Viscous Fluid Flow
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Module - 11
Turbulent Flows - I
Lecture - 03
External Turbulent Flows

Hello, everyone. In last class we derived the Reynolds average Navier Stoke equations and we have seen that there are additional 9 terms appearing due to the velocity fluctuations. Out of these 9 additional components, we have 6 unknowns. Today, we will consider external flows, flow over flat plate and we will simplify these equations and we will try to model these unknowns.

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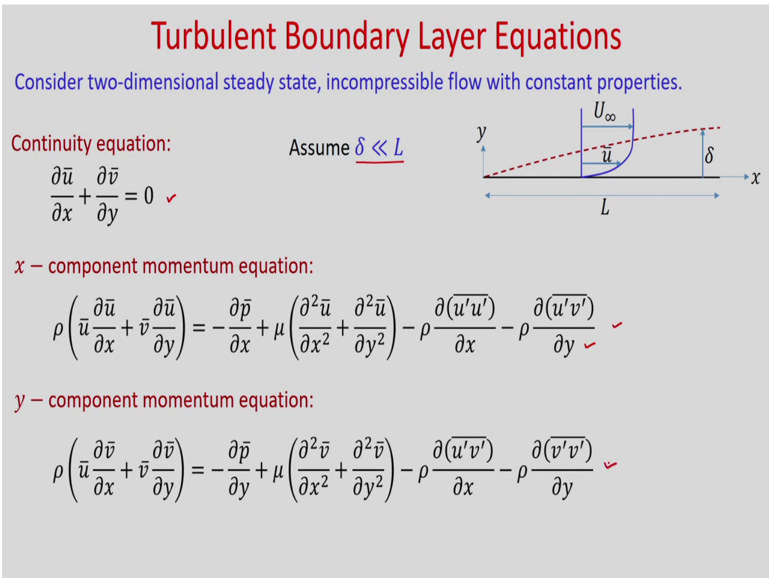

So, consider two-dimensional steady state incompressible flow with constant properties. So, this is a flat plate, flow over flat plate is taking place, this is the boundary layer thickness delta, L is the length of the flat plate, u bar is the time averaged velocity and it is varying in the boundary layer and outside we have free stream velocity U infinity. When you are considering turbulent boundary layer equations so; obviously, delta will be much much smaller than L.

So, this is the continuity equations we have derived in last class and this is the x component of momentum equations as you are considering two-dimensional situation. So, we have two additional terms in the x component of momentum equations. Similarly, in y component of momentum equations, we have two additional terms. If we consider turbulent boundary layer flow, so, let us check which are the terms, actually we can drop from these equations.

As we derived the boundary layer equation for laminar flow, we will follow the similar scale analysis and we will try to see that which is term is very very small compared to the other. So, here the scale will take or u r as U infinity x as L and y as boundary layer thickness delta. And; obviously, as you are considering boundary layer flow delta is much much smaller than L.

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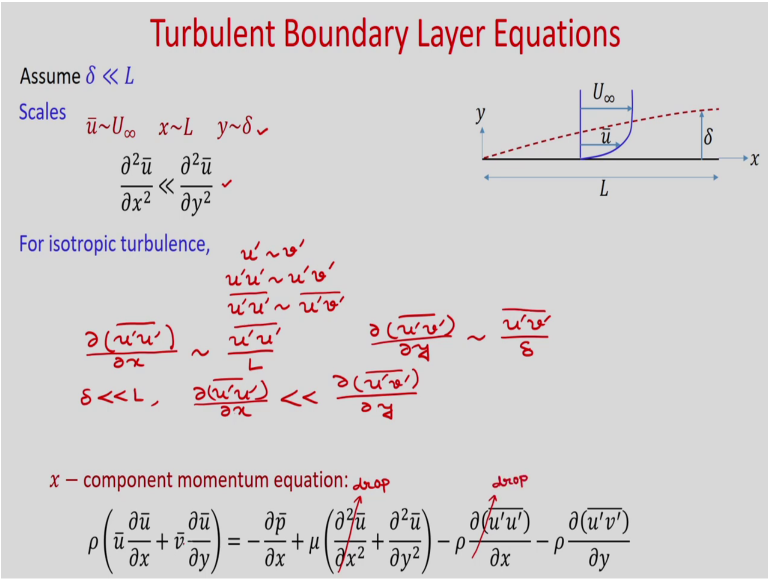

So, using this scale; obviously, you can see that del 2 u bar by del x square will be much much smaller than del 2 u bar by del y square.

Now, if we consider isotropic turbulence then; obviously, this u prime will be order of v prime ok. So, if it is so, then you can write u prime u prime will be order of u prime v prime. Now, if you take the time average, then u prime u prime time average will be order of u prime v prime time average. So, now, if you take the gradient then, we can write that del u prime u prime bar of del x will be order of so, you can write u prime u prime and x is order of L.

