Viscous Fluid Flow
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Module - 02
Steady One-dimensional Rectilinear Flows
Lecture - 02
Plane Poiseuille Flow

Hello everyone, so today we will study Plane Poiseuille Flow. What is plane Poiseuille flow? Plane Poiseuille flow is the flow when a fluid is forced to flow between 2 infinite parallel stationary flat plates under constant pressure gradient and 0 gravity. In the last class we have already studied plane Coquette flow and we know how we get the fully developed flow.

So, we will have the assumptions of fully developed flow in the case of plane Poiseuille flow and we will derive the velocity distribution, shear stress distribution and volume flow rate per unit width.

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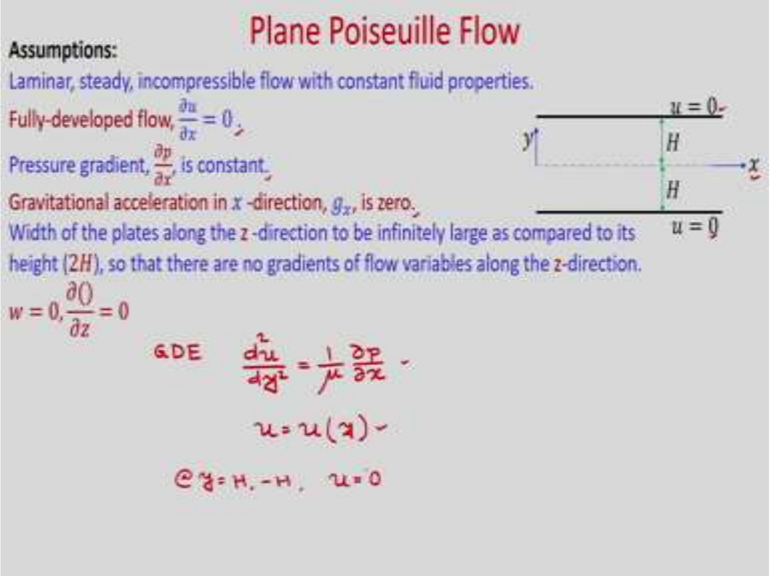

So, let us consider two parallel plates these are stationary, which means velocity is 0 and the distance between two parallel plates is 2H, we have taken the axis in the mid of this plane so this is your center line and x is measured in the axial direction and y is measured from the center line.

We have taken this coordinate here for convenient, because we know that the velocity distribution is symmetric about the center line. So, what are the assumptions we are taking? We are assuming laminar steady incompressible flow with constant fluid properties, fully developed flow that means $\frac{\partial u}{\partial x}$ is 0, pressure gradient is constant gravitational acceleration in x direction is 0 and we are taking width of the plates along the z direction to be infinitely large as compared to its height 2 H.

So, that there are no gradients of flow variables along the z direction. That means, in z direction W velocity is 0 and $\frac{\partial ()}{\partial z}$ of any flow variables is 0, so obviously as z direction is infinite. So, end effects are neglected and we know for the fully developed flow condition that from continuity equation putting $\frac{\partial u}{\partial x}$ is 0 and W as 0; you can so that b is equal to 0 everywhere.

So, with these assumptions you can write the Navier Stokes equations, specially the x momentum equation as

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu}\frac{\partial p}{\partial x}$$

So, we know that u is a function of y only right. So, it is one-dimensional flow because we are considering fully developed flow and u is a function of y only.

So, if you integrate twice you will be able to find the velocity distribution with proper boundary conditions. What are the boundary conditions? So, you can see from here that at y is called to H as well as at minus H you have velocity 0.

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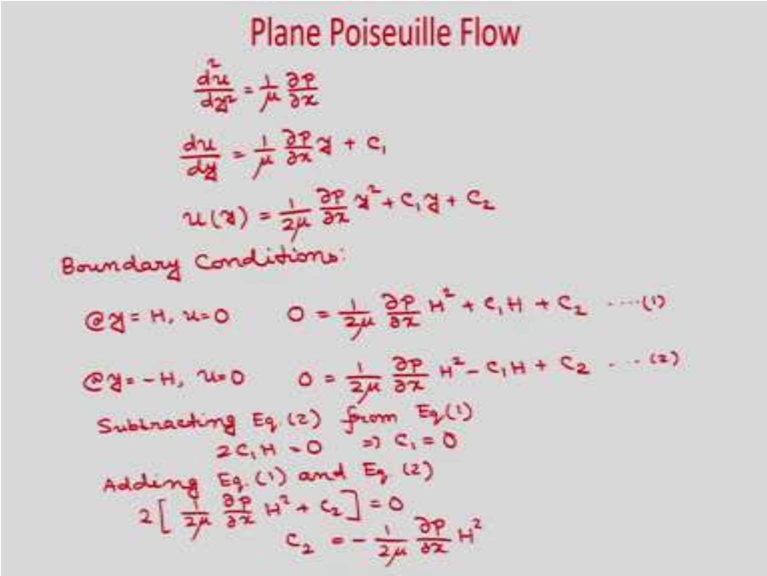

