

$$\frac{d^2 u_A}{dy^2} = \frac{1}{\mu_A}\frac{\partial p}{\partial x}$$

$\frac{\partial p}{\partial x}$ is a constant rate.

So, for this fluid A layer, you can write the velocity distribution, which is y in between -H to 0. So, this you can write as

$$\frac{du_A}{dy} = \frac{1}{\mu_A}\frac{\partial p}{\partial x}y + C_{1A}$$

And

$$u_A = \frac{1}{2\mu_A}\frac{\partial p}{\partial x}y^2 + C_{1A}y + C_{2A}$$

So, we have the boundary conditions at y is equal to $-H_A$, u is equal to 0. So, you can see if you put it here, you will get

$$0 = \frac{1}{2\mu_A}\frac{\partial p}{\partial x}H_A^2 - C_{1A}H_A + C_{2A}$$

So, you can express it as

$$C_{2A} = C_{1A}H_A - \frac{H_A^2}{2\mu_A}\frac{\partial p}{\partial x}$$

So now, if you put it in this expression of this velocity distribution in the fluid a domain; so you can write

$$u_A = \frac{1}{2\mu_A}\left(-\frac{\partial p}{\partial x}\right)(H_A^2 - y^2) + C_{1A}(y + H_A)$$

Now, similarly let us find the velocity distribution in layer fluid B.

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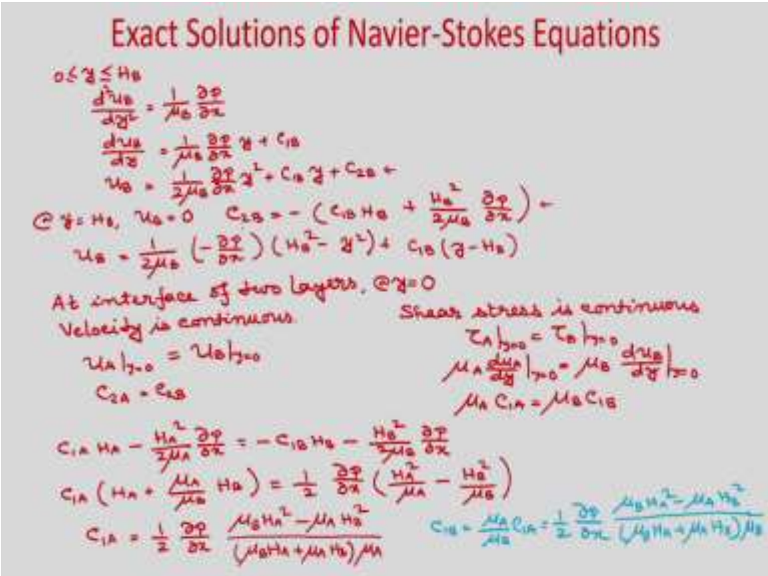

So, in the range of y 0 to H_B, we have the equation

$$\frac{d^2u_B}{dy^2} = \frac{1}{\mu_B}\frac{\partial p}{\partial x}$$

So, integrating twice, you will get

$$\frac{du_B}{dy} = \frac{1}{\mu_B}\frac{\partial p}{\partial x}y + C_{1B}$$

$$u_B = \frac{1}{2\mu_B}\frac{\partial p}{\partial x}y^2 + C_{1B}y + C_{2B}$$

Again we have the boundary condition at y is equal to H_B, u_B is equal to 0. So, from this expression, you can write

$$C_{2B} = -\left(C_{1B}H_B + \frac{H_B^2}{2\mu_B}\frac{\partial p}{\partial x}\right)$$

Then you can write

$$u_B = \frac{1}{2\mu_B}\left(-\frac{\partial p}{\partial x}\right)(H_B^2 - y^2) + C_{1B}(y - H_B)$$

Now, we need two boundary conditions. For this, we will use the interface conditions. So, at the interface you know that velocity is continuous as well as shear stress is continuous. So, from these two conditions, we will be able to find these two constants. So, at the interface of two layers ok, velocity is continuous and shear stress is also continuous.

