Viscous Fluid Flow
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Module - 11
Turbulent Flows - I
Lecture - 02
Derivation of Reynolds Average Navier-Strokes Equations

Hello everyone. So, today we will derive Ran's equation. Ran's stands for Reynolds average Navier-Stoke to equations. This equation is equation for turbulent flows, we have already discussed that these velocities we can decompose into two quantities, one is mean velocity and one is fluctuating velocity.

Due to these presence of fluctuating velocities there will be some additional stresses in the governing equations. So, we will derive today this Reynolds average Navier stoke equation using some averaging property.

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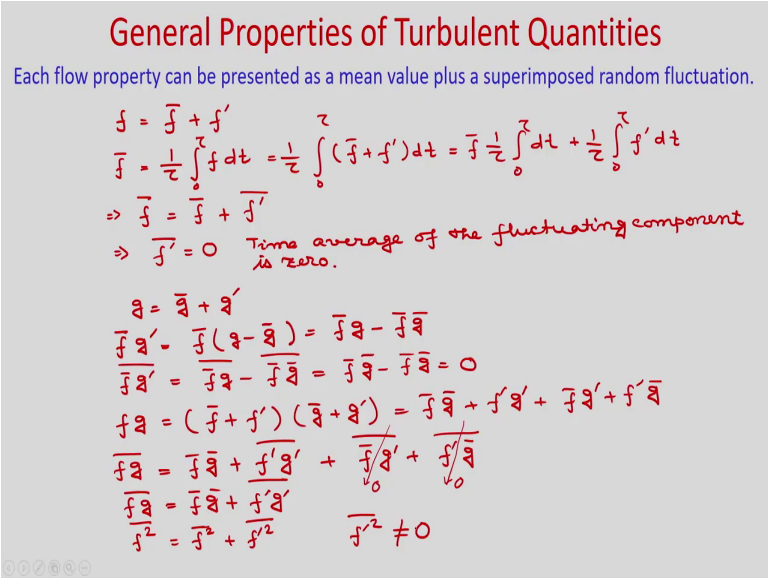

First let us consider two turbulent quantities f and g, then we will apply these rules of averaging to this turbulent quantities f and g. So, as we know that each flow property can be presented as a mean value plus a super imposed random fluctuation. If we consider two turbulent quantities f and g, we can write f is equal to f bar plus f prime. So, you can see this f bar is mean value and this f prime is the fluctuating component.

So obviously, f bar we can write as 1 by tau integral 0 to tau f dt right. This is the averaging quantity f bar. So, were averaging over a time 0 to tau. You can see in the right hand side this f we can substitute with mean value plus fluctuating value. So, you can write 1 by tau 0 to tau f bar plus f prime dt ok. So obviously, this is averaging quantity. So, we can take it outside the integral. So, you can write f bar 1 by tau to 0 to tau to dt and this we can write as 1 by tau 0 to tau f prime dt.

So, you can see that we can write f bar is equal to this will become f bar plus this now it is averaging of f prime bar. So, f prime bar so you can see from here we can show that averaging or fluctuating quantities is 0. So, we can write that time average of the fluctuating component is 0. Similarly, we can write g is equal to g bar plus g prime ok. So, now, if we do f bar g prime ok, so, f bar so, g prime we can write as g minus g bar ok. So, we can write as f bar g minus f bar g bar ok.

So, if you take the time averaging of this quantity. So, you can write f bar g prime bar is equal to f bar g time averaging minus f bar g bar. So obviously, you can see that this will become f bar g bar this will also become f bar g bar so, this will be 0. So, you can see this quantity is 0.

Similarly f into g if you write then f bar plus f prime g bar plus g prime. So, you can write f bar g bar plus f prime g prime plus f bar g prime plus f prime g bar, ok. Now if you take the average of this quantity time average. So, we can write a write as f bar g bar plus f prime g prime bar plus f bar g prime bar plus f prime g bar bar.

So, you can see from here this quantity is 0. So, these two quantities will become 0 ok. So, you can see that we can write f g bar is equal to f bar g bar plus f prime g prime bar. Similarly, we can write f square bar if you put g is equal to f, then we can write here f bar square plus f prime square bar and you can see from here that f prime bar square is not 0 f prime square bar not equal to 0 f prime square bar not equal to 0.