So, let us find the velocity distribution from starting from this ordinary differential equation. So, our ordinary differential equation is

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu}\frac{\partial p}{\partial x}$$

So, if you integrate this equation you will get

$$\frac{du}{dy} = \frac{1}{\mu}\frac{\partial p}{\partial x}y + C_1$$

Again if you integrate then you will get

$$u(y) = \frac{1}{2\mu}\frac{\partial p}{\partial x}y^2 + C_1 y + C_2$$

So now, let us invoke those 2 boundary conditions because there are 2 unknowns so we need 2 boundary conditions and find the integration constants C_1 and C_2.

So, boundary conditions are at y is equal to H. So, you can see this will be 0 right velocity is 0, u is 0, so you will get

$$0 = \frac{1}{2\mu}\frac{\partial p}{\partial x}H^2 + C_1 H + C_2$$

And if you invoke the other boundary condition at the rate of y is equal to –H, u is equal to 0. So, you will get

$$0 = \frac{1}{2\mu}\frac{\partial p}{\partial x}H^2 - C_1 H + C_2$$

So if this equation is 1 and if this equation is 2. So, if you subtract these two equations. So, if you subtract equation 2 from 1 then you will get, so subtracting equation 2 from equation 1. So, if you subtract you can see so these two1 terms are same, so it will get cancelled C 2 will get cancelled you will get $2C_1H$ is equal to 0, that means C_1 is equal to 0. If you add these 2 equations so you will get adding equation 1 and equation 2. What you will get? So, you can see you will get

$$2\left[\frac{1}{2\mu}\frac{\partial p}{\partial x}H^2 + C_2\right] = 0$$

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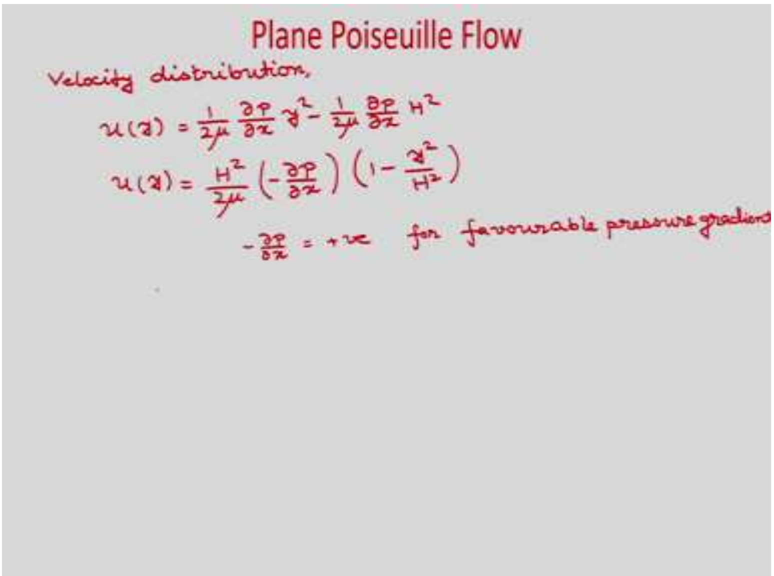

Now, let us put these two integration constants C_1, C_2 in this equation and get the velocity profile, so your velocity distribution. What is your velocity distribution after invoking the constants? So,

$$u(y) = \frac{1}{2\mu}\frac{\partial p}{\partial x}y^2 - \frac{1}{2\mu}\frac{\partial p}{\partial x}H^2$$

So, if you rearrange you will get

$$= \frac{H^2}{2\mu}\left(-\frac{\partial p}{\partial x}\right)\left(1-\frac{y^2}{H^2}\right)$$

So, this is your velocity distribution. So, you know that in this particular case $\frac{\partial p}{\partial x}$ is constant and we have favourable pressure gradient. So, if it is favourable pressure gradient, then your flow will occur in the positive x direction.

So, in this case $-\frac{\partial p}{\partial x}$ is a positive quantity right. So, in this case $-\frac{\partial p}{\partial x}$ is a positive quantity for favourable pressure gradient. So now we can see this velocity distribution is parabolic ok. Now, if you want to calculate the mean velocity, so first we will calculate the volumetric flow rate.

