Viscous Fluid Flow
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Module - 02
Steady One-dimensional Rectilinear Flows
Lecture - 01
Plane Couette Flow

Hello, everyone. So, in the last module using the Reynolds transport theorem we derived the continuity equation and Navier-Stokes equation. In today's lecture, we will try to find the analytical solution of Navier-Stokes equations for the simplified problem and simple geometry.

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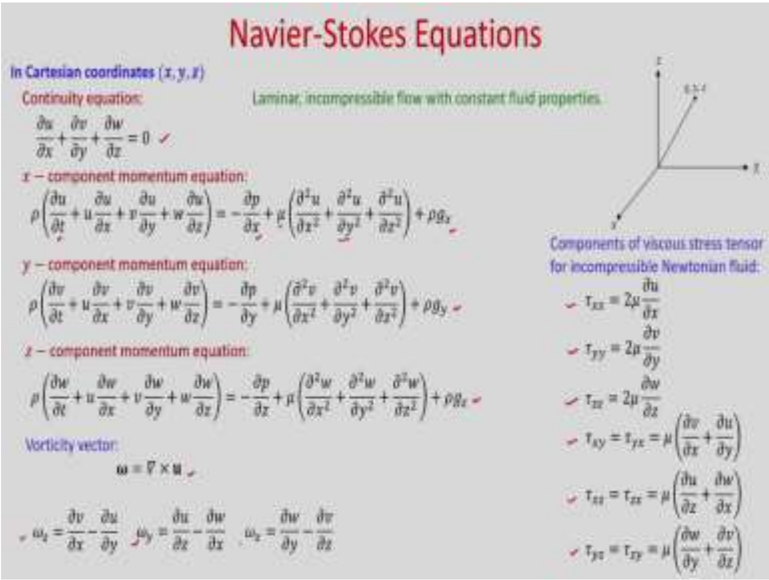

You can see that in the last class we derived the continuity equation for Cartesian coordinate. So, this is the continuity equation for laminar incompressible flow with constant fluid properties. This is the x - momentum equation,

$$\rho\left[\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right]=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+\rho g_x$$

this is the temporal term, ρ is the density of the fluid and you can see this is the convective term which is non-linear. This is the pressure gradient term, μ is the viscosity of the fluid and this is the viscous term and this is the gravity term and which is known as the body force term.

So, similarly, we derived the y-component of momentum equation and z-component of momentum equation.

$$\rho\left[\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right]=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)+\rho g_y$$

$$\rho\left[\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right]=-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)+\rho g_z$$

So, you can see these equations are coupled and non-linear. You can also find the components of viscous stress tensor for an incompressible Newtonian fluid. So, these are the normal stresses and these are the shear stresses. Also, you can find the vorticity component. So, this is the vorticity factor curl of the velocity vector and these are the components of this vorticity ω_z, ω_y and ω_x.

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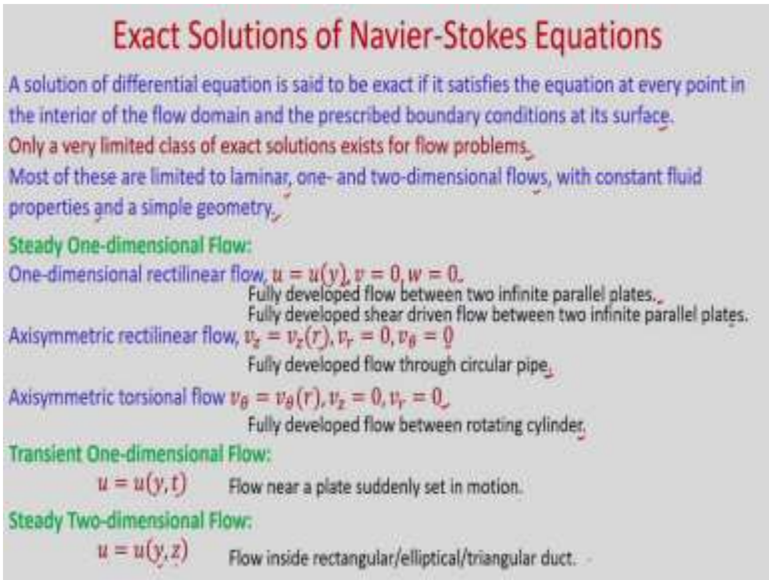

So, what is the exact solution of Navier-Stoke equations? So, that we can see that a solution of a differential equation is said to be exact if it satisfies the equation at every point in the interior of the flow domain and the prescribed boundary conditions at its surface.

As we told that only a very limited class of exact solutions exist for flow problems and most of these are limited to laminar one and two-dimensional flows with constant properties and simple geometry.

So, you can see that we can have the analytical solution for steady one-dimensional flow. So, we can have a one-dimensional rectilinear flow. So, it represents the flow in the Cartesian coordinate where axial velocity u is a function of y and other velocity components v and w are 0.

So, the examples are fully developed flow between two infinite parallel plates; which is known as plane Poiseuille flow fully developed shear driven flow between two infinite parallel plates which is known as plane Couette flow. We can also have the axisymmetric rectilinear flow; what is axisymmetric flow?

In the axisymmetric, the velocity in theta direction is 0 and the gradient of any velocity component or pressure in the direction of theta is 0. So, v_z is a function of r and v_r is 0. So, we can have the solution for fully developed flow through a circular pipe which is known as Hagen Poiseuille flow and we can also have an axisymmetric torsional flow where v_θ is a function of r only and v_z and v_r are 0. So, the example is a fully developed flow between the rotating cylinder.

We can also have the tangent one-dimensional flow where velocity u is a function of one space coordinate and time flow near a plate suddenly set in motion is an example of this tangent one-dimensional flow and steady two-dimensional flow where velocity is a function of two spatial coordinates y and z. So, the examples are flowing inside rectangular or elliptical or triangular duct with uniform cross-section.

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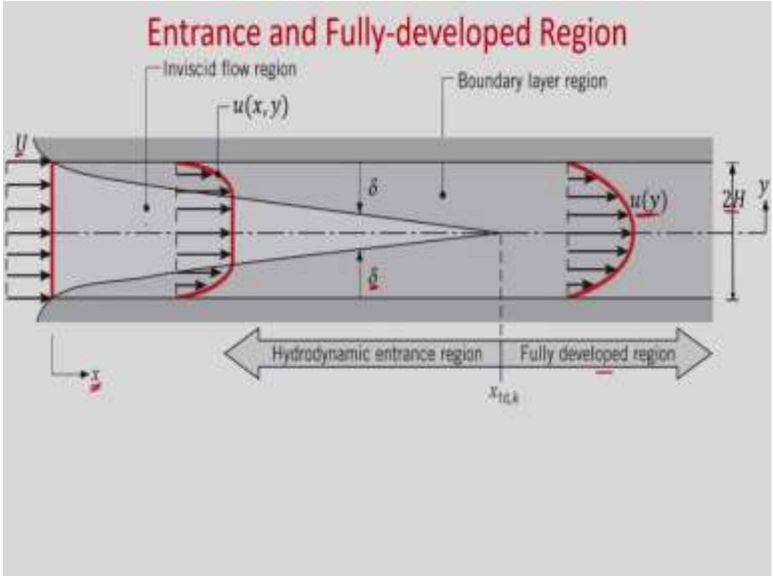

So, now let us consider flow inside two infinite parallel plates. At inlet, we have uniform velocity inlet and you know that when it comes into contact with the parallel plates obviously, there will be the formation of boundary layer.