So, if velocity is continuous, then you can write. So, this is at y is equal to 0 right; at the interface means, at y is equal to 0. So, we can write

$$u_A|_{y=0} = u_B|_{y=0}$$

So, at y is equal to 0; you will get C_{2A} is equals to C_{2B}. And if shear stress is continuous, then you can write

$$\tau_A|_{y=0} = \tau_B|_{y=0}$$

$$\mu_A \frac{du_A}{dy}\bigg|_{y=0} = \mu_B \frac{du_B}{dy}\bigg|_{y=0}$$

$$\mu_A C_{1A} = \mu_B C_{1B}$$

$$C_{1A}H_A - \frac{H_A^2}{2\mu_A}\frac{\partial p}{\partial x} = -C_{1B}H_B - \frac{H_B^2}{2\mu_B}\frac{\partial p}{\partial x}$$

So, now you can write

$$C_{1A}\left(H_A - \frac{\mu_A}{\mu_B}H_B\right) = \frac{1}{2}\frac{\partial p}{\partial x}\left(\frac{H_A^2}{\mu_A} - \frac{H_B^2}{\mu_B}\right)$$

$$C_{1A} = \frac{1}{2}\frac{\partial p}{\partial x}\frac{\mu_B H_A^2 - \mu_A H_B^2}{(\mu_B H_A + \mu_A H_B)\mu_A}$$

$$C_{1B} = \frac{\mu_A}{\mu_B} C_{1A} = \frac{1}{2}\frac{\partial p}{\partial x}\frac{\mu_B H_A^2 - \mu_A H_B^2}{(\mu_B H_A + \mu_A H_B)\mu_B}$$

And if you put in the corresponding velocity distribution, you will get the velocity distribution in two-fluid layers.

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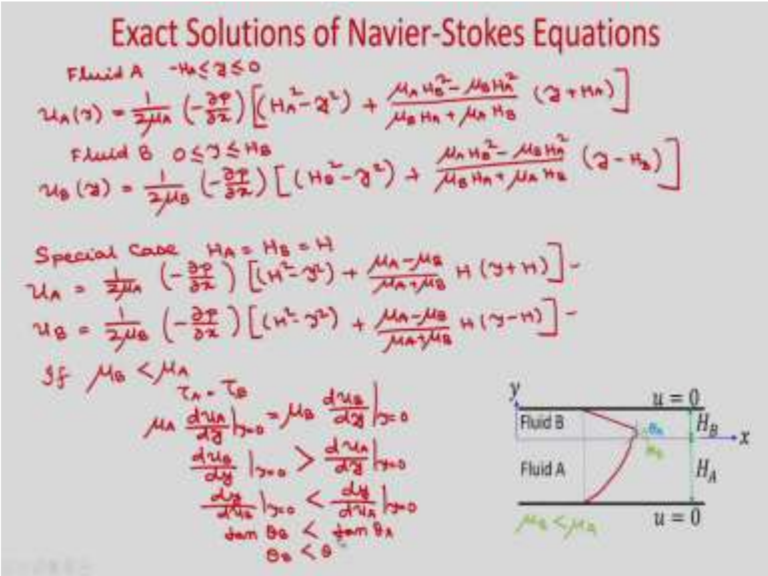

So, after putting the expression of these constants in the velocity distribution, let us write the final velocity distribution in two different layers. So, in fluid A domain, where y varies from -H_A to 0; fluid velocity is u_A which is

$$u_A(y) = \frac{1}{2\mu_A}\left(-\frac{\partial p}{\partial x}\right)\left[(H_A^2 - y^2) + \frac{\mu_A H_A^2 - \mu_B H_B^2}{\mu_B H_A + \mu_A H_B}(y + H_A)\right]$$

Similarly, in fluid B layer, where y varies between 0 and H_B; we can write the velocity distribution u_B as

$$u_B(y) = \frac{1}{2\mu_B}\left(-\frac{\partial p}{\partial x}\right)\left[(H_B^2 - y^2) + \frac{\mu_A H_B^2 - \mu_B H_A^2}{\mu_B H_A + \mu_A H_B}(y - H_B)\right]$$

So, now, you can see here, what velocity distribution we are found; now at the interface obviously, the gradient will change and at the interface the velocity is continuous as well as shear stress is continuous. So, how will draw the velocity profile? So, here you can see that obviously for this particular case you can see that velocity distribution will look like this.