And, similarly you can see that for the other term in the x component momentum equation, we have del u prime v prime bar del y ok. So, this also we can write here it will be bar. So, will be order of u prime v prime bar and y scale is delta. Now, you compare these two ok. So, you can see that already we have shown that u prime u prime bar will be order of u prime v prime bar; that means, in the numerator these are of same order right.

So, you can see if delta is much much smaller than L ok; obviously, del u prime u prime del x will be much much smaller than del u prime v prime bar del y ok. So, obviously, you can see that this term is very very small compared to this term. So, now, if you see the x component momentum equation, so, from the viscous term ok you can see here. So, del 2 u bar by del x square is much much smaller than del two u bar by del y square.

So, we can drop this term ok. And if you compare these two terms then; obviously, this is very small compared to the other. So, we can drop this term ok. So, you can see that we have simplified this turbulent boundary layer equation ok, for the x component of momentum equation as this. Now, let us consider the y component of momentum equation, we know that v bar is very very small compared to the u bar so, that we can show from the continuity equation.

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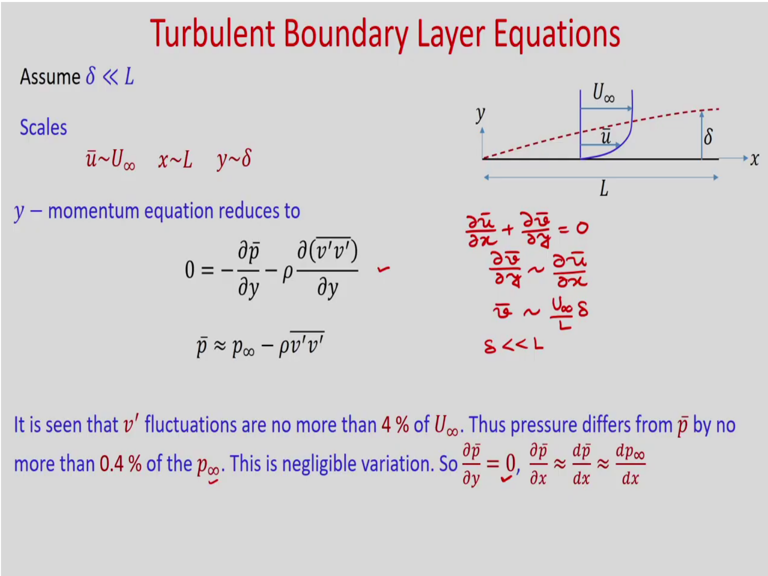

So, what is our continuity equation? Del u bar by del x plus del v bar by del y is equal to 0. So, from here you can see del v bar by del y will be order of del u bar by del x. So, v bar will be order of so, this will be u infinity ok, x is L and y is delta ok. And; obviously, delta is much much smaller than L.

So, delta by L is much much smaller than 1. So, v bar will be very very small compared to u infinity ok. So, you can see that from here the terms in the y component momentum equation, the viscous term and the inertia terms you can neglect and you will get this y component of momentum equation as this ok. And from here you can see that p bar you can write as p infinity minus rho v prime v prime bar ok.

So, it is seen that v prime fluctuations are no more than 4 percent of U infinity ok. Thus pressure differs from p bar by no more than 0.4 percent of the p infinity. So, this is negligible variation. So, we can write del p bar by del y is equal to 0. So, if del p bar by del y is equal to 0. So, del p bar by del x we can write d p bar by d x, which we can write also d p infinity by d x where p infinity is the free stream pressure ok outside the boundary layer.

So, using scale analysis, we have dropped some terms in the x momentum equations and from y component of momentum equations we have shown that normal pressure gradient is 0.

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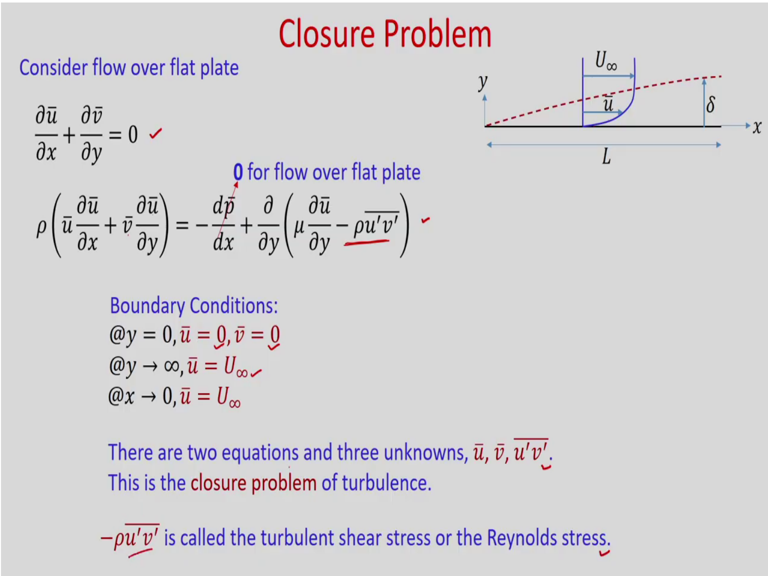

Now, we can write the simplified equations as this is the continuity equation and if we consider flow over flat plate. So, d p bar by d x will be 0. So, and we will have this term ok.