Due to the viscous effect you can see that these hydrogen boundary layers will start developing in near to the plate region and the thickness of the boundary layer will grow gradually. And, outside this boundary layer near to the central region, the flow will be inviscid and there will be no viscous effect.

So, you can see here this is the flow inside two parallel plates where inlet velocity is u with uniform velocity, it is entering and you can see the thickness of this boundary layer is gradually growing and this region where the viscous effect is not there that region is known as inviscid flow region. And, velocity is a function of x, y inside the boundary layer and outside obviously, it is constant.

And, this is known as core region velocity and you can see this core velocity will increase at the different axial locations as you have the boundary layer near to the wall. After a certain distance, you can see these boundary layers will merge in the central region and after that, there will be no change of the velocity profile in the flow direction.

So, these region is known as fully developed region, where the velocity profile remains the same. It does not vary in the flow direction and if x is the axial direction then in this case y is measured from the axis and u is function of y only ok. And, you can see if the distance between two parallel plates is 2H, then these maximum hydrodynamic boundary layer thickness will be H ok.

And, this region is known as hydrodynamic entrance region where the thickness of the hydrodynamic boundary layer increases and up to the point where it merges.

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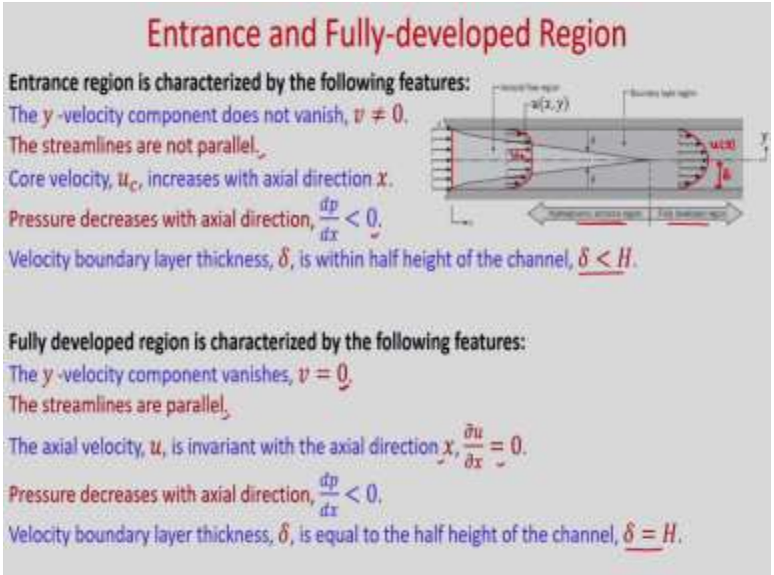

So, you can see here the entrance region is characterized by the following features. The y-velocity component does not vanish; that means, v not equal to 0, in the hydrodynamic entrance region. The streamlines are not parallel. Core velocity, u_c, increases with axial direction x. So, this is the u_c it increases to maintain the conservation of mass at every cross-section.

Pressure decreases with axial direction which means, dp/dx is less than 0; that means, the flow takes place from high-pressure region to low pressure region. Velocity boundary layer thickness delta is within half-height of the channel; that means, in the hydrodynamic entrance region δ will be less than H.

A fully developed region is characterized by the following features. So, you know that this is the fully developed region. The y-velocity component vanishes and v is equal to 0 in the fully developed region. The streamlines are parallel. The axial velocity u is invariant with the axial direction x; that means, $\frac{\partial u}{\partial x}$ will be 0, because these velocity profiles u is a function of y only and it does not change in the direction of the flow. So, $\frac{\partial u}{\partial x}$ will be 0 in the fully developed region.

Pressure decreases with axial direction; that means, $\frac{dp}{dx}$ will be less than 0 and velocity boundary layer thickness δ is equal to the half-height of the channel; that means, δ will be H. So, in the fully developed region, as the parallel plates are separated by a distance 2H, so, hydrodynamic boundary layer thickness will be H in the fully developed region.

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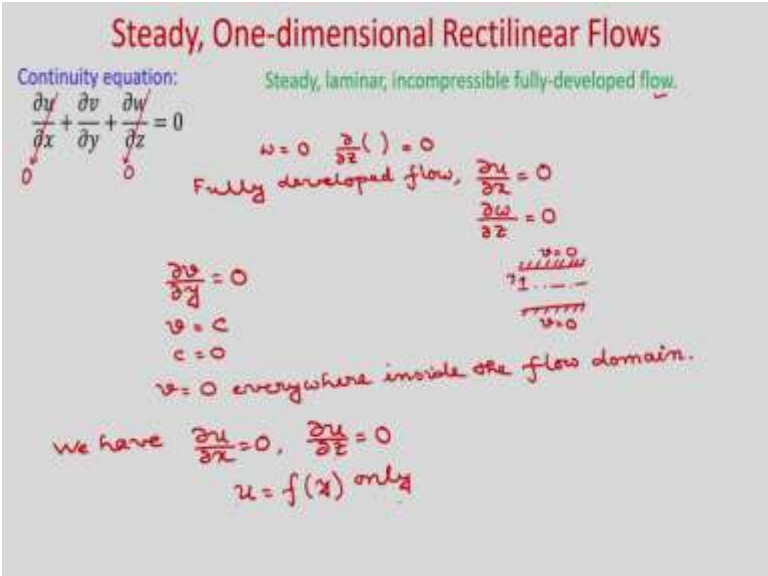

Now, let us consider steady one-dimensional rectilinear flow and now, we will simplify the continuity equation and Navier-Stoke equation, so that we can have the exact solutions.

So, first, we are assuming that it is a steady laminar incompressible fully developed flow. So, you can see that we are considering one-dimensional rectilinear flow. And in the third direction let us say that in the z-direction it is infinite and w velocity is 0 and the gradient of any quantity in the direction of z is 0 ok. So, $\frac{\partial ()}{\partial z}$ of any quantity is 0 as it is infinite in the z-direction.

Now, as it is a fully developed flow so, we have $\frac{\partial u}{\partial x}$ is 0. Velocity profile u does not change in the direction of the flow. So obviously, you can see that we have $\frac{\partial u}{\partial x}$ is 0 and also $\frac{\partial w}{\partial z}$ is 0, ok. So, from this continuity equation, you can see we have this is 0, as it is fully developed flow, this is 0. So, we have $\frac{\partial v}{\partial y}$ is equal to 0; that means if you integrate it v will be constant ok.

And, as you are considering let us say that flow between two parallel plates so, these are non-porous plates. So, if these are non-porous, then v will be 0 at the plate. So, if you can see that if v is 0 at the plate and let us say y is measured from here, so, obviously, if v is 0 at the plates then integration constant will be 0 and v will be 0 everywhere inside the flow domain ok.

And, now we can see that del of del z of any quantity is 0. So, that means, we have $\frac{\partial u}{\partial x}$ is 0 and we have $\frac{\partial u}{\partial z}$ is equal to 0; because the third direction is infinite and the gradient of any quantity is 0 in the z direction. So, obviously, u is not a function of x, u is not function of z. So, that means, u is function of y only and at as it is a steady flow, so obviously, it is not function of time. So, u is a function of y only.

Now, let us consider the x component of the momentum equation, and let us simplify this equation invoking the assumptions.