So, some gradient will be there at the interface. So, this will be θ_A, ok. And at the wall, it is 0. And if you draw from in fluid B; so there will be some velocity distribution maybe like this, the velocity distribution will look like this and here you can see the tangent will be this one and this is θ_B.

So, obviously it depends on how the velocity will look like depending on θ_A is greater than θ_B or θ_A is less than θ_B; it depends on the viscosity μ_A and μ_B. So, if μ_B is less than μ_A, obviously you may get the velocity distribution like this.

Now, let us take a special case, where you have the same height of the two layers; that means H_A is equal to H_B. So, in this particular case, special case, where H_A is equal to H_B is equal to H; then you can write the velocity profile

$$u_A(y) = \frac{1}{2\mu_A}\left(-\frac{\partial p}{\partial x}\right)\left[(H^2 - y^2) + \frac{\mu_A - \mu_B}{\mu_A + \mu_B} H(y + H_A)\right]$$

$$u_B(y) = \frac{1}{2\mu_B}\left(-\frac{\partial p}{\partial x}\right)\left[(H^2 - y^2) + \frac{\mu_A - \mu_B}{\mu_A + \mu_B} H(y + H_A)\right]$$

So, you can see that the velocity distribution is given for the equal height of the layers. Now, if μ_B is less than μ_A. Then at the interface, shear stress is continuous.

So, τ_A is equal to τ_B. So,

$$\mu_A \left.\frac{du_A}{dy}\right|_{y=0} = \mu_B \left.\frac{du_B}{dy}\right|_{y=0}$$

So, now, you can see from here that, if μ_B is less than μ_A; then obviously

$$\left.\frac{du_A}{dy}\right|_{y=0} > \left.\frac{du_B}{dy}\right|_{y=0}$$

So, from here you can see

$$\left.\frac{dy}{du_B}\right|_{y=0} < \left.\frac{dy}{du_A}\right|_{y=0}$$

So, this is the representation at that interface of the gradient, so that is nothing but tan θ_B is less than tan θ_A. So, hence θ_B is less than θ_A. So, obviously, if μ_B is less than μ_A; velocity distribution will look like this, such that at the interface θ_B will be less than θ_A. So, now, let us find the volume flow rate per unit width.

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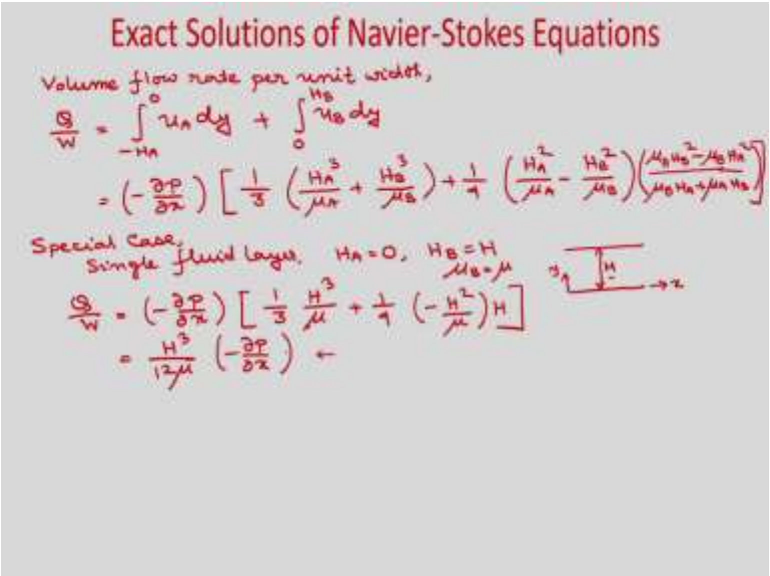