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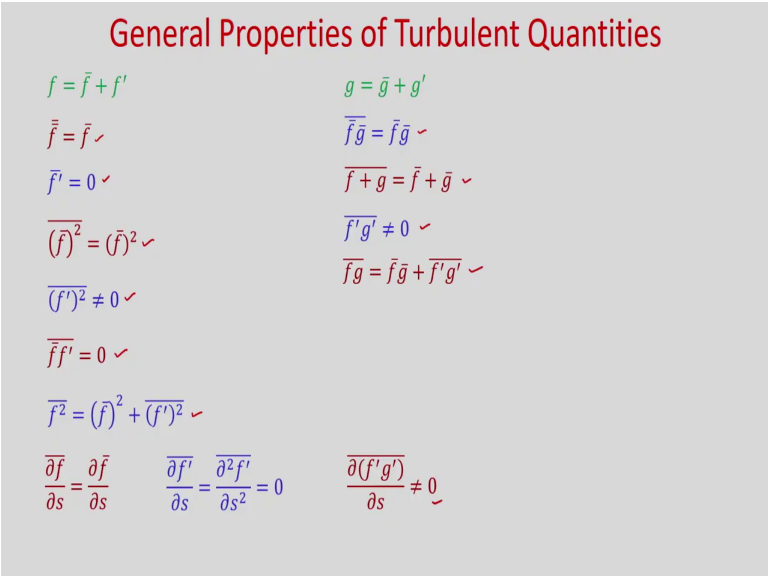

So, similarly way we can show other rules also. So, you can see that these are already we have derived these two. And this quantity you can see f bar square bar you can write f bar whole square and f bar square bar is not equal to 0, f bar f prime bar is equal to 0 and we have already derived this, f square is equal to f bar square plus f prime square bar.

Similarly, g is equal to g prime plus g bar. So, this we have already shown this we can write and f prime g prime bar not equal to 0 and this we have already shown. And similarly if you take the derivative, then we can write this time average of this derivative. So, we can write del f bar del f by del s bar is equal to del f bar by del s. Similarly, we can write del f bar by del s bar is equal to del 2 f prime by del s square bar is equal to 0. And del of f prime g prime by del s bar not equal to 0.

So, this rules we will use while deriving the Reynolds average Navier Stroke equations. So, to derive Reynolds average Navier Stroke equation, first we will use Reynolds decomposition; that means, any turbulent quantity we can decompose into 2; one is mean component and another is fluctuating component.

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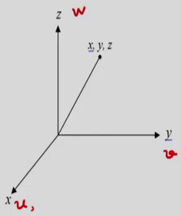
Reynolds Averaging of Conservation Equations

Consider incompressible flow with constant properties.

In Cartesian coordinates (x, y, z)



Continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

x – component momentum equation:

$$\frac{\partial u}{\partial t} + \frac{\partial (uu)}{\partial x} + \frac{\partial (uv)}{\partial y} + \frac{\partial (uw)}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

y – component momentum equation:

$$\frac{\partial v}{\partial t} + \frac{\partial (uv)}{\partial x} + \frac{\partial (vv)}{\partial y} + \frac{\partial (vw)}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

z – component momentum equation:

$$\frac{\partial w}{\partial t} + \frac{\partial (uw)}{\partial x} + \frac{\partial (vw)}{\partial y} + \frac{\partial (ww)}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

First let us consider the Navier Stroke equation in Cartesian coordinate for incombustible flow with constant properties. So, this is the continuity equation u v w at the velocities in x direction, y direction and z direction. And this is the x component of momentum equation, we have written in terms of conservative form. So, this inertia term we have written in conservative form and this is the viscous term and this is the pressure gradient term and nu is the kinematic viscosity.

Similarly, this is the y momentum equation and this is the z component of momentum equation. So, we will start from these equations and we will use this Reynolds decomposition and we will derive the Reynolds average Navier Stroke equations.