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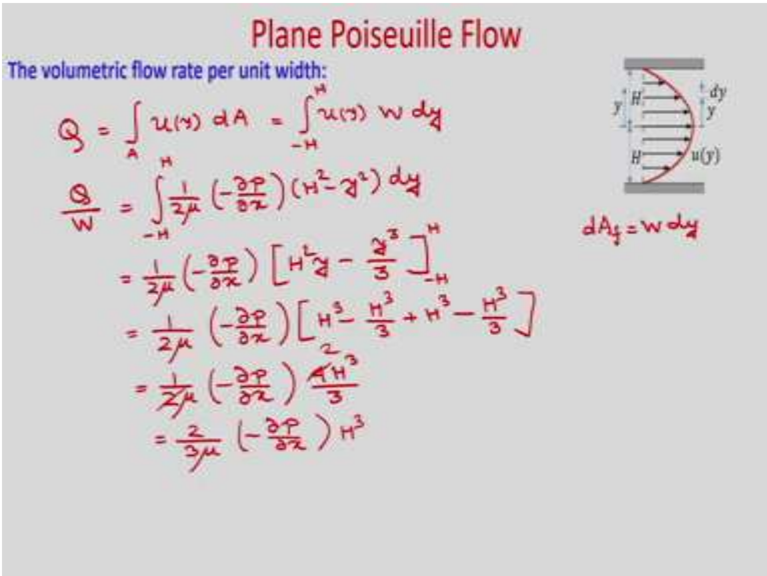

So, first, let us calculate the volumetric flow rate per unit width. So, if we consider in the z-direction the width is W, then we will calculate the volumetric flow rate Q/W. And how we will calculate? So obviously, at any section, you need to calculate the flow area and this flow area into the mean velocity will give you the volumetric flow rate.

So, in this case if you see that at a distance y we have taken one infinitesimal distance dy and in z-direction if the width is W, then you will get the flow area as Wdy. So, in this particular case now volumetric flow rate Q will be

$$Q = \int_A u(y)\, dA$$

And now if you integrate from -H to H, then you will get

$$= \int_{-H}^{H} u(y) W dy$$

$$\frac{Q}{W} = \int_{-H}^{H} \frac{1}{2\mu}\left(-\frac{\partial p}{\partial x}\right)(H^2 - y^2) dy$$

$$= \int_{-H}^{H} \frac{1}{2\mu}\left(-\frac{\partial p}{\partial x}\right)\left[H^2 y - \frac{y^3}{3}\right]_{-H}^{H}$$

$$= \frac{1}{2\mu}\left(-\frac{\partial p}{\partial x}\right)\left[H^3 - \frac{H^3}{3} + H^3 - \frac{H^3}{3}\right]$$

$$= \frac{2}{3\mu}\left(-\frac{\partial p}{\partial x}\right) H^3$$

So, this is the volumetric flow rate. So, you can see that the volumetric flow rate is proportional to the pressure gradient and inversely proportional to the viscosity. In most application the mean velocity is provided in the channel and we are interested to find what is the pressure gradient. So, if we want to calculate the mean velocity we will be able to calculate from the expression of this volumetric flow rate.

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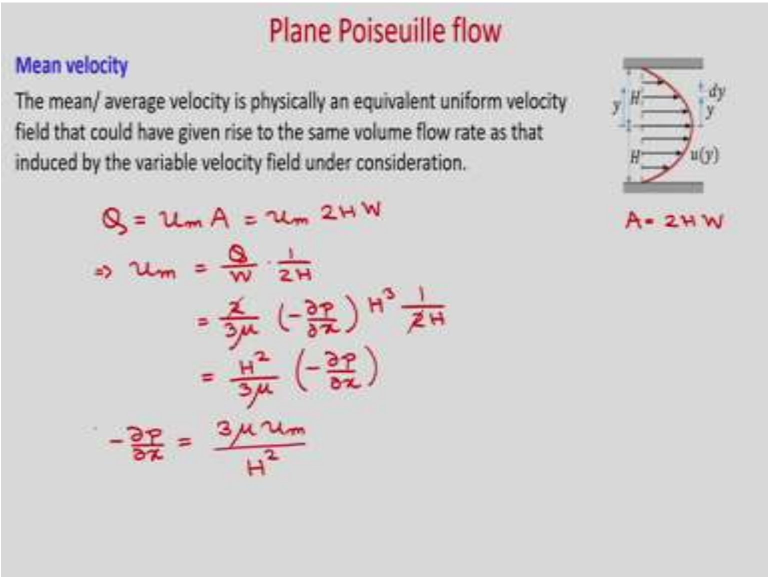

So, what is mean velocity? The mean or average velocity is physically an equivalent uniform velocity field that could have given rise to the same volume flow rate as that induced by the variable velocity field under consideration.