So, we know the velocity distribution. So, we can integrate it at a particular cross-section and we can find the volume flow rate per unit width. So,

$$= u_{av}A = \int_A u\, dA$$

$$\frac{Q}{W} = \int_{-H_A}^{0} u_A dy + \int_{0}^{H_B} u_B dy$$

So, if you put the velocity distribution here u_A and u_B and if you do the integration and putting the limits, you will get

$$= \left(-\frac{\partial p}{\partial x}\right)\left[\frac{1}{3}\left(\frac{H_A^3}{\mu_A} + \frac{H_B^3}{\mu_B}\right) + \frac{1}{4}\left(\frac{H_A^2}{\mu_A} - \frac{H_B^2}{\mu_B}\right)\left(\frac{\mu_A H_B^2 - \mu_B H_A^2}{\mu_B H_A + \mu_A H_B}\right)\right]$$

Now, if you take as a special case, where you have a single fluid layer, let us say H_A is equal to H; then obviously you will get the volume flow rate same as for the plane Poiseuille flow, where the distance between two parallel plates is H. So, a special case where you have a single fluid layer, where we can put H_A is equal to 0 and H_B is equal to H.

Then you will get a single fluid layer and it is a representation of plane Poiseuille flow, where y is measured from the bottom plate and x is the axial direction and this is H. So, if you put H_A is equal to 0 here and H_B is equal to H, then

$$\frac{Q}{W} = \left(-\frac{\partial p}{\partial x}\right)\left[\frac{1}{3}\frac{H^3}{\mu} + \frac{1}{4}\left(-\frac{H^2}{\mu}\right)H\right]$$

$$= \frac{H^3}{12\mu}\left(-\frac{\partial p}{\partial x}\right)$$

So, you can see, this is the same expression as you will get for the plane Poiseuille flow, where the distance between two parallel plates is H, and it is $\frac{H^3}{12\mu}\left(-\frac{\partial p}{\partial x}\right)$.

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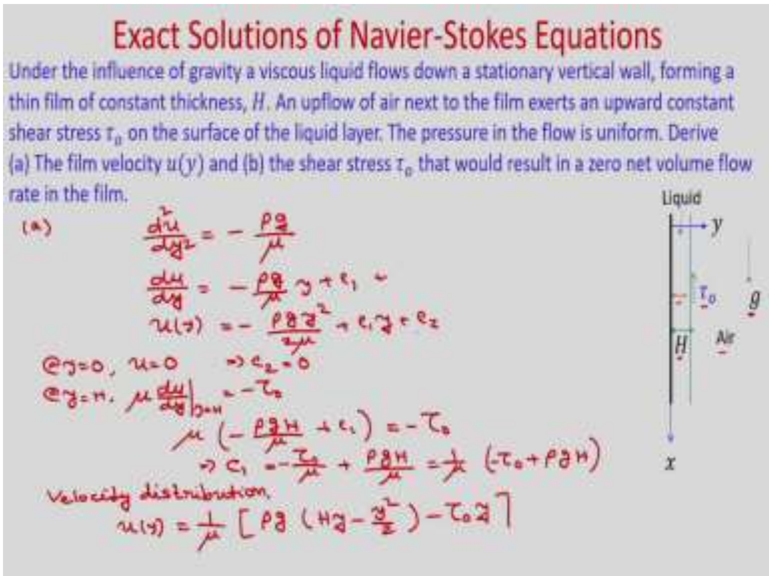

Now, let us take another example problem. Under the influence of gravity, a viscous liquid flows down a stationary vertical wall, forming a thin film of constant thickness, H. An up-flow of air next to the film exerts an upward constant shear stress τ_o on the surface of the liquid layer. The pressure in the flow is uniform. Derive a the film velocity u as a function of y and b the shear stress τ_o that would result in a zero net volume flow rate in the film.

So, here you can see, this is the thin liquid film of thickness H and x is in the downward direction, y is measured perpendicular to this vertical plate. And in the airside, it is told in the question that, it exerts upward constant shear stress is τ_o. So, at the interface, the τ_o is acting in the upward direction and gravity obviously is acting in the positive x-direction in this case. So, this is g.

So, we have to find the velocity distribution inside the film as well as the shear stress τ_o that would result in a zero net volume flow rate in the film. So, you have to find the τ_o value, such that at any location in the liquid, net volume flow rate will be 0. So, we will start from the ordinary differential equation; you can see in this particular case, it is gravity-driven flow, there is no imposed pressure gradient.