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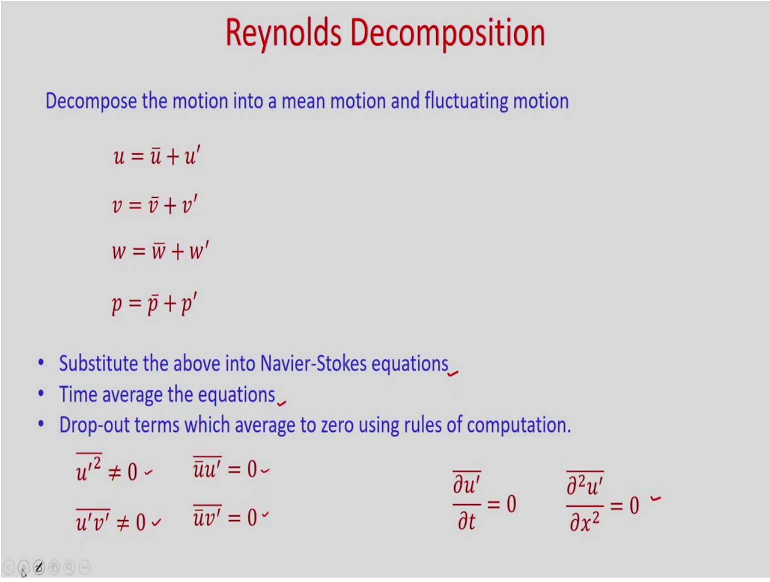

So, decompose the motion into a mean motion and fluctuating motion. So, this the u velocity u bar plus u prime, b is equal to b bar b prime, w is equal to w bar plus w prime and p is equal to p bar plus p prime.

So, now, we will follow 3 steps first we will substitute this decomposition into the governing equations, then we will use the time averaging and then we will use the rules of averaging to drop out some terms. So, first substitute the above into Navier Stroke equations then time average the equations then drop out terms, which average to 0 using rules of computation. So, these are the few rules we will use while deriving the Ran's equation. So, first let us start with the continuity equation.

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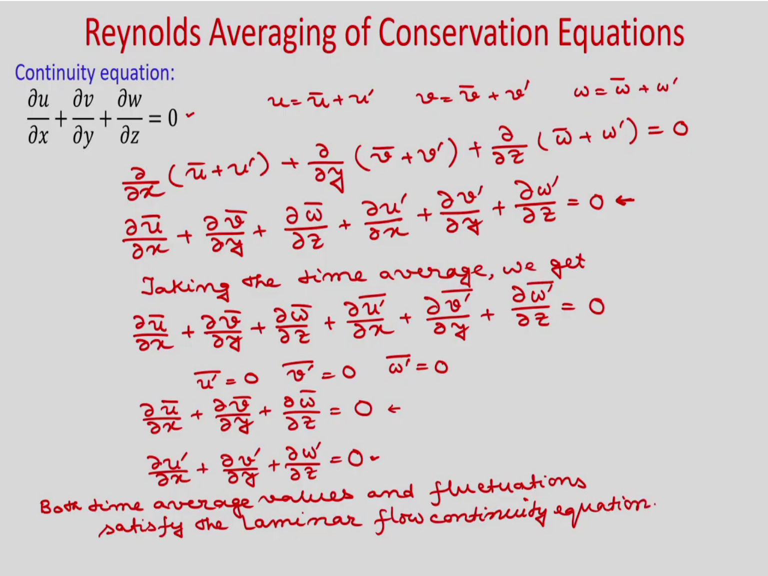

So, you can see this is the continuity equation. Now, if you substitute this u is equal to u bar plus u prime, v is equal to v bar plus v prime and w is equal to w bar plus w prime. So, you can see this equation we can write so, let us substitute this here.

So, del of del x u bar plus u prime plus del of del y v bar plus v prime and del of del z w bar plus w prime is equal to 0. So, now, we can write this as del u bar by del x plus del v bar by del y plus del w bar by del z. And these fluctuating component derivative we can write, del u prime by del x plus del b prime by del y plus del w prime by del z is equal to 0.

So, you can see in this equation; obviously, if you take the time average of this equations. So, what you will get? So, taking the time average we get. So, you use the rules of averaging, then we can write as del u bar by del x plus del v bar by del y plus del w bar by del z plus. So, this will become del u prime bar by del x plus del v prime bar by del y plus del w prime bar by del z is equal to 0.