So, what are the boundary conditions? You can see at y is equal to 0; obviously, u bar is equal to 0 v bar is equal to 0 and y tends to infinity, u bar is equal to infinity and x tends to 0 near to the leading edge u bar is equal to u infinity ok.

So, now, you can see, we have two equations right, continuity equation and this simplified x component momentum equation ok. And how many unknowns are there? So, you can see we have u bar v bar and this u prime v prime bar.

So, there are two equations and three unknowns ok. So, this is the closure problem of turbulence, because these we have lesser number of equation than unknowns. So, this term minus rho u prime v prime bar is called the turbulent shear stress or Reynolds stress.

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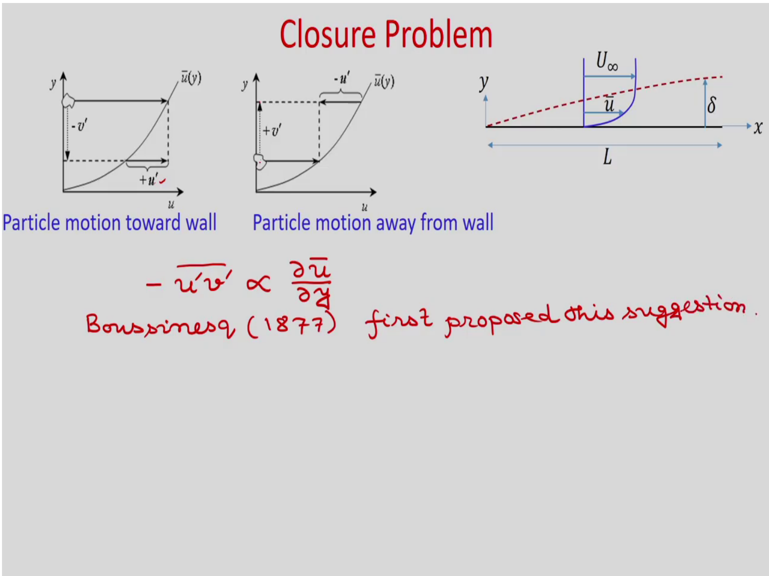

Now, question is how to model this term? So, this unknown term, let us see that whether we can model using the velocity gradient ok. Considered a fluid particle fluctuation imposed on some average velocity profile ok. So, you can see that at this location; obviously, it is having some velocity, if this fluid particle comes to this position ok in the negative y direction so;

obviously, it will have some higher velocity. So, what it will do. So, this fluid particle will try to accelerate or move this particle at a velocity u prime plus u prime ok.

So, you can see that relative to the local velocity the fluid particle has a velocity that is higher by a value of plus u prime. So, the velocity fluctuation in y direction minus v prime, results in a velocity fluctuation in x direction plus u prime. So, it appears that magnitude of u prime depends on the slope of the mean velocity profile. So, you can see that we can write these component u prime v prime in terms of mean velocity gradient.

Similar way you consider this fluid particle at this position which is having lesser velocity than at this point. So, due to this fluctuation plus v prime these particle moves here then; obviously, it will try to retire the flow with a velocity minus u prime. So, you can see that when minus v prime we have this velocity fluctuation, we have this plus u prime. So, now, we can write that like viscous shear stress time average fluctuations is also proportional to the velocity gradient.

So, that we can write that minus u prime v prime bar is proportional to the average velocity gradient.

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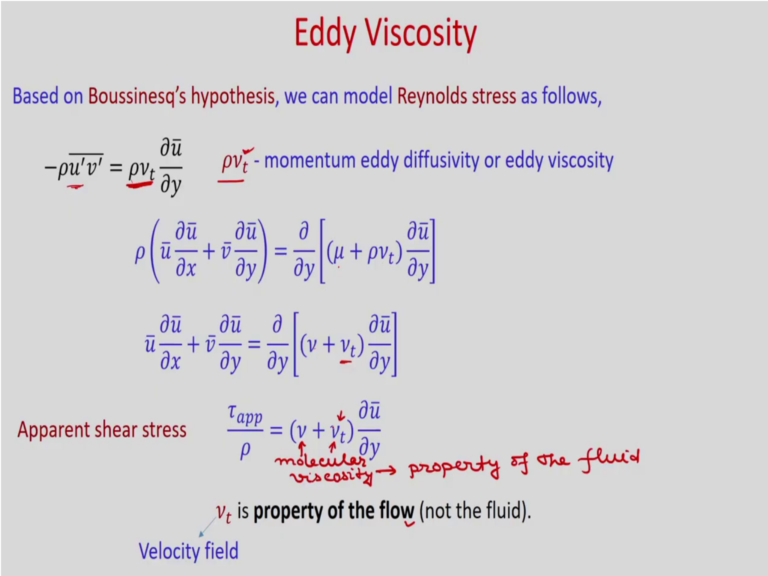