So, you know that we have already calculated the volume flow rate and this will be your mean velocity into the flow area. So, in this case you can see what is the flow area? So, flow area is your 2HW. So, it will be

$$Q = u_m A = u_m 2HW$$

$$u_m = \frac{Q}{W} \cdot \frac{1}{2H}$$

So, if you rearrange it you will get

$$= \frac{2}{3\mu}\left(-\frac{\partial p}{\partial x}\right) H^3 \frac{1}{2H}$$

$$= \frac{H^2}{3\mu}\left(-\frac{\partial p}{\partial x}\right)$$

$$-\frac{\partial p}{\partial x} = \frac{3\mu u_m}{H^2}$$

So, you can see in the right hand side μ is constant and positive quantity u_m is also positive it is a mean velocity H is also dimension. So obviously, right side term is positive hence $-\frac{\partial p}{\partial x}$ is positive.

Now, let us calculate the maximum velocity. Where we will get the maximum velocity you can see it is a parabolic profile; obviously, due to symmetry at the center line you will get the maximum velocity. You can also see that du/dy will be 0 if you put 0, then obviously at y is equal to 0 you will get the maximum velocity.

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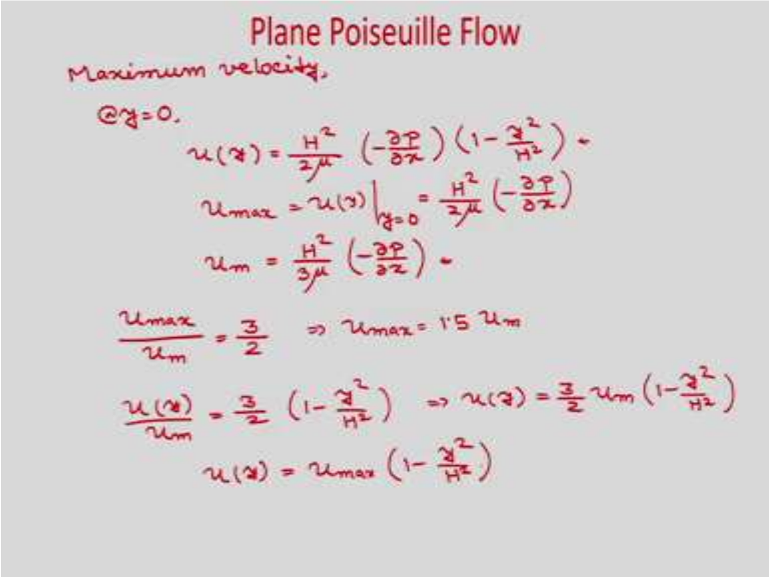

So, maximum velocity you will get at y is equal to 0 at y is equal to 0 and you will get u_{max}. So, what is the velocity profile your velocity profile

$$u(y) = \frac{H^2}{2\mu}\left(-\frac{\partial p}{\partial x}\right)\left(1 - \frac{y^2}{H^2}\right)$$

$$u_{max} = u(y)|_{y=0} = \frac{H^2}{2\mu}\left(-\frac{\partial p}{\partial x}\right)$$

$$u_m = \frac{H^2}{2\mu}\left(-\frac{\partial p}{\partial x}\right)$$

$$\frac{u_{max}}{u_m} = \frac{3}{2}$$

$$u_{max} = 1.5u_m$$

So, if you consider plane Poiseuille flow; that means, flow between two infinite parallel plates.

Then you will get maximum velocity as 1.5 times the mean velocity. So, let us calculate the ratio $u(y)/u_m$. So, you can see this is your mean velocity and this is your velocity expression. So, if you divide then you will get

$$\frac{u(y)}{u_m} = \frac{3}{2}\left(1 - \frac{y^2}{H^2}\right)$$

And from here you can see that your u_{max} is $\frac{3}{2}u_m$; that means, also you can write the velocity distribution

$$u(y) = u_{max}\left(1 - \frac{y^2}{H^2}\right)$$

So, now we want to find what is the shear stress distribution inside the flow domain as well as what is the wall shear stress. So, you know in the case of fully developed flow your shear stress τ_{yx} will be just μdu/dy.