So, we have already shown that the time averaging of this fluctuating component is 0, right. So, these are 0. So, if these are 0 then these three terms are 0. So, we can write del u bar by del x plus del v bar by del y plus del w bar by del z is equal to 0 ok. So, you can see this is the continuity equation for mean velocities for turbulent flows.

So, if you satisfy these in this equation. So, if you use this equation and if you satisfy this here, then you will get del u prime by del x plus del v prime by del y plus del w prime by del z is equal to 0. So, this is 0 so; obviously, these three terms together is 0 from this equation. So, you will get just del u prime by del x plus del v prime by del y plus del w prime by del z is equal to 0.

So, you can see that this fluctuating components also satisfy the continuity equation. So, you can write, that both time average values and fluctuations satisfy the laminar flow continuity equation.

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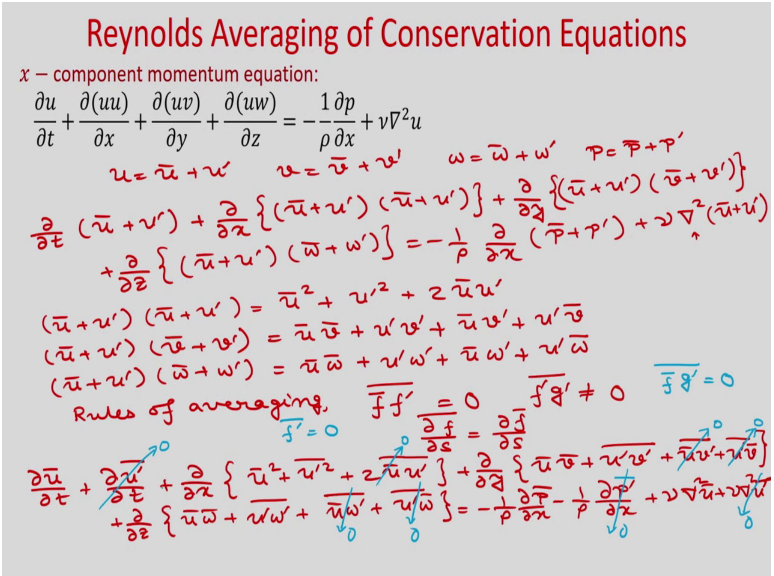

So, now, we will consider the momentum equation and we will derive the Reynolds average Navier Stoke equations. So, first let us consider x component of momentum equation, and the other equations similarly you can derive. So, this is the x component of momentum equation, where nabla square u is the diffusion term. So, we can now use these u is equal to u bar plus u prime, v is equal to v bar plus v prime, w is equal to w bar plus w prime and p is equal to p bar plus p prime ok.

So, these velocities and pressure we are writing in and summation of mean quantity and the fluctuating quantity. So, you substitute it here. So, what you will get del of del t u bar plus u prime plus del of del x. So, we will get u bar plus u prime u bar plus u prime, then del of del y we will get u bar plus u prime into v bar plus v prime, then del of del z u bar plus u prime w bar plus w bar prime is equal to minus 1 by rho del of del x p bar plus p prime plus nu nabla square u bar plus u prime ok.

So, now, you can see these two terms what you can write? u bar plus u prime u bar plus u prime ok. So, this we can write as u bar square plus u prime square plus twice u bar u prime.

Similarly, u bar plus u prime, v bar plus v prime we can write u bar v bar plus u prime v prime plus u bar v prime plus u prime v bar and u bar plus u prime into w bar plus w prime we can write as, u bar w bar plus u prime w prime plus u bar w prime plus u prime w bar ok. So, what we will do now. So, we will take the time average of this equation and we will use these rules of averaging ok.

So, now if you use these rules of averaging say like f bar f prime bar is equal to 0 f prime g prime bar not equal to 0, then we will write del f by del s bar is equal to del f bar by del s ok. So, all these rules will use after time averaging of this equation ok.

So, if you see that we will write del u bar by del t plus del u prime bar by del t plus del of del x. So, you can write as u bar square plus u prime square bar twice u bar u prime bar, then we have del of del y u bar v bar plus u prime v prime bar plus u bar v prime bar plus u prime v bar bar.