So, this was actually proposed by a Boussinesq in 1877 first proposed this suggestion. Now, based on this Boussinesq hypothesis, we can model Reynolds stress as follows. So, you can write minus rho u prime v prime bar ok. So, you can write so, this rho u prime v prime bar is proportional to del u bar by del y.

So, we can write with some constant nu t rho nu t del u bar by del y, where rho nu t is known as momentum eddy diffusivity or eddy viscosity ok. So, this rho nu t is momentum eddy diffusivity and this nu t; obviously, is unknown ok. So, if we model using this Boussinesqs hypothesis like this, then the in the right hand side we can write minus rho u prime v prime bar as rho nu t del u bar by del y and already we have molecular viscosity.

So, now, you have written this equation just dividing the rho in the right hand side. So, we can write u bar del u bar by del x plus v bar del u bar by del y is equal to del of del y nu. So, mu by rho is nu t and this will be just nu t del u bar by del y.

So, you can see that using Boussinesqs hypothesis these fluctuations we have written like this and here only unknown term is nu t ok. So, here we can write this term as apparent shear stress. So, tau app by rho is equal to nu plus nu t del u bar by del y. So, nu; obviously, you can see that this is molecular viscosity and it is a property of the fluid. However, nu t is not property of the fluid, nu t depends on the velocity field and nu t is property of the flow ok.

So, here this is your molecular viscosity ok so; obviously, this is property of fluid; however, this nu t term is property of the flow. Now, you can see that how to model this nu t right. Because it is unknown and in boundary layer equation we do not have any exact solutions, we can either use some numerical techniques or we can use some integral method with some approximation.

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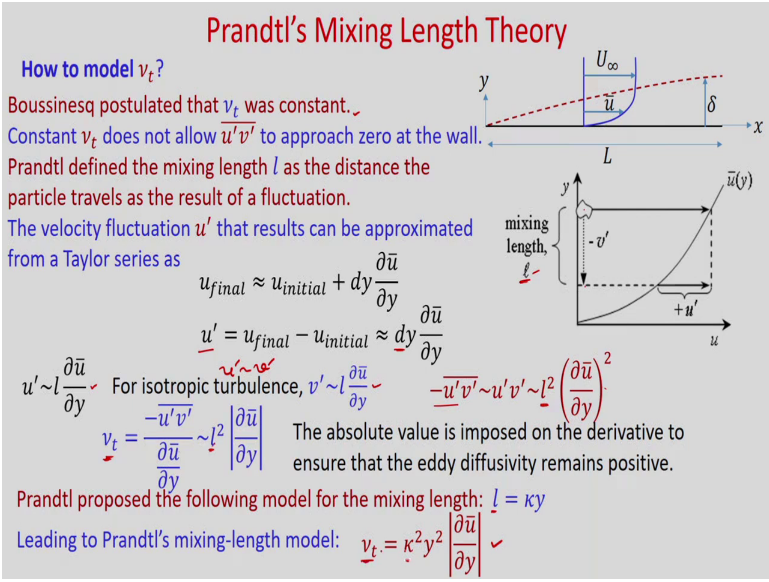

So, to model this nu t Boussinesq postulated that nu t was constant ok. So, constant nu t does not allow u prime v prime bar to approach zero at the wall. So, Prandtl defined the mixing length l as the distance the particle travels as the result of fluctuation.

So, you can see that when this fluid particle comes to this position with a fluctuation of minus v prime ok it results in a velocity of plus u prime. So, you can see that this we can actually use this length as mixing length and Prandtl define this mixing length l as the distance the particle travels as result of fluctuation.

So, these velocity fluctuation u prime that results can be approximated from a Taylor series ok. So, if it is u final here this is u initial. So, u final we can use Taylor series expansion and can write u initial plus dy del u bar by del y ok. So, obviously, the difference in the velocity u final minus u initial is u prime that we can write as dy del u bar by del y and this d y, we can write as a mixing length l. So, you can write u prime is order of l del u bar by del y ok.

And for isotropic turbulence, we know that u prime is order of v prime. So, v prime will be ordered of l del u bar by del y. And, here you can see that if you take the time average of this minus u prime v prime bar. So, we will get this as l square del u bar by del y square. Because these two terms and we can see that; obviously, this minus u prime v prime bar will be positive quantity because, you can see that when v prime is negative u prime is positive right.