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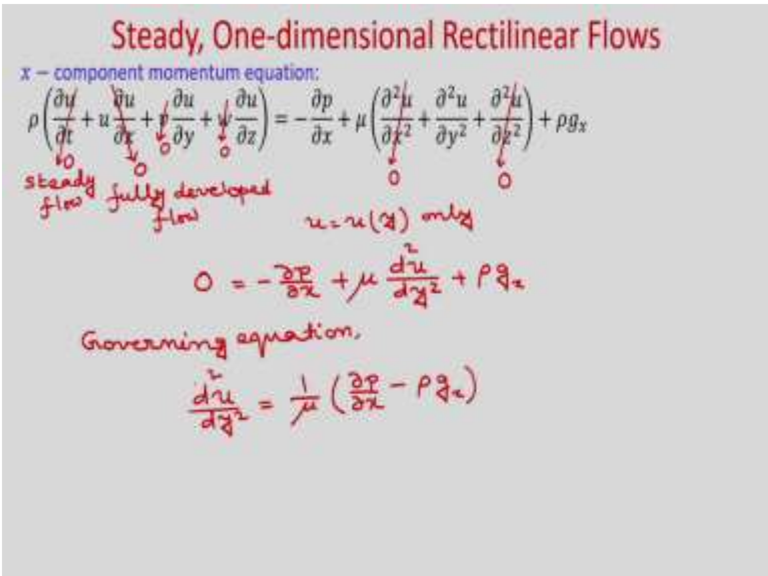

So, you can see that it is a steady flow. So, $\frac{\partial u}{\partial t}$ is 0 because it is a steady flow it is a fully developed flow so, this is 0. We have seen that v is 0, w is 0. So, you can see left hand side all terms are 0 and as $\frac{\partial u}{\partial x}$ is 0 everywhere so, $\frac{\partial^2 u}{\partial x^2}$ is 0; $\frac{\partial u}{\partial z}$ is 0 everywhere so, $\frac{\partial^2 u}{\partial z^2}$ also will be 0 and already we have shown that u is a function of y only.

So, now these partial derivatives we can write as ordinary derivative keeping that $\frac{\partial p}{\partial x}$ maybe it is constant. So, we can write

$$0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} + \rho g_x$$

g_x is the component of the g in the x direction. So, we can have the governing equation to find the velocity profile

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu}\left(\frac{\partial p}{\partial x} - \rho g_x\right)$$

So, in most of the flows, we get that $\frac{\partial p}{\partial x}$ is constant. So, this is the pressure gradient is constant except the pulsatile flow. So, in the right-hand side you can see this will be constant so, you can integrate this governing equation and satisfy the boundary condition to get the velocity profile.

Once you find the velocity profile then you will be able to calculate the shear stress. In this particular case, when we consider flow inside two infinite parallel plates we will have one non-zero shear stress that is τ_{yx} because in this case u is a function of y only and the other components of velocity is v and w are 0.

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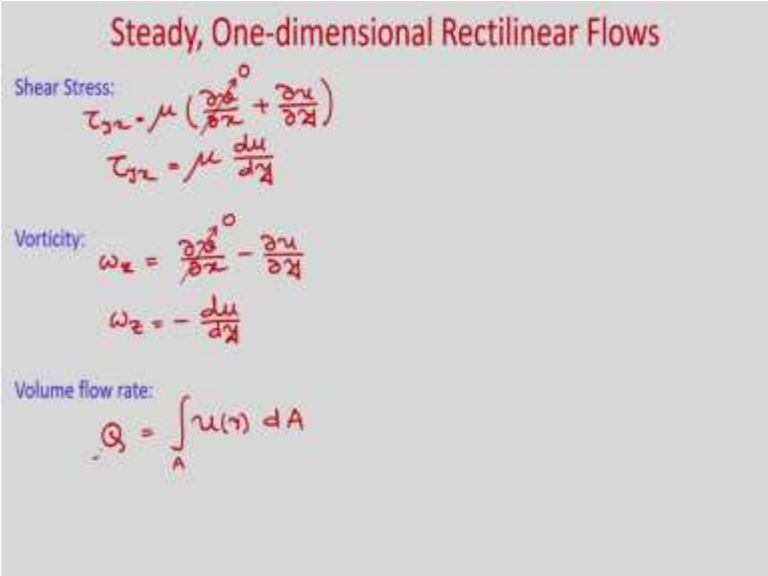

So, we have shear stress component

$$\tau_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

So, in this particular case v is 0. So,

$$\tau_{yx} = \mu\frac{\partial u}{\partial y}$$

Similarly, vorticity you can write one component will be non-zero,

$$\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

So, this is 0. So,

$$\omega_z = -\frac{\partial u}{\partial y}$$

So, if you can find the velocity distribution u then obviously, you will be able to calculate the shear stress and the vorticity component.

Now, to find the volume flow rate at a particular cross-section then you can find Q as

$$Q = \int_A u(y)\, dA$$

So, once you integrate then you will be able to find the volume flow rate at a particular cross-section.

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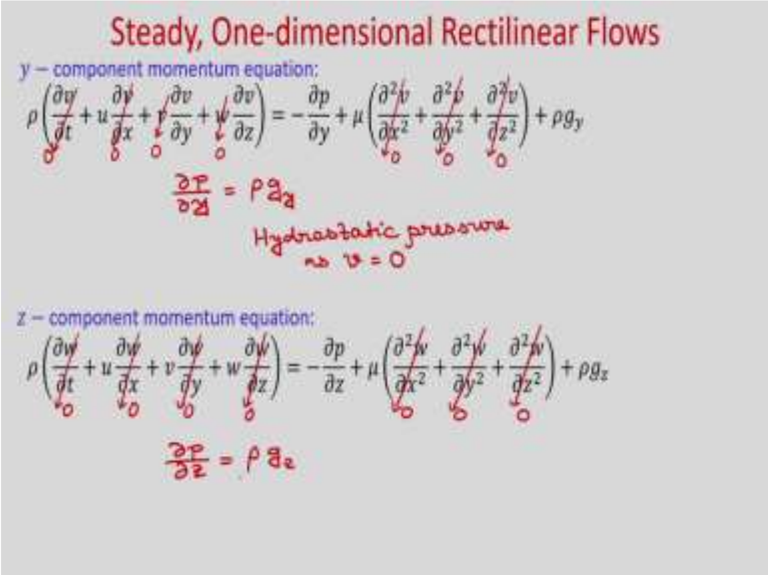

Now, for this particular case when we are considering flow inside two infinite parallel plates let us simplify y and z-component of momentum equations. So, you can see this is the y component of the momentum equation

$$\rho\left[\frac{\partial \mathrm{v}}{\partial \mathrm{t}}+u\frac{\partial \mathrm{v}}{\partial \mathrm{x}}+v\frac{\partial \mathrm{v}}{\partial \mathrm{y}}+w\frac{\partial v}{\partial \mathrm{z}}\right]=-\frac{\partial \mathrm{p}}{\partial \mathrm{y}}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)+\rho g_y$$

as it is a steady flow $\frac{\partial \mathrm{v}}{\partial \mathrm{t}}$ is 0; as v is 0, left side all the terms are 0 and we have obviously, v is 0 so, all these terms are 0 in the viscous term.