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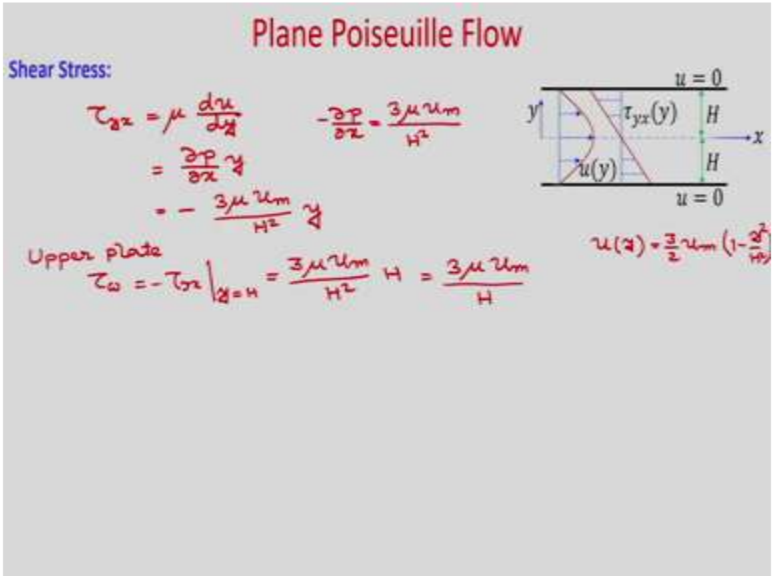

So, shear stress τ_{yx} you will get only $\mu\frac{du}{dy}$. So, you know the velocity distribution. So, if you put the values you will get actually this as $\frac{\partial p}{\partial x}y$ and we have already found that

$$-\frac{\partial p}{\partial x} = \frac{3\mu u_m}{H^2}$$

So, you are going to get

$$\tau_{yx} = -\frac{3\mu u_m}{H^2}y$$

So, you can see from this expression that τ_{yx} will vary inside the flow domain linearly from y is equal to -H to H, because it is a function of y, so linear function of y.

So if you want to find the wall shear stress τ_W, so obviously this will be

$$\tau_w = -\tau_{yx}|_{y=H} = \frac{3\mu u_m}{H^2} H = \frac{3\mu u_m}{H}$$

So, you can see the how your velocity distribution will look like as we have seen that

$$u(y) = \frac{3}{2} u_m \left(1 - \frac{y^2}{H^2}\right)$$

So obviously at y is equal to -H you can see this will be 0, y is equal to plus H velocity will be 0 and we have seen that at y is equal to 0 you will get maximum velocity and it is a parabolic in nature and shear stress distribution linearly it varies.

So now, we will find the skin friction coefficient, so the skin friction coefficient is the dimension less representation of wall shear stress.

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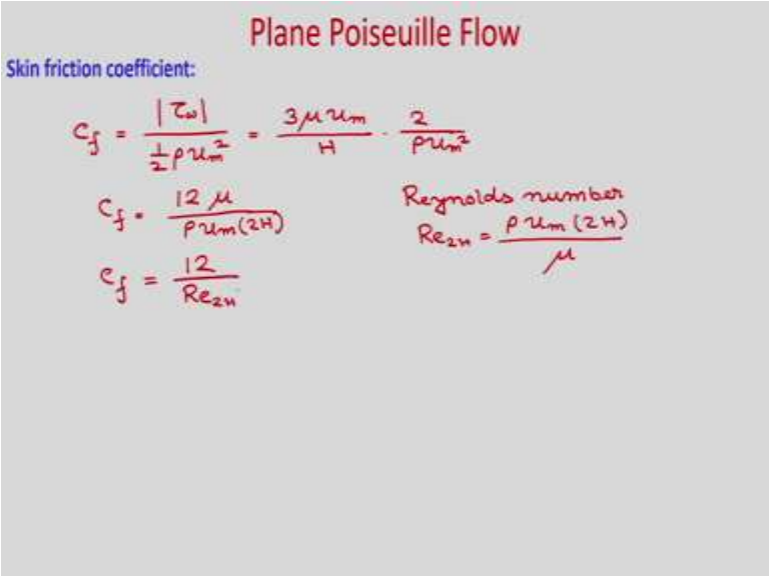

So, you can see that we can define skin friction coefficient C_f as

$$C_f = \frac{|\tau_w|}{\frac{1}{2}\rho u_m^2} = \frac{3\mu u_m}{H} \frac{2}{\rho u_m^2}$$