So, when we consider minus u prime v prime due to this negative sign; obviously, this will be a positive quantity. Similar way we have also discussed that when this fluid particle goes in the upward direction with a fluctuation v prime, it results a velocity in minus u prime. So, you can see that this product of u prime v prime will be negative. So, this quantity itself will be a positive.

So, now as you have used this Prandtl's mixing length theory to express this minus u prime v prime bar as l square del u bar by del y square, where l is the mixing length. And now, we can write nu t that minus u prime v prime bar divided by del u bar by del y right. So, this term we have now used this expression. So, we can see we can write it as l square del u bar by del y ok.

So, we are use this absolute value because nu t is positive ok. So, the absolute value is imposed on the derivative to ensure that the eddy diffusivity remains positive ok. So, now, Prandtl propose the following model for the mixing length so, because l is also unknown right. So, Prandtl propose this mixing length l as equal to kappa y ok and leading to Prandtl's mixing length model. So, nu t we can write l as kappa y. So, kappa square y square absolute value of del u bar by del y.

So, this is the expression of this eddy viscosity using Prandtl's mixing length theory. Where kappa is unknown that is to be determined from the experimental results. We are considering flow over plate so, it is nearly parallel flow.

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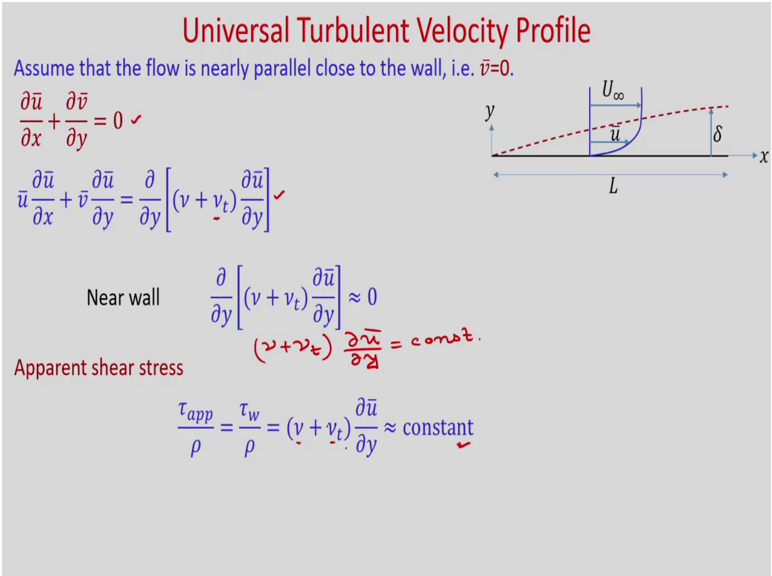

So, we can assume that v bar as 0. So, you can see that we have this continuity equation and this is the turbulent boundary layer momentum equation ok.

Where nu t is eddy viscosity and that using Prandtl's mixing length hypothesis, we have expressed as kappa square y square del u bar by del y. So, now, if you consider near wall region ok so; obviously, u bar and v bar will be 0. So, from here you can see that if v bar is 0. So, this term will become 0 and u bar will be very very small. So, this term will be 0. So, in near wall we can write this term is equal to 0.

So; obviously, you can see that if this term is 0, then nu plus nu t if you integrate it ok. So, it will be constant ok. So, obviously, then we have already expressed this apparent viscosity by rho ok.

So, that we can write as at the wall; obviously, we can write tau w by rho is equal to nu plus nu t del u bar by del y which is constant. Now, let us see that when we can drop one of these two terms because this is your molecular viscosity, ok which is very prominent near to the wall and nu t you can see that it is due to the fluctuating components.

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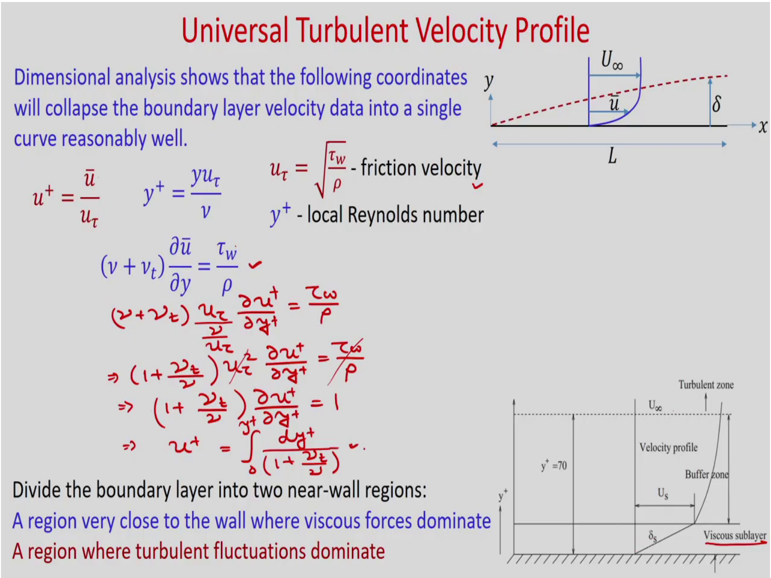