So, we will have only

$$\frac{\partial \mathrm{p}}{\partial \mathrm{y}}=\rho g_y$$

So, obviously, this is nothing but the hydrostatic pressure right. And, similarly, in the z-component of momentum equation

$$\rho\left[\frac{\partial \mathrm{w}}{\partial \mathrm{t}}+u\frac{\partial \mathrm{w}}{\partial \mathrm{x}}+v\frac{\partial \mathrm{w}}{\partial \mathrm{y}}+w\frac{\partial w}{\partial \mathrm{z}}\right]=-\frac{\partial \mathrm{p}}{\partial \mathrm{z}}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)+\rho g_z$$

$\frac{\partial w}{\partial t}$ is 0 as it is steady state w is 0, so, these terms are 0. The viscous term is also 0 as w is 0. So, you will get

$$\frac{\partial p}{\partial z} = \rho g_z$$

So, this is also hydrostatic pressure as w is equal to 0.

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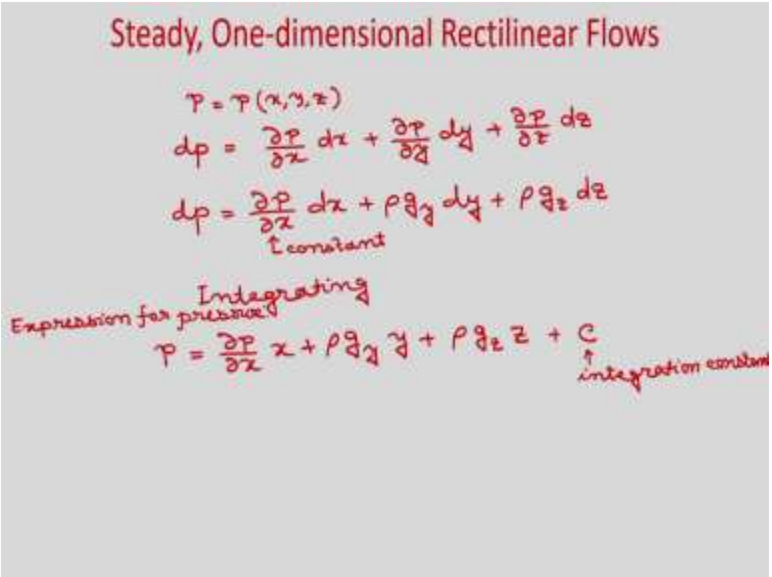

So, now we know that pressure is function of x, y and z. So, we can write

$$dp = \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy + \frac{\partial p}{\partial z} dz$$

So, we can write

$$dp = \frac{\partial p}{\partial x} dx + \rho g_y dy + \rho g_z dz$$

So, this is constant let us assume. So, if it is not versatile flow, then obviously, $\frac{\partial p}{\partial x}$ will be constant inside pipe flow. So, you can see now if you integrate it keeping $\frac{\partial p}{\partial x}$ is constant, we can write p as

$$p = \frac{\partial \mathrm{p}}{\partial \mathrm{x}} x + \rho g_y y + \rho g_z z + c$$

Where c is the integration constant. So, using this expression you will be able to find the pressure distribution inside the flow field. So, this is the expression for pressure.

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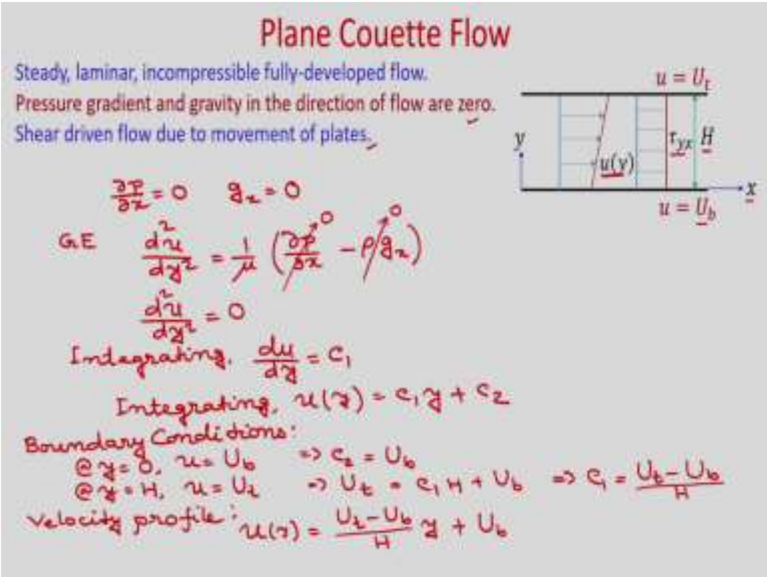

Now, let us consider plane Couette flow which is known as shear-driven flow. So, flow inside two parallel plates where one plate is moving with respect to the other. So, we are assuming steady laminar incompressible fully developed flow and we are assuming for this particular case as we are considering plane Couette flow, pressure gradient and gravity in the direction of the flow are 0 and it is shear driven flow due to movement of plates.

So, in general, we are considering two infinite parallel plates separated by a distance H, x is the axial direction and y is measured from the bottom plate and let us consider that bottom plate is moving with a constant velocity U_b and the upper plate is moving with a constant velocity U_t, where t represents top and b represents bottom.

So, in general, first we will find the velocity distribution and the shear stress distribution and then we will calculate the volume flow rate. So, as we have seen that $\frac{\partial \mathrm{p}}{\partial \mathrm{x}}$ is 0 and g_x is 0. So, whatever governing equation we have derived you can see that

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu}\left(\frac{\partial \mathrm{p}}{\partial \mathrm{x}} - \rho g_x\right)$$

So, these are 0 for this particular case. So, we have the governing equation

$$\frac{\partial^2 u}{\partial y^2} = 0$$

Now, integrating this equation what you will get?

$$\frac{\partial \mathrm{u}}{\partial \mathrm{y}} = C_1$$

And again if you integrate you will get u which is function of y as

$$u(y) = C_1 y + C_2$$

Now, let us find these integration constants C_1, C_2 invoking the boundary conditions. So, we have the velocities known at bottom and top plates.

So, what are the boundary conditions boundary conditions? So, we have at y is equal to 0 u is equal to U_b. So, at y is equal to 0 if you put is u is equal to U_b, this will give

$$C_2 = U_b$$

And at y is equal to H we have u is equal to U_t. So, this will give

$$U_t = C_1 H + U_b$$

So, that will give

$$C_1 = \frac{U_t - U_b}{H}$$

So, now if we put the values of C_1 and C_2 we will get the velocity profile as

$$u(y) = \frac{U_t - U_b}{H} y + U_b$$

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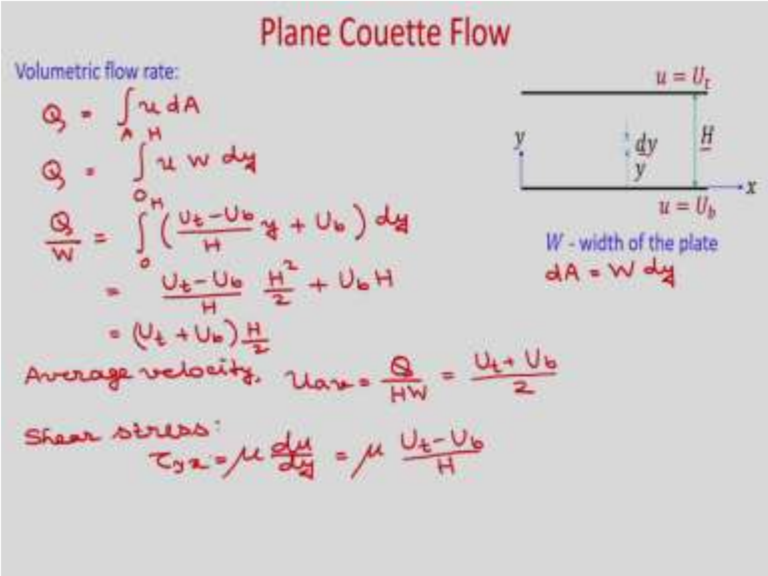

Now, let us find the volumetric flow rate. So, we can find

$$Q = \int_A u(y)\, dA$$

So, let us consider that W is the width of the plates. So, in the z-direction say let us say that we have the width W which is very long and we are considering one elemental strip dy at a distance y from the bottom plate.