Then we have del of del z u bar w bar plus u prime w prime bar plus u bar w prime bar plus u prime w bar bar, then we have minus 1 by rho del p bar by del x minus 1 by rho del p prime by del x plus nu grad square u bar plus nu grad square u prime bar. So, now, you can see that this term is 0 because, we know that f prime bar is equal to 0 right.

So, this is 0, this term is 0 from here you can see that, this will be 0, then you can write f bar g prime bar is equal to also 0. So, if this is 0 then this is 0 this is 0 this is 0 this is 0. So, this is bar. So, this is 0 and this is 0. So, using these rules of averaging all these terms are becoming 0.

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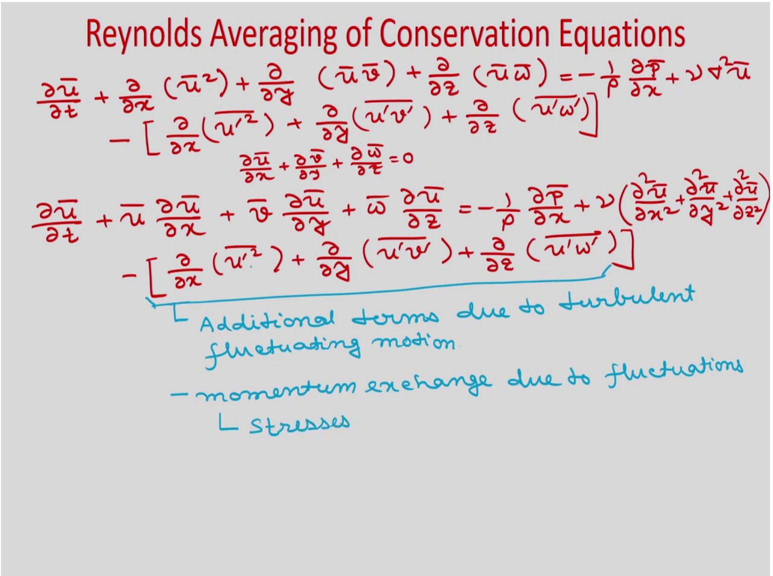

So, finally, you can write this equation as del u bar by del t plus del of del x u bar square plus del of del y u bar v bar plus del of del z u bar w bar is equal to minus 1 by rho del p bar by del x plus nu grad square u bar and we have some additional terms from the convective terms ok.

So, that will we are taking in the right side. So, it will become minus. So, we will get del of del x u prime square bar plus del of del y u prime v prime bar plus del of del z u prime w prime bar. So, you can see these are the additional terms coming due to these fluctuating component. So, this is known as Reynolds apparent stress ok.

So, we can write now in non conservative form del u bar by del t plus u bar del u bar by del x plus v bar del u bar by del y plus w bar del u bar by del z is equal to minus 1 by rho del p bar by del x plus nu del 2 u bar by del x square plus del 2 u bar by del y square plus del 2 u bar by del z square minus del of del x u prime square bar plus del of del y u prime v prime bar plus del of del z u prime w prime bar ok.

So, you can see that this you can write just invoking the continuity equation del u bar by del x plus del v bar by del y plus del w bar by del z is equal to 0. So, this equation you can write and you can see, these are the additional terms due to turbulent fluctuating motion.

So, these are the additional terms due to turbulent fluctuating motion and here you can see these are actually exchanging the momentum due to fluctuation. So, momentum exchange due to fluctuations and this will actually rise to stresses due to this fluctuating components.

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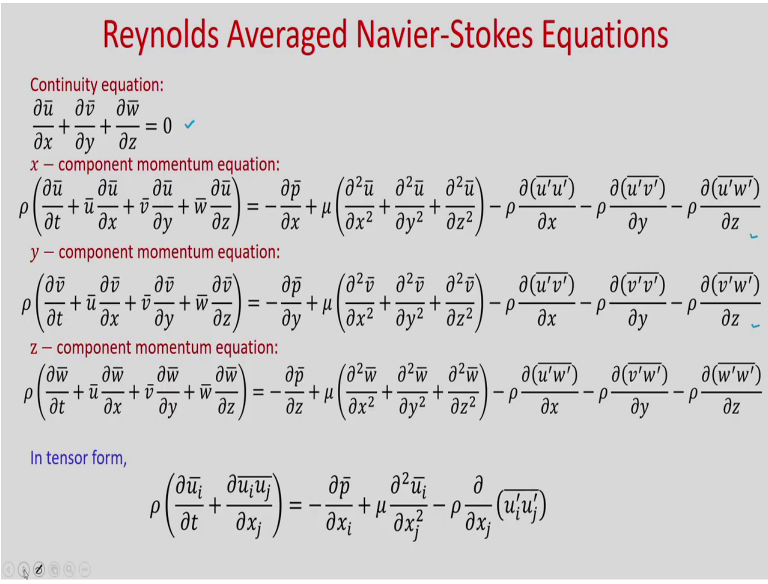