So, away from the wall this nu t will dominate the flow. Dimensional analysis shows that the following coordinates will collapse the boundary layer velocity data into a single curve reasonably well. So, we will use these non-dimensional parameters u plus which is the ratio of mean velocity u bar divided by u tau, where u tau is the friction velocity ok which is root tau w by rho ok. And y plus it is the non-dimensional y coordinate which is equal to y u tau by nu, where u tau is the friction velocity and nu is the molecular viscosity ok.

So, from here you can see that this equation already we have written in terms of the wall shear stress tau w. So, now, you can see that we can using these non-dimensional parameters, we can write this equation as. So, nu plus nu t u bar. So, this u bar is u plus u tau. So, you can write u tau del u bar by del y ok. So, del y if you write. So, it will be nu y plus by u tau. So, it will be del y plus, but in the denominator we will have nu by u tau and in the right hand side we have tau w by rho.

So, now if you divide nu here so, you will get 1 plus nu t by nu and you can see it will be u tau square right, it will be u tau square sorry, this will be u plus. So, it will be del u plus by del y plus is equal to tau w by rho. Now, here you can see that u tau square is tau w by rho. So, these will get cancel ok. So, we can write 1 plus nu t by nu del u plus by del y plus is equal to 1 ok.

So, now, from here if you integrate and you can find the u plus as integral 0 to y plus d y plus divided by 1 plus nu t by nu ok. So, integrating this you can find the value of u plus. Now, we will divide the boundary layer into two near-wall regions ok. So, if you consider flow over flat plate near to the wall; obviously, this molecular viscous force will dominate ok. And if you consider very far away from the wall then; obviously, this fluctuating component will dominate the flow.

So, due to that; obviously, the eddy viscosity will dominate the flow. So, if we divide the boundary layer into two regions very near to the wall and very far away from the wall, then you can see that here if this is the flow over a flat plate. So, this is y plus and now, if you see that very near to the wall ok this viscous force dominate. So, this is known as viscous sub layer. And if you consider away from the wall. So, let us say here it is fully turbulent zone and this region turbulent fluctuation dominate ok.

So, these are the two extreme cases you can consider and in between we can consider one over lap player or which is known as buffer zone also, where we will have the effect of both these viscous force as well as turbulent fluctuations ok. So, now, this equation ok let us simplify for viscous sub layer and this fully turbulent layer. Now, let us consider the viscous sub layer which is very close to the surface of the flat plate then; obviously, in this region you can see viscous sub layer dominates and the velocity fluctuations will be very small.

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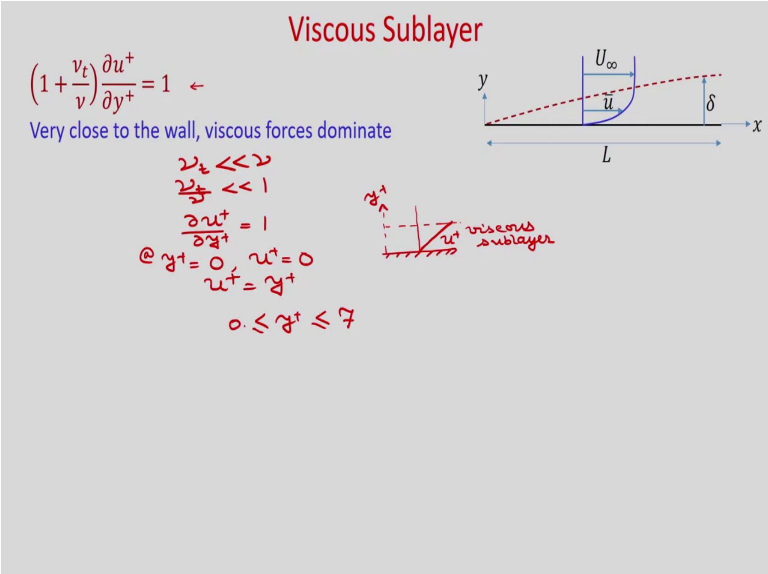

So, obviously, you can see that nu will be much much greater than nu t ok. With this simplification, we can write this equation, you can see that very close to the wall viscous force dominate. So, we will get nu t ok is much much smaller than nu ok. So, nu t by nu will be much much smaller than 1. So, if it is much much smaller than 1 than, this equation we can simplify and write del u plus by del y plus is equal to 1 ok.

So, now, if you apply the boundary conditions that at y plus is equal to 0 u plus is equal to 0. So, from here you can write u plus is equal to y plus ok. And so, in the viscous sub layer region ok, we can write u plus is equal to y plus which is a linear profile that means this u plus ok.