And what are the boundary conditions, at y is equal to 0, it is a stationary plate; so u is equal to 0 and at the interface, you have the shear stress which is actually equal to $-\tau_o$. At the fluid layer, it will be equal to minus the imposed shear stress from the air side.

So, in this particular case, it is a balance between the viscous force and the gravity force. So, it will be

$$\frac{d^2u}{dy^2} = -\frac{\rho g}{\mu}$$

$$\frac{du}{dx} = -\frac{\rho g}{\mu}y + C_1$$

$$u(y) = -\frac{\rho g y^2}{2\mu} + C_1 y + C_2$$

At y is equal to 0, these are the boundary condition; at u is equal to 0, so that will give C_2 is equal to 0.

And at y is equal to H, the shear stress in the fluid side will be equal to $-\tau_o$. So, we can write

$$\mu \frac{du}{dy}\bigg|_{y=H} = -\tau_o$$

So,

$$\mu\left(-\frac{\rho g H}{\mu} + C_1\right) = -\tau_o$$

So,

$$C_1 = -\frac{\tau_o}{\mu} + \frac{\rho g H}{\mu} = \frac{1}{\mu}(-\tau_o + \rho g H)$$

So, final velocity distribution,

$$u(y) = \frac{1}{\mu}\left[\rho g\left(\frac{H^3}{2} - \frac{H^3}{6}\right) - \frac{\tau_o H^2}{2}\right]$$

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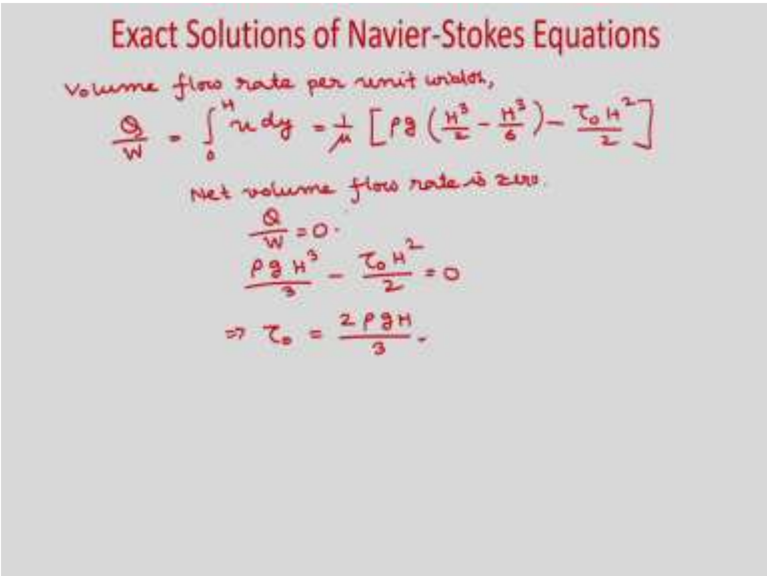

So, this is the velocity distribution. Now, let us calculate the volume flow rate first; then we will put the volume flow rate at 0 and we will find the value of shear stress τ_o. So, the volume flow rate per unit width Q/W we can write as

$$Q = 0$$

$$\frac{\rho g H^3}{3} - \frac{\tau_o H^2}{2} = 0$$

$$\tau_o = \frac{2\rho g H}{3}$$

Due to gravity there will be a viscous force and that viscous force will be balanced by the shear stress on the wall and the net volume flow rate will be 0. Now, let us consider the next problem.