So, you can see if you consider these elemental flow areas so, you will get dA as elemental flow area as Wdy. So, now, we can write

$$Q = \int_0^H uWdy$$

So, let us calculate the volumetric flow rate per unit width; that means, Q/W. So, it will be

$$\frac{Q}{W} = \int_0^H \left(\frac{U_t - U_b}{H} y + U_b\right) dy$$

$$= \frac{U_t - U_b}{H} \frac{H^2}{2} + U_b H$$

$$= (U_t + U_b)\frac{H}{2}$$

So, average velocity now you can calculate. So,

$$U_{av} = \frac{Q}{HW} = \frac{U_t + U_b}{2}$$

So, now, let us calculate the shear stress. So, we have

$$\tau_{yx} = \mu\frac{du}{dy} = \mu\frac{U_t - U_b}{H}$$

So, you can see that τ_{yx} is constant because U_t, U_b, H and μ are constant. So, this is constant inside the flow domain.

So, we have found the velocity profile, the volumetric flow rate and the shear stress, if you see the expression for velocity profile you can see it is a linear profile and velocity will vary from U_b from the bottom plate to U_t at the top plate and shear stress is constant. So, you can see that this is your variation of velocity as a function of y and shear stress will be constant.

Now, we will consider four different cases. The first case we will consider that the bottom plate is stationary and the upper plate is moving with a constant velocity u.

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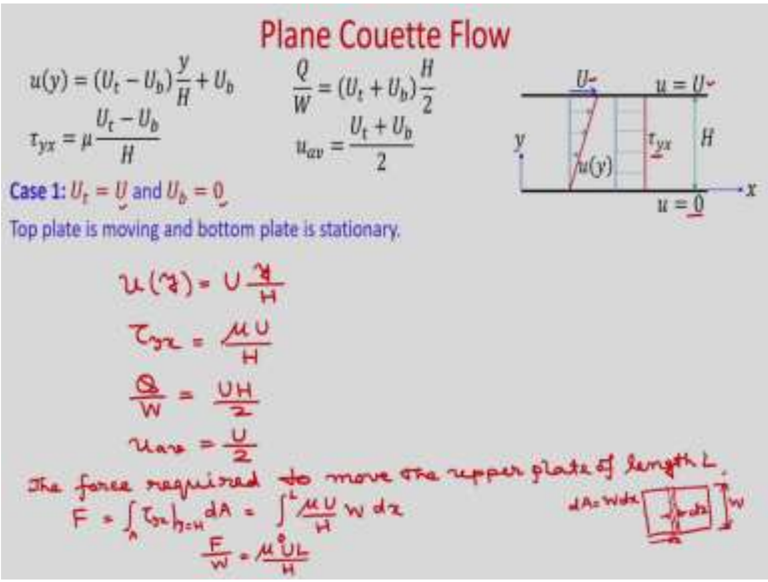

So, this is the special case where bottom plate is stationary u is equal to 0 and upper plate is moving with a constant velocity U. So, we can see whatever expression we have for velocity distribution, shear stress and the volumetric flow rate just let us put U_t is equal to U and U_b is equal to 0. Then we will get the velocity profile u(y) so,

$$u(y) = U\frac{y}{H}$$

So, it will vary from 0 at the bottom plate to U at the upper plate. So, this is the velocity profile.

Now, if you calculate the shear stress τ_{yx} it will be

$$\tau_{yx} = \frac{\mu U}{H}$$

And

$$\frac{Q}{W} = \frac{UH}{2}$$

And

$$u_{av} = \frac{U}{2}$$

So, now if we want to calculate the force required to move the upper plate of length L, then you can see that if you have an upper plate of width W.

So, let us say that this is W, top view we are seeing and at a distance x if you take one elemental strip of distance dx then the area elemental area dA will be just Wdx over the plate. So, obviously, F you can calculate as

$$F = \int_A \tau_{yx}\,|_{y=H} dA = \int_0^L \frac{\mu U}{H} W dx$$

$$\frac{F}{W} = \frac{\mu UL}{H}$$

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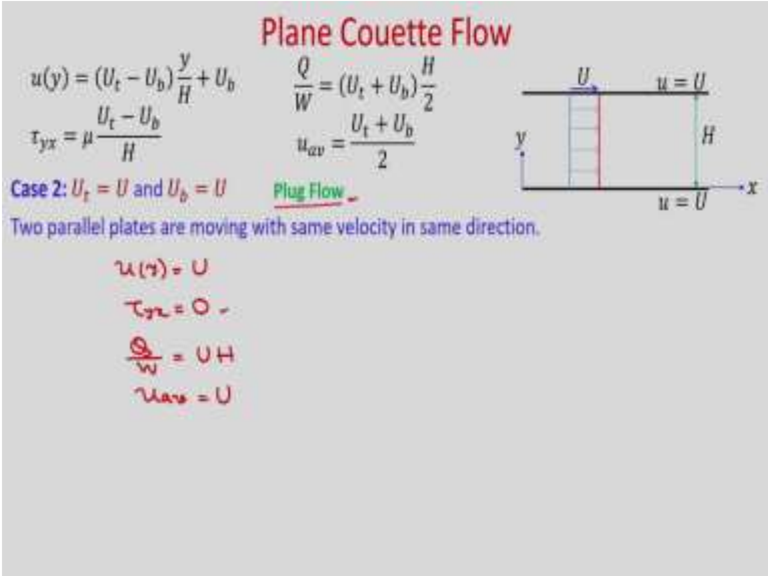

Now, let us consider the 2nd case where both the plates are moving with the same velocity in the same direction ok. So, that means, U_t and U_b is equal to U. So, now, we are considering U_t is equal to U, U_b is equal to U. So, this is known as plug flow because from here you can see that your velocity u_y; if you put u here so, this will become 0.

So, it will be just u and τ_{yx} obviously, from this expression you can see it will be 0. Q/W will be from this expression you can see it will be U_H and u_{av} obviously, it will be U. So, you can see that the fluid will have the motion as a solid body because the whole fluid will move with a constant velocity U and hence there will be no shear stress so, τ_{yx} is 0. So, it is known as plug flow.