So, if you rearrange it you will get

$$C_f = \frac{12\mu}{\rho u_m (2H)}$$

So now, we will define Reynolds number based on 2H. So, you know Reynolds number is the ratio of inertia force to the viscous force. So, Reynolds number based on twice H we are writing as

$$Re_{2H} = \frac{\rho u_m (2H)}{\mu}$$

ρ into mean velocity into twice H divided by μ. So, you can see you can write

$$C_f = \frac{12}{Re_{2H}}$$

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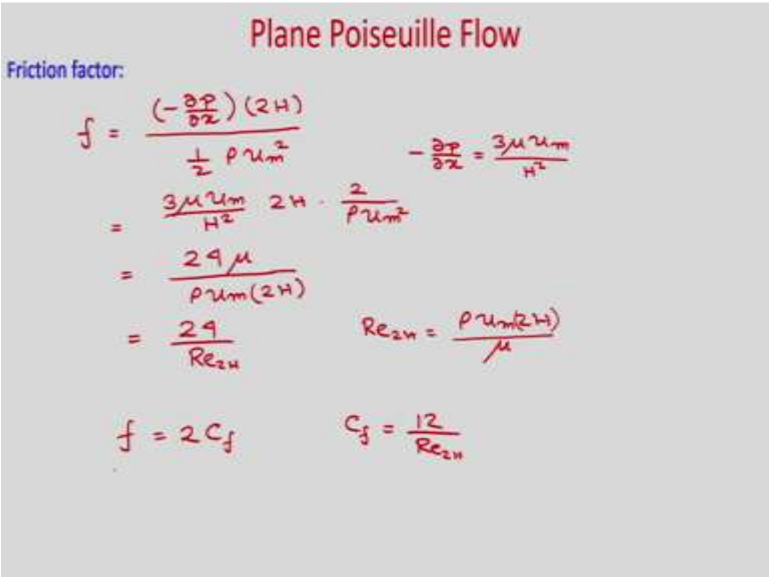

Similarly, we can represent the dimensionless pressure gradient in terms of Friction factor. So, we can define friction factor as

$$f = \frac{\left(-\frac{\partial p}{\partial x}\right)(2H)}{\frac{1}{2}\rho u_m^2}$$

So, this is the expression in terms of mean velocity.

$$f = \frac{3\mu u_m}{H} 2H \, \frac{2}{\rho u_m^2}$$

$$= \frac{24\mu}{\rho u_m (2H)}$$

$$= \frac{24}{Re_{2H}}$$

So, hence you can write

$$f = 2C_f$$

Now, we will consider another case that is your plane Poiseuille flow between inclined plates.

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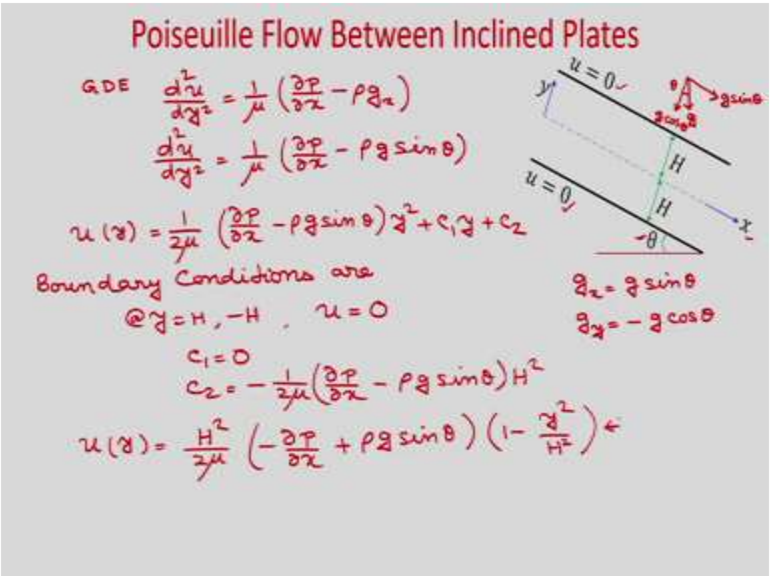

So, we let us consider two parallel plates which are inclined with the horizontal with angle θ. In this case the plates separated by a distance 2H and along the center line we are measuring x and y is measured from the center line. So, these are stationary plates now you can see in this particular case your gravity g is acting in this direction. It will have two components one is in this direction and other is in this direction.

So obviously, you can see this is your θ, so you will get gcosθ in this direction and gsinθ in axial direction x

So, when we will write the governing equation you can see obviously in the x-direction x momentum equation the gravity factor will be gsinθ and in y component of the momentum equation, your gravity will be gcosθ. That means, g_x is your gsinθ and this is in the positive x direction. However, g_y will be -gcosθ ok.