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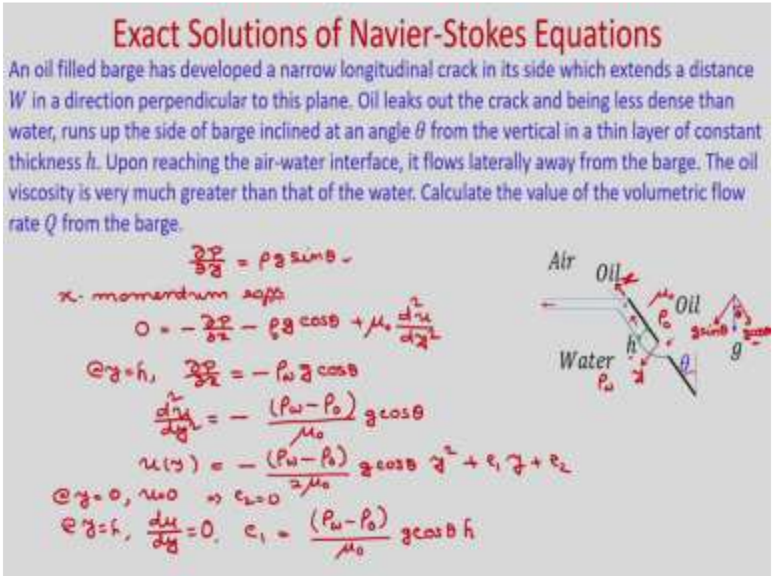

An oil-filled barge has developed a narrow longitudinal crack in its side which extends a distance W in a direction perpendicular to this plane. Oil leaks out the crack and being less dense than water runs up the side of the barge inclined at an angle θ from the vertical in a thin layer of constant thickness h.

Upon reaching the air-water interface, it flows laterally away from the barge. The oil viscosity is very much greater than that of the water. Calculate the value of volume flow rate Q from the barge. So, you can see this is the oil-filled barge; so there is a crack and from this crack, this oil is flowing inside. And as water is here, it is forming a thin film of thickness h and this oil is going up and here the air is there; so it is flowing over this water in this direction.

So, this plate you can see, it is having this angle θ with the vertical direction. So, we need to find what is the Q or volume flow rate of this oil, and the gravity is in this direction. So, you can see, we can take the x-axis along this direction, and y we can measure perpendicular to this direction.

So, we will have two components of this g. So, one is this direction and another is in this direction. So, obviously you can see, this will be θ and you will get gcosθ in the negative x-direction and in the positive y-direction you will get gsinθ. Now, obviously from the y component momentum equation, you can write

$$\frac{\partial p}{\partial x} = \rho g \sin\theta$$

This is the hydrostatic pressure.

From the x momentum equation, you will get

$$0 = -\frac{dp}{dx} - \rho g \cos\theta + \mu \frac{d^2u}{dy^2}$$

Now, if you consider at y is equal to H at the interface; so obviously you can see that you will get water is stationery. And at this, you will get

$$\frac{\partial p}{\partial x} = -\rho_w g \cos\theta$$

So, we can denote ρ_a, ρ_o as oil density and ρ_w oil water and μ is the oil viscosity. So, from here you can see this we can denote as ρ_w. So, from here if $\frac{\partial p}{\partial x}$ is acting at the interface and it will be imposed inside; because $\frac{\partial p}{\partial y}$ is just constant, so it will be just imposed inside. So, $\frac{\partial p}{\partial x}$ will be the same inside the oil this, inside the thin film.

So, obviously $\frac{\partial p}{\partial x}$ value we can put it here as $-\rho_w g\cos\theta$. So, we can write

$$\frac{d^2u}{dy^2} = -\frac{\rho_w - \rho_o}{\mu_o} g \cos\theta$$

So, now this ordinary differential equation, you integrate twice; find the velocity distribution inside the thin film and from there, we can calculate the volume flow rate. So, if you integrate twice, you will get velocity distribution

$$u(y) = -\frac{(\rho_w - \rho_o)}{2\mu_o} g \cos\theta \, y^2 + C_1 y + C_2$$

And boundary conditions are at y is equal to 0, it is a stationary plate, so u is equal to 0.