So, if this is the laminar sub layer or viscous sub layer, then you can see that u plus will vary linearly with y plus. So, if this is y plus then you can see that here this velocity will vary linearly ok. So, this is your u plus ok. And generally this viscous sub layer will be, in the range of 0 y plus less than equal to 7 ok.

And, you can see that laminar sub layer is of importance for turbulent flow calculations, since the shear stress at the wall is determined by the thickness of the laminar sub layer. Now, if we consider the region far away from the wall then these viscous force will be very very small and the velocity fluctuations will dominate; that means, Reynolds stress will dominate ok.

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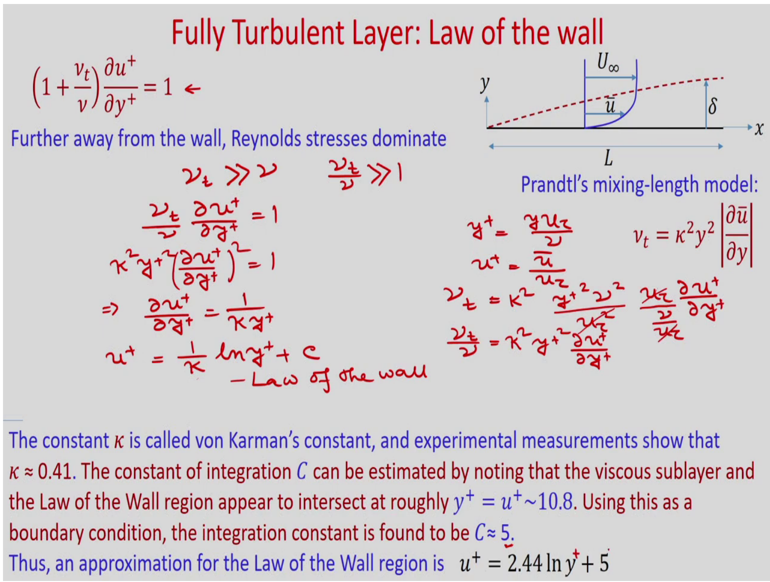

So, again we consider this equation. So, nu t will be much much greater than nu so; obviously, you can see that in the fully turbulent layer ok far away from the wall nu t is much much greater than nu as Reynolds stresses dominate. So, that means, nu t by nu will be much

much greater than 1. So, from this equation, we can write nu t by nu del u plus by del y plus is equal to 1 ok.

Now, nu t by nu ok what is nu t? So, from Prandtl's mixing length model, we know nu t is equal to kappa square y square del u bar by del y. So, we can write and we have also y plus is equal to y u tau by nu and u plus is u bar by u tau ok. So, now, nu t, we can write from here you can see, kappa square ok y square ok. So, from here you can see y square you can write, y plus square nu square divided by u tau square ok.

And, we have del u bar by del y. So, these del u bar we can write u tau del u plus by del y. So, del y again so, we can write del y plus, but in the denominator we will have nu by u tau ok. In the denominator we have u tau so; this will be u tau square. So, this will get cancelled and 1 nu will be there and we can write nu t by nu as kappa square y plus square del u plus by del y plus ok.

So, now, this let us write as kappa square y plus square del u plus by del y plus square is equal to 1.

So, this we can write as del u plus by del y plus is equal to 1 by kappa y plus. So, now, if we integrate this, we can write u plus is equal to 1 by kappa ln y plus plus some constant C ok. So, this is known as law of the wall ok and which is valid in fully turbulent layer. Now, what is the value of kappa and C? So, it is very difficult to determine so, these are actually determined from the experimental results ok.

So, we have some empirical relations and from there you can find the value of kappa and C. This constant kappa is called a Von Karman's constant and experimental measurements. So, that kappa is order of 0.41 and the constant of integration C can be estimated by noting that the viscous sub layer and the law of the wall region appeared to intersect at roughly y plus is equal to u plus at 10.8. Using these as a boundary condition the integration constant is found to be 5 ok.

So, C is equal to 5. So, now, if you put kappa is equal to 0.41 and C is equal to 5, then the approximation for the law of the wall region is u plus is equal to 2.44 ln y plus plus 5. Now, we have another layer which actually in between the viscous sub layer and fully turbulent layer, that layer is known as overlap layer or buffer layer, where viscous force and the fluctuations dominate the flow. So, there are many laws proposed by different scientists.