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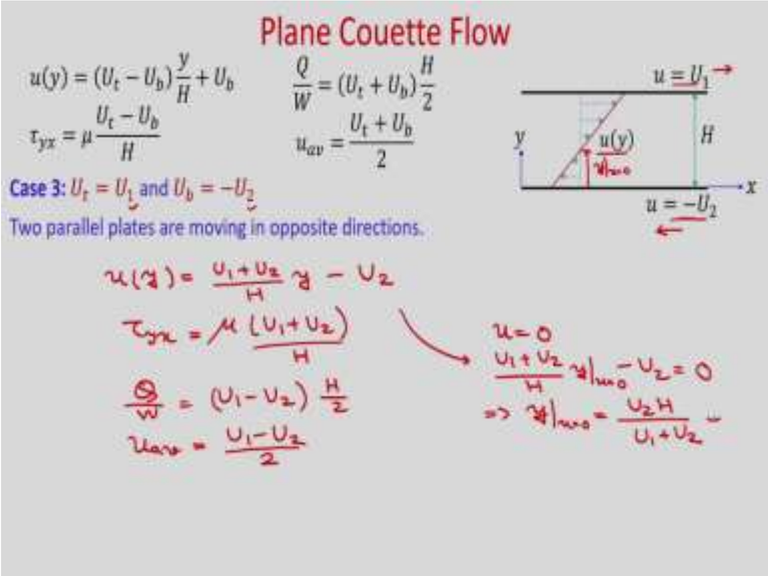

The 3rd special case we are considering that the top plate is moving in the x-direction at velocity constant velocity U_1 and the bottom plate is moving in the opposite direction as $-U_2$ ok. So, U is equal to $-U_2$ at the bottom plate and at the top plate U is equal to plus U_1. So, this is moving in this direction and the upper plate is moving in the positive x-direction.

So, if you put this expression U_t is equal to U_1 and U_b is equal to $-U_2$, the velocity profile you will get

$$u(y) = \frac{U_1 + U_2}{H} y - U_2$$

Shear stress τ_{yx} will be

$$\tau_{yx} = \frac{\mu(U_1 + U_2)}{H}$$

$$\frac{Q}{W} = (U_1 - U_2)\frac{H}{2}$$

and

$$u_{av} = \frac{(U_1 - U_2)}{2}$$

So, now you can see in this particular case obviously, inside the flow domain somewhere the velocity will become 0. So, at what distance y you will get the velocity 0 let us find. So, you can see u will be 0. So, from this expression you can see if you put it. So,

$$\frac{U_1 + U_2}{H} y|_{u=0} - U_2 = 0$$

So, that will give

$$y|_{u=0} = \frac{U_2 H}{U_1 + U_2}$$

So, you can see the velocity distribution will look like this it is a linear profile and at this distance the velocity will become 0 at this distance.

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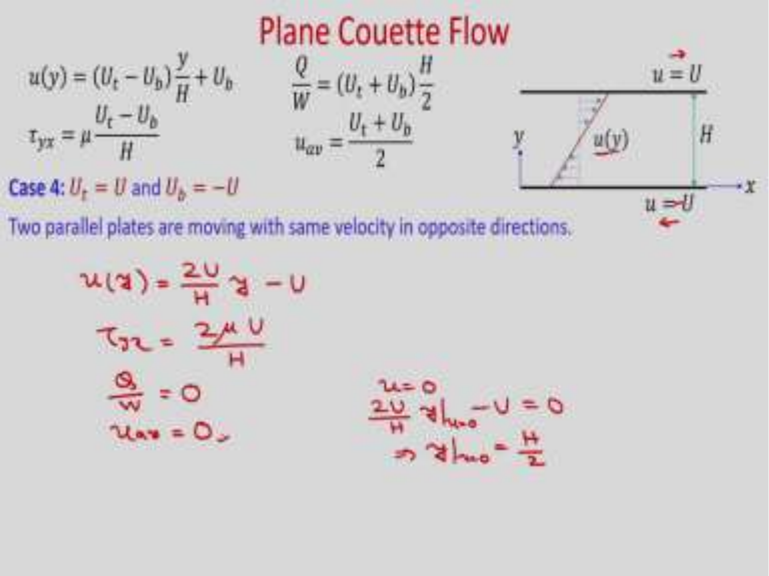

Next let us consider that top and bottom plates are moving with a same velocity, but in opposite direction ok. So, we can see that it is moving with minus U, so that means, in this direction and u is equal to U in this direction. So, obviously, the velocity profile u(y) will be

$$u(y) = \frac{2U}{H} y - U$$

$$\tau_{yx} = \frac{2U\mu}{H}$$

$$\frac{Q}{W} = 0$$

$$u_{av} = 0$$

So, from here you can see that u will be 0, when

$$\frac{2U}{H} y|_{u=0} - U = 0$$

So,

$$y|_{u=0} = \frac{H}{2}$$

So, this is the velocity profile and as you can see that you have from the here upper side you have velocity in the positive direction and the bottom side it is in the negative direction and you will get average velocity as 0.

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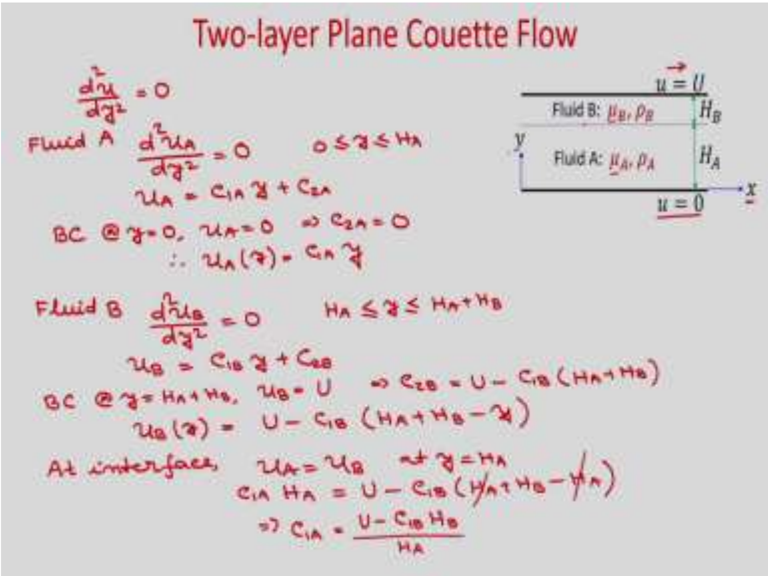

Now, let us consider two-layer Couette flow where we have two different fluids inside these two parallel plates where upper plate is moving with respect to the bottom plate these two fluids are immiscible and having different viscosity.

So, if you consider here x is the axial direction, y is measured from the bottom plate; the bottom plate is stationary, the upper plate is moving with a constant velocity u in the positive x-direction. And, this is fluid A where viscosity is μ_A, this is fluid B where viscosity is μ_B and these fluids are immiscible. So, the interface is located at a distance H_A from bottom plate and at a distance H_B from the top plate.

Now, we are considering steady incompressible fluid flow. So, we have the same governing equations. So,

$$\frac{\partial^2 u}{\partial y^2} = 0$$

So, for fluid A just we will write this equation as

$$\frac{\partial^2 u_A}{\partial y^2} = 0$$

This is in the range 0 to H_A and you will get the u_A as

$$u_A = C_{1A} y + C_{2A}$$

So, if you put boundary condition at y is equal to 0, u_A is equal to 0. So, that will give C_{2A} is equal to 0 hence you will get

$$u_A(y) = C_{1A} y$$

Now, for fluid B, we can write the same governing equation where U_B is the velocity profile inside the domain for fluid B. So,

$$\frac{\partial^2 u_B}{\partial y^2} = 0$$

And this is valid in the range H_A to H_A+H_B. So, we will get the velocity profile u_B as

$$u_A = C_{1B} y + C_{2B}$$

Now, we will apply the boundary condition at y is equal to H_A+H_B , u_B is equal to U. So, that will give

$$C_{2B} = U - C_{1B}(H_A + H_B)$$

So, if you put these value in this expression so, the velocity profile u_B we will get as

$$u_B(y) = U - C_{1B}(H_A + H_B - y)$$

So, now, at the interface we will apply the interface condition ok. So, at the interface you know that velocity is continuous velocity will be the same and the shear stress is continuous ok. So, at the interface,

$$u_A = u_B \text{ at } y = H_A$$

So, if you put it so, from this expression you can see

$$C_{1A}H_A = U - C_{1B}(H_A + H_B - H_A)$$

So, we will get

$$C_{1A} = \frac{U - C_{1B}H_B}{H_A}$$

Now, let us write at interface that shear stress is continuous. So, at y is equal to H_A; that means, at interface we have shear stress is continuous shear stress is continuous.