So, from here you will get C_2 is equal to 0. And at y is equal to H; you can see here it is written that the oil viscosity is very much greater than that of the water. So, using that and at the interface obviously, the shear stress du/dy will be 0; that we have already discussed earlier. So, du/dy is 0; so

$$C_1 = \frac{(\rho_w - \rho_o)}{\mu_o} g \cos\theta \;\, h$$

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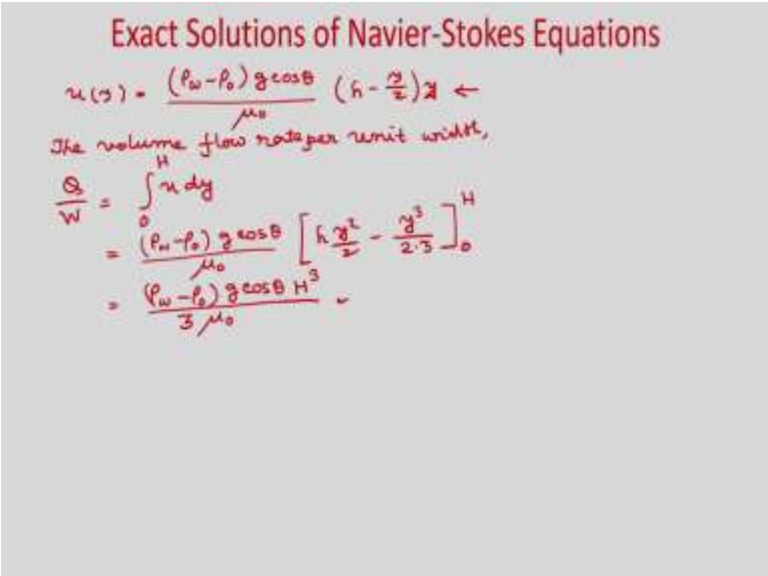

So, now if you put these constants in this expression, we will get

$$u(y) = \frac{(\rho_w - \rho_o) g \cos\theta}{\mu_o} \left(h - \frac{y}{2}\right)$$

Now,

$$\frac{Q}{W} = \int_0^H u dy$$

$$= \frac{(\rho_w - \rho_o) g \cos\theta}{\mu_o} \left[h\frac{y^2}{2} - \frac{y^3}{2.3} \right]_0^H$$

$$= \frac{(\rho_w - \rho_o) g \cos\theta \; H^3}{3\mu_o}$$

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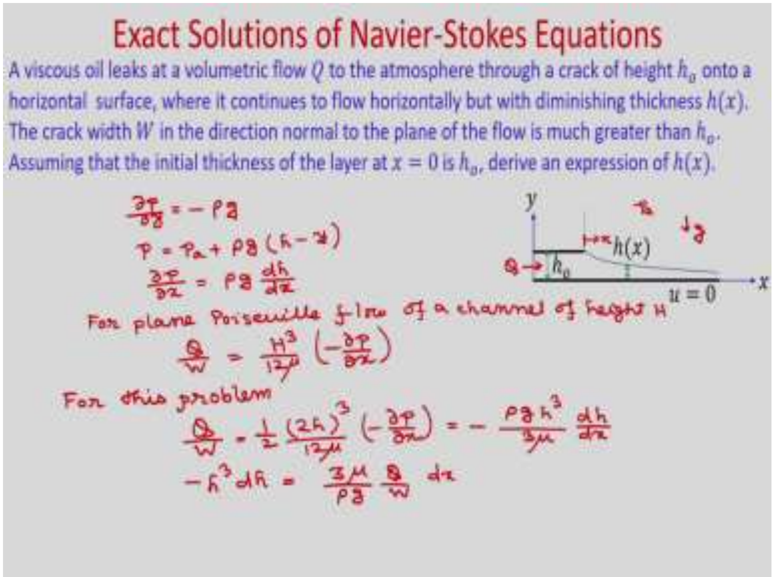

Now, let us consider another problem, viscous oil leaks at a volumetric flow Q to the atmosphere through a crack of height h_o onto a horizontal surface, where it continues to flow horizontally, but with diminishing thickness h which is a function of x. The crack width W in the direction normal to the plane of the flow is much greater than h_o. Assuming that the initial thickness of the layer at x equal to 0 is equal to h_o, derive an expression of h(x).

So, you can see a viscous oil it is leaking into the atmosphere through this crack. And the initial height of this liquid is h_o and y is measured from the bottom and x is in this axial direction and obviously, the plate is stationary. So, u is equal to 0; but the thickness is a function of x from here.