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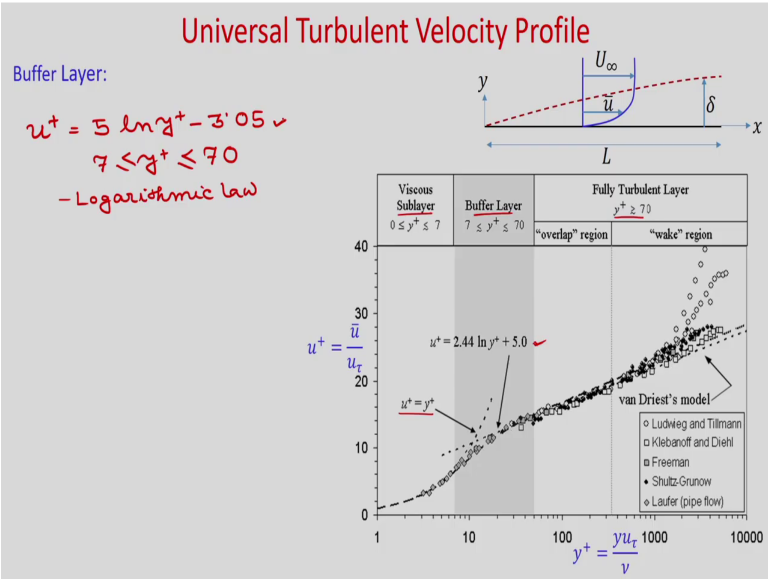

So, we can write one of these law as u plus is equal to 5 ln y plus minus 3.05and which is in the layer of 7 y plus less than equal to 70 ok.

So, in this buffer layer laminar and turbulent motion coexists ok and this is your logarithmic law. You can see that y plus up to 7, it is viscous sub layer y plus greater than 70 it is fully turbulent layer and in between y plus 7 and 70 we have buffer layer or overlap layer.

So, these two actually are known as overlap layer or buffer layer. Now, you can see that u plus bar y plus if plot. So, here this will be u plus is equal to y plus. So, you can see that up to y plus 7 ok, you will get almost linear profile and in fully turbulent layer. We have the logarithmic and from the experimental values different scientists they actually plotted this u plus versus y plus and you can see these symbols are from these results from these papers.

And we have this u plus is equal to y plus an d we have u plus is equal to 2.44 ln y plus plus 5 that is in fully turbulent layer and in between you can join and you can write this u plus is equal to 5 ln y plus minus 3.05.

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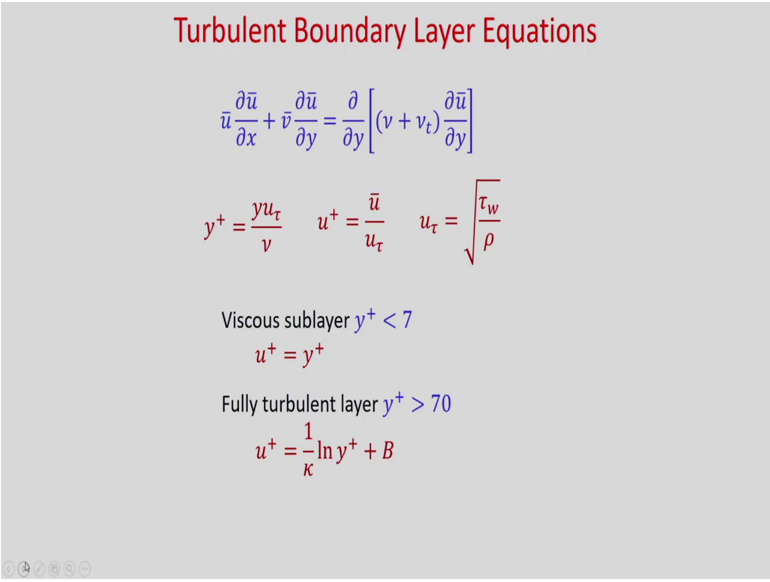

So, now, let us summarize whatever we have studied in today's class. In today's class, we first started with the Reynolds average Navier-Stoke equations for steady state two-dimensional flow. Using scale analysis, we have derived the turbulent boundary layer equations dropping some terms.

Then we used Prandtl's mixing length hypothesis to find the unknown minus rho u prime v prime, but and that we have represented in terms of mixing length l. So, in the apparent shear stress, we have contribution from the molecular viscosity as well as eddy viscosity, but eddy viscosity is unknown and using Prandtl's mixing length hypothesis, we have represented this mixing length l in terms of kappa and y. Finally, this eddy viscosity is represented in terms of kappa and the velocity gradient.

So, obviously, we have seen that these values can be found using some experimental result in the turbulent boundary layer, we have three distinct regions, one is very near to the wall which is known as viscous sublayer or laminar sublayer, then we have far away from the wall that is known as fully turbulent layer and in between we have buffer layer or overlap layer.

So, using these non-dimensional parameters u plus and y plus, we have found the u plus value in terms of y plus in these three regions. We have seen that viscous sub layer will be y plus from 0 to 7 and y plus greater than 70, we have fully turbulent layer and y plus in between 7 and 70 we have buffer layer.

Thank you.