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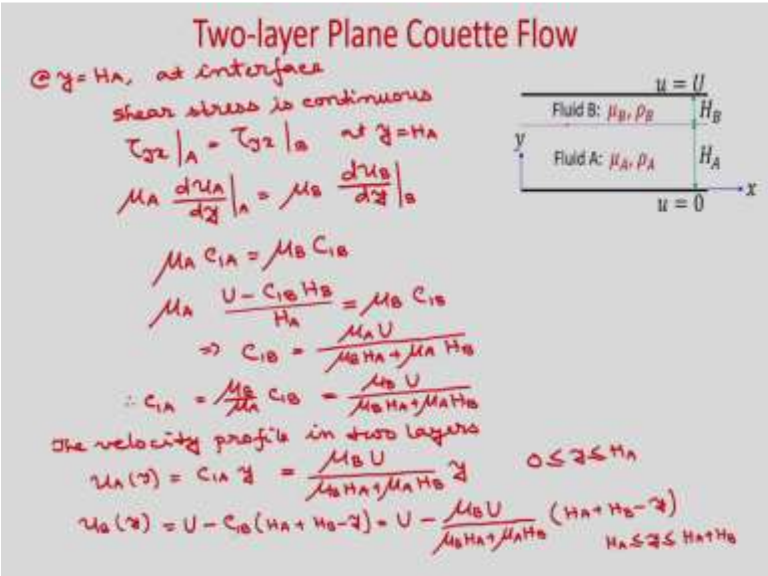

So, you can see that at this point the stress from side fluid A will be equal to the stress from fluid side B. So, τ_{yx} from fluid A will be equal to shear stress from fluid B at y is equal to H_A; that means, at the interface.

So, you can see

$$\mu_A \left.\frac{du_A}{dy}\right|_A = \mu_B \left.\frac{du_B}{dy}\right|_B$$

So, you will get

$$\mu_A C_{1A} = \mu_B C_{1B}$$

So, you can write that

$$\mu_A \frac{U - C_{1B} H_B}{H_A} = \mu_B C_{1B}$$

$$C_{1B} = \frac{\mu_A U}{\mu_B H_A + \mu_A H_B}$$

So,

$$C_{1A} = \frac{\mu_B}{\mu_A} C_{1B} = \frac{\mu_B U}{\mu_B H_A + \mu_A H_B}$$

So, the velocity profile in the two layers we will get

$$u_A(y) = C_{1A} y = \frac{\mu_B U}{\mu_B H_A + \mu_A H_B} y$$

In the range y between 0 and H_A.

Similarly,

$$u_B(y) = U - C_{1B}(H_A + H_B - y) = U - \frac{\mu_B U}{\mu_B H_A + \mu_A H_B}(H_A + H_B - y)$$

In the range of y H_A less than equal to y less than equal to H_A+H_B.

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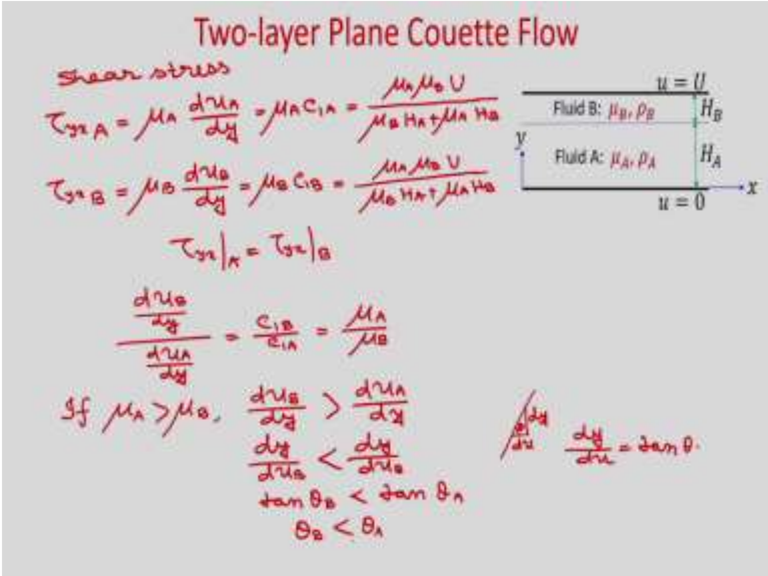

Now, let us find the shear stress distribution in the fluid domain A and B. So, shear stress you can find

$$\tau_{yx_A} = \mu_A \frac{du_A}{dy} = \mu_A C_{1A} = \frac{\mu_A \mu_B U}{\mu_B H_A + \mu_A H_B}$$

In fluid domain b you will get

$$\tau_{yx_B} = \mu_B \frac{du_B}{dy} = \mu_B C_{1B} = \frac{\mu_A \mu_B U}{\mu_B H_A + \mu_A H_B}$$

So, if we look into the expression you can see that τ_{yx} in the fluid A and fluid B are same and constant; that means, the shear stress will be constant and same value in entire fluid domain . So, that means,

$$\tau_{yx}|_A = \tau_{yx}|_B$$

So, now from this expression you can see the if you want to compare the velocity gradient say

$$\frac{\frac{du_B}{dy}}{\frac{du_A}{dy}} = \frac{C_{1B}}{C_{1A}} = \mu_A/\mu_B$$

So, if you consider that the viscosity in fluid domain A is greater than the viscosity in the fluid domain B. So, what will happen? So, from this expression you can say that μ_A if it is greater than μ_B then obviously, $\frac{du_B}{dy}$ will be greater than $\frac{du_A}{dy}$. So, obviously, you can see that the fluid velocity is linear inside domain fluid domain. So, obviously, this will be constant in the entire fluid domain.

So, obviously, you can write

$$\frac{dy}{du_B} < \frac{dy}{du_A}$$

\ So, if your velocity is varying linearly so, you can see that this is your dy and this is your du. And, if this is the angle θ so, you can see you can write dy/du as tan θ ok. So, you can see $\frac{dy}{du_B}$ will be just $\tan\theta_B$ will be less than $\tan\theta_A$; that means, θ_B will be less than θ_A.

So, depending on the values of μ_A and μ_B you can see that if μ_A is greater than μ_B, you will get θ_B as less than θ_A.