So, when you are considering Poiseuille flow between two inclined parallel plates, then obviously as you have considered x as the axial direction. So, our governing equation will remain same only we are not neglecting the gravitational acceleration ok.

So, we can write the governing equation

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu}\left(\frac{\partial p}{\partial x} - \rho g_x\right)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu}\left(\frac{\partial p}{\partial x} - \rho g \sin\theta\right)$$

S So obviously you can see if we integrate twice then you will get the velocity distribution. So, the velocity distribution u which is a function of y you can write it as

$$u(y) = \frac{1}{2\mu}\left(\frac{\partial p}{\partial x} - \rho g \sin\theta\right) y^2 + C_1 y + C_2$$

So, you can see the boundary conditions. So, boundary conditions are at y is equal to H and -H your u is equal to 0. So, if u is equal to 0 you put it here and you will get the constants C_1 and C_2. So, C_1 you will get as 0 as earlier and C_2 you will get

$$C_2 = -\frac{1}{2\mu}\left(\frac{\partial p}{\partial x} - \rho g \sin\theta\right) H^2$$

So, if you put this C_2 and C_1 values here you will get the velocity distribution,

$$u(y) = \frac{H^2}{2\mu}\left(-\frac{\partial p}{\partial x} + \rho g \sin\theta\right)\left(1 - \frac{y^2}{H^2}\right)$$

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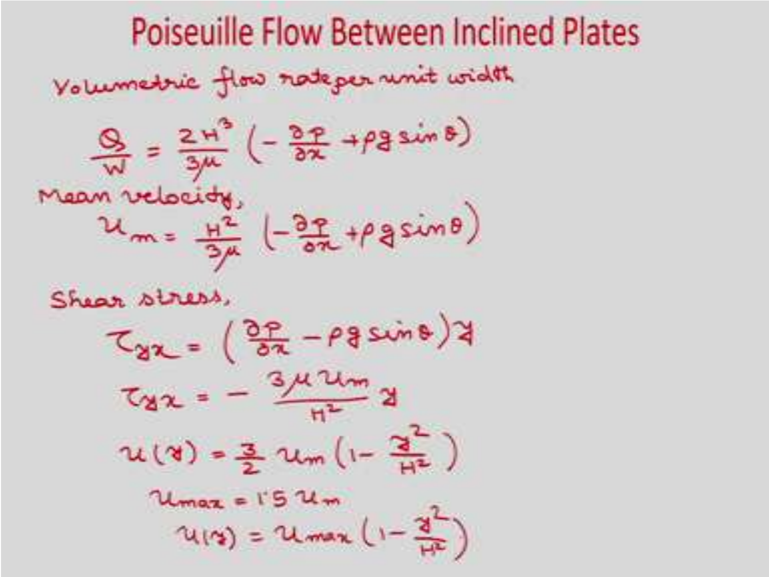

And now if you want to calculate the volumetric flow rate and the shear stress it will be similar to the plane Poiseuille flow except your $-\frac{\partial p}{\partial x}$ will be replaced with $-\frac{\partial p}{\partial x} + \rho g \sin\theta$.

So, your volumetric flow rate will be

$$\frac{Q}{W} = \frac{2H^3}{3\mu}\left(-\frac{\partial p}{\partial x} + \rho g \sin\theta\right)$$

Here you can see that $\rho g \sin\theta$ is constant; because ρ is density and it is constant for incompressible flow g is the gravitational acceleration. So, all these quantities are constant. So, $-\frac{\partial p}{\partial x}$ is also constant. So, these two terms actually are constant. So, you will get the similar expression .

In place of $-\frac{\partial p}{\partial x}$ for in case of plane Poiseuille flow we are substituting here. So, in this particular case we are writing $-\frac{\partial p}{\partial x} + \rho g \sin\theta$ in place of $-\frac{\partial p}{\partial x}$ for this plane Poiseuille flow. So, similarly you can find the mean velocity u_m as

$$u_m = \frac{H^2}{3\mu}\left(-\frac{\partial p}{\partial x} + \rho g \sin\theta\right)$$