So, you can see, if you measure x from here. So, at x equal to 0, you can put h is equal to h_o. So, h is measured from this place. So, at the interface how you can see the flow as half of the plane Poiseuille flow. So, if you have a plane Poiseuille flow of height h; then obviously at h/2 this is the interface and the shear stress we can put as 0.

So, we have a constant pressure gradient. So,

$$\frac{\partial p}{\partial y} = -\rho g$$

So, g is acting in this direction and the pressure if you integrate it, you will get atmospheric pressure. So, this is your p_a, then you can write

$$p = p_a + \rho g(h - y)$$

So, from here you can see that

$$\frac{\partial p}{\partial x} = \rho g \frac{dh}{dx}$$

So, the velocity distribution in this layer will be the same as that in the lower half of the plane Poiseuille flow in a channel of height 2h, where there is 0 shear stress at the mid-distance h above the lower surface, ok. So, we can write for plane Poiseuille flow, we know the volume flow rate, plane Poiseuille flow of a channel of height H.

So, we know that

$$\frac{Q}{W} = \frac{H^3}{12\mu}\left(-\frac{\partial p}{\partial x}\right)$$

So, now, this thickness we are considering as half-width of the or half-thickness of the plane Poiseuille flow. So, obviously in this case then H will be of 2H and volume flow rate, obviously, it is it will be half.

So, for this problem, we can write

$$\frac{Q}{W} = \frac{1}{2}\frac{(2h)^3}{12\mu}\left(-\frac{\partial p}{\partial x}\right) = -\frac{\rho g h^3}{3\mu}\frac{dh}{dx}$$

So, h is function of x, so dh/dx. So, now using this expression, you can integrate it and after putting this boundary condition at x is equal to 0, the height is h_o.

So, you can find the, this height in as a function of x. If you rearrange it, so you will get

$$-h^3\,dh = \frac{3\mu}{\rho g}\frac{Q}{w}\,dx$$

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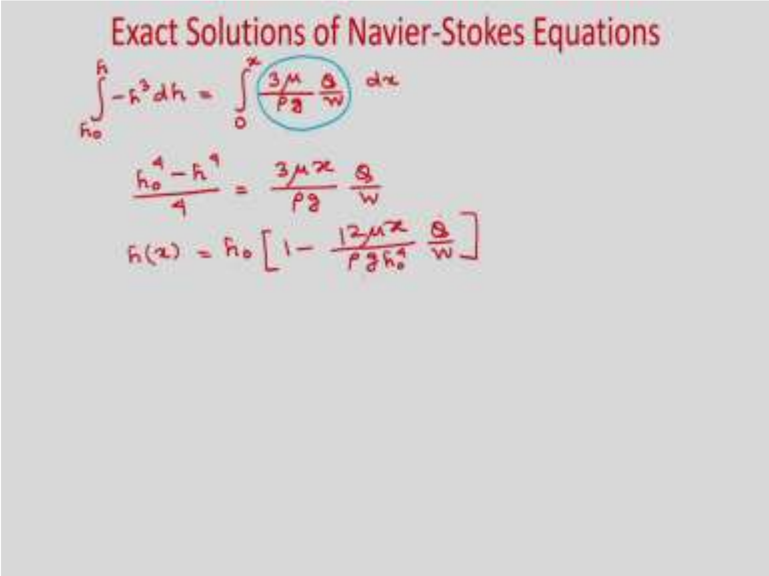

Now, if you integrate it

$$\int_{h_o}^{h} -h^3\,dh = \int_0^x \frac{3\mu}{\rho g}\frac{Q}{w}\,dx$$

$$\frac{h_o^4 - h^4}{4} = \frac{3\mu x}{\rho g}\frac{Q}{w}$$

$$h(x) = h_o\left[1 - \frac{12\mu x}{\rho g h_o^4}\frac{Q}{w}\right]$$

So, in today's lecture, we have solved several problems, where we used the knowledge of this exact solution of Navier Stokes equations, which we have already learned in this module. We have seen how the engineering problems can be solved using the knowledge of this Viscous Fluid Flow. You can solve several problems from any viscous fluid flow book, which we have already referred to in this course.

Thank you.