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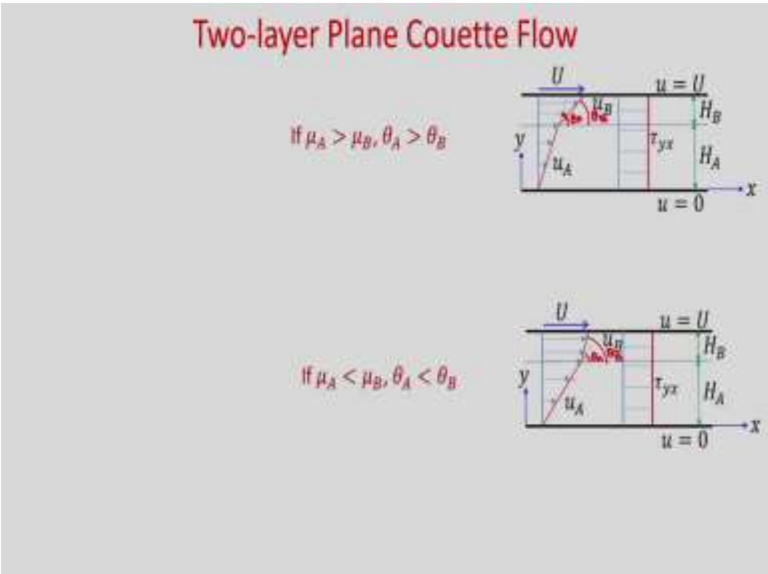

So, now, let us plot the velocity profile for two different conditions. So, we have already found that if μ_A is greater than μ_B then θ_A will be greater than θ_B and obviously, you can see that this is the angle for the velocity profile in fluid domain A. So, this will be θ_A and this is θ_B. So, θ_A will be greater than θ_B.

So, you can see that velocity is linearly varying in the fluid domain A after that. So, θ_B will be less than θ_A. So, your velocity profile will look like this u_B in the fluid domain B and if you consider that μ_A less than μ_B . So, then you can so that θ_A will be less than θ_B.

So, you can see this is θ_A and this is θ_B. So, θ_A obviously, is less than θ_B. So, your velocity profile will be like this after that θ_B will be greater than θ_A. So, u_B velocity profile will look like this. So, in this way you can plot the velocities in the two different fluid domains.

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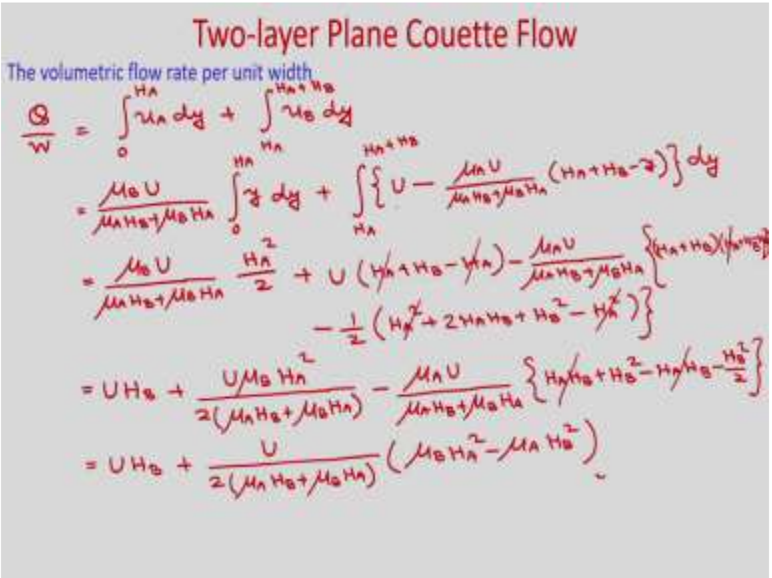

Now, let us find what is the volumetric flow rate for these two layer Couette flow. So, in each fluid domain we need to integrate this integral udA to find the Q. So,

$$\frac{Q}{W} = \int_0^{H_A} u_A \, dy + \int_{H_A}^{H_A+H_B} u_B \, dy$$

$$= \frac{\mu_B U}{\mu_A H_B + \mu_B H_A} \int_0^{H_A} y \, dy + \int_{H_A}^{H_A+H_B} \left\{ U - \frac{\mu_A U}{\mu_A H_B + \mu_B H_A} (H_A + H_B - y) \right\} dy$$

So, now if you integrate it, it will be

$$= \frac{\mu_B U}{\mu_A H_B + \mu_B H_A} \frac{H_A^2}{2} + U H_B - \frac{\mu_A U}{\mu_A H_B + \mu_B H_A} \{(H_A + H_B) H_B\} - \frac{1}{2} (H_B^2 + 2 H_A H_B)$$

$$= U H_B + \frac{\mu_B U H_A^2}{2(\mu_A H_B + \mu_B H_A)} - \frac{\mu_A U}{\mu_A H_B + \mu_B H_A} \left\{ H_B^2 - \frac{H_B^2}{2} \right\}$$

$$= UH_B + \frac{U}{2(\mu_A H_B + \mu_B H_A)}(\mu_B H_A^2 - \mu_A H_B^2)$$

So, this is the expression for volumetric flow rate per unit width for two layer plane Couette flow.

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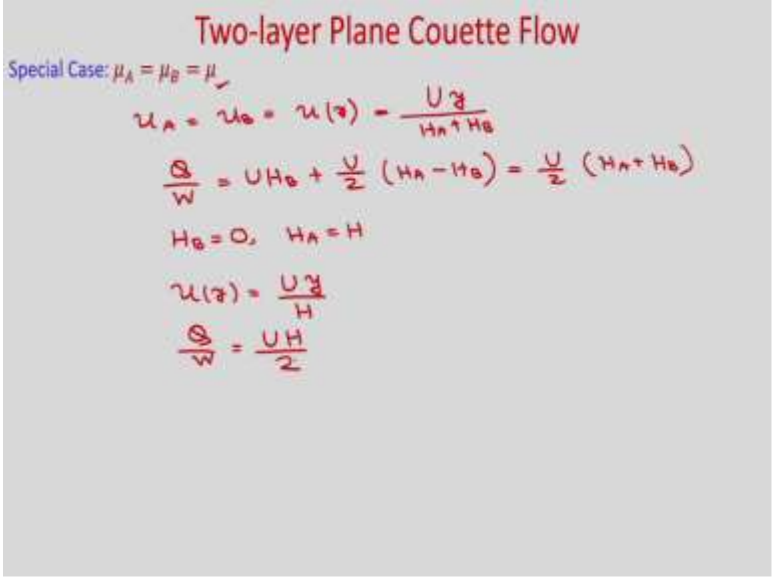

Now, let us see the special case where we have μ_A is equal to μ_B is equal to μ; that means, we have same fluid in region A and B. So, in this case you will get obviously, you can see

$$u_A = u_B = u(y) = \frac{Uy}{H_A + H_B}$$

and

$$\frac{Q}{W} = UH_B + \frac{U}{2}(H_A - H_B) = \frac{U}{2}(H_A + H_B)$$

And, let us say that H_B is equal to 0 and H_A is equal to H then you will get plane Couette flow. So, you see the velocity profile will be same as the plane Couette flow whatever we have derived

$$u(y) = \frac{Uy}{H}$$

And

$$\frac{Q}{W} = \frac{UH}{2}$$

So, in today's class first, we simplified the Navier-Stoke equation invoking the assumptions so that we can have the analytical solution. So, what we did? We converted the partial differential equation to an ordinary differential equation invoking the assumptions. Then, we derived the velocity profile, shear stress distribution, the volume flow rate and the average velocity for plane Couette flow considering different cases.

Then we considered two-layer Couette flow where we have two immiscible fluids of different viscosities μ_A and μ_B. In this particular case, we found the velocity distribution in fluid layer A and fluid layer B.

Then, we calculated the shear stress distribution and as we have the interface condition that shear stress is continuous at the interface and in each fluid layer the shear stress is constant, hence the shear stress is the same and constant for the whole fluid domain. Then we calculated the volumetric flow rate inside the flow domain for two-layer Couette flow.

Thank you.