And the shear stress distribution you will get

$$\tau_{yx} = \left(\frac{\partial p}{\partial x} - \rho g \sin\theta\right) y$$

And you can see here also you will get the linear profile and if you put the expression of $-\frac{\partial p}{\partial x} + \rho g \sin\theta$, then you can write

$$\tau_{yx} = -\frac{3\mu u_m}{H^2} y$$

And this is the same expression as in case of plane Poiseuille flow. So, you can write the velocity distribution

$$u(y) = \frac{3u_m}{2}\left(1 - \frac{y^2}{H^2}\right)$$

And in this case also you will get

$$u_{max} = 1.5u_m$$

So obviously

$$u(y) = u_{max}\left(1 - \frac{y^2}{H^2}\right)$$

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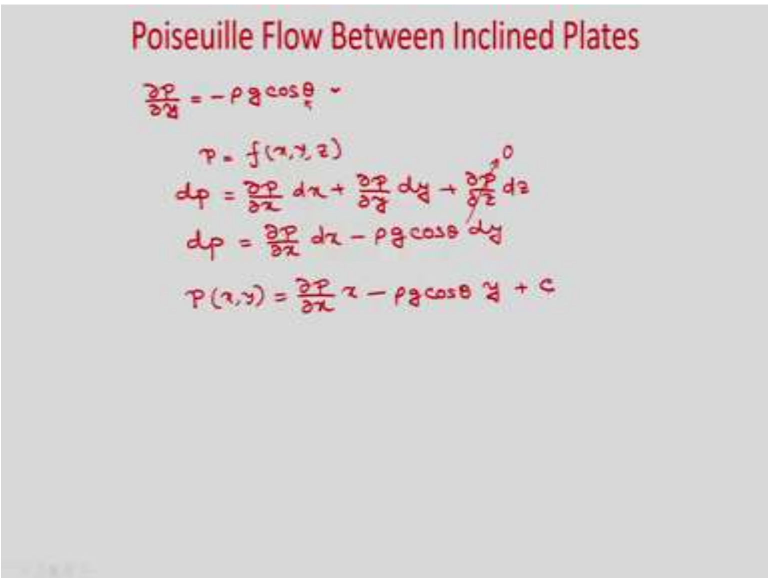

So, if you put the gravitational acceleration in the y momentum equation, then obviously you know that v is equal to 0, y momentum equation will boils down to

$$\frac{\partial p}{\partial y} = -\rho g \cos\theta$$

So obviously $\frac{\partial p}{\partial y}$ you can see this is your constant and this is nothing but the hydrostatic pressure right. We know that p is function of x y and z, so we can write

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz$$

So, you can see that $\frac{\partial()}{\partial z}$ of any flow variable is 0 because we consider z direction is infinite. So, this will be 0 and $\frac{\partial p}{\partial y}$ is constant this is the expression and $\frac{\partial p}{\partial x}$ also is constant for Poiseuille flow.

So, you can see that dp will be

$$dp = \frac{\partial p}{\partial x}dx - \rho g \cos\theta dy$$

So, if we integrate it you will get the pressure distribution as

$$p(x, y) = \frac{\partial p}{\partial x}x - \rho g \cos\theta\ y + C$$

So, this integration constant can be found just by putting a value of pressure at any location you can find the constant C. So, in today's class we considered plane Poiseuille flow which is the flow between 2 parallel plates and with a constant pressure gradient.

So, in that case we have found the velocity distribution and we have shown that it is parabolic. Then we calculated the volumetric flow rate and we have seen that Q is proportional to the pressure gradient and inversely proportional to the viscosity of the fluid. Then we calculated the mean velocity and we have shown that your pressure gradient $-\frac{\partial p}{\partial x}$ is a positive quantity.

And we have represented in terms of mean velocity then we have shown that your maximum velocity will occur at center line and this is equal to 1.5 times the mean velocity.

Then we calculated the shear stress and shear stress distribution is linear inside the flow domain.

And from there we have expressed the dimensionless shear stress which is your skin friction coefficient and we have also expressed the pressure gradient in terms of non dimensional quantity and that is known as friction factor. And we have shown that friction factor is twice the skin friction coefficient. Later we considered Poiseuille flow inside inclined parallel plates and in this case your gravity in the x direction is ρgsinθ.

So, in this case your pressure gradient will be $-\frac{\partial p}{\partial x} + \rho g \cos\theta$ will be added. And velocity distribution is again parabolic in this particular case and it will be the similar profile which you get in the plane Poiseuille flow, only the pressure will differ because $-\frac{\partial p}{\partial x}$ is replaced with $-\frac{\partial p}{\partial x} + \rho g \sin\theta$ ok.

So, but your velocity distribution will be similar as the horizontal case, where θ is equal to 0. And then we have expressed the mean velocity and the shear stress distribution and also we have calculated the pressure distribution inside the flow domain.

Thank you.