So, you can follow the similarly procedure and you can derive the y and z component of momentum equation. So, finally, we will get this set of Reynolds average Navier Stroke equation. So, you can see this is the continuity equation, this is the x component of momentum equation we have already derived and these are some additional terms and y component of momentum equation you will get there also will get three additional terms and z component of momentum equation, you will get another three additional terms.

So, all these three equations, you can write in tensor form like this rho del u i bar by del t plus del u i u j bar by del x j is equal to minus del p bar by del x i plus mu del 2 u i bar by del x j square minus rho del of del x j u i prime u j prime bar ok.

So, you can see that these up to this term this resembles with the Navier Stoke equation, but due to this fluctuating components we have this additional term, which actually gives rise to the stresses.

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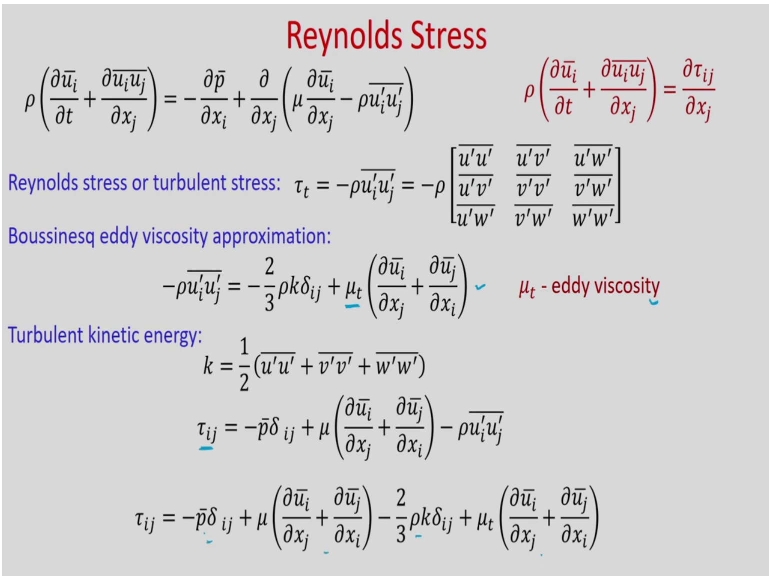

And so, now, if you write these 2 terms together, then we can write it as del of del x j mu del u i bar by del x j minus rho u prime u j prime bar ok. So, you can seethat this we can represent as stress. So, we can write these term as del x delta y j by del x j, where tau i j is the stress including both laminar and turbulent stress, ok.

So, you can see that tau i j actually is the stress and this having one additional stress that is known as Reynolds stress or turbulent stress due to the fluctuating components. So, that is minus u i prime u j prime bar and this you can see there are total 9 components ok, but out of these 9 components these two are same; similarly these two are same and these two are same. So, we have 6 unknowns here ok.

So, now you can see that we have total 4 governing equations right one continuity equation and three momentum equations, but how many unknowns are there. So, we have u v w p this 4 flow parameters and here 6 unknowns, which are actually in the arising in the Reynolds stress right.

So, 6 plus 4 10. So, there are 10 unknowns, but we have 4 equations. So, this problem is known as closure problem. So, to avoid that, now we need to write these fluctuating components in terms of the velocity component. So, these fluctuating components, we need to write in terms of the velocity gradient. So, now, you can see that Boussinesq actually did this approximation, which is known as Boussinesq eddy viscosity approximation. So, this stress minus rho u i prime u j prime bar he wrote in terms of the velocity gradient.

So, he wrote as minus 2 by 3 rho k delta i j plus mu t del u i bar by del x j plus del u j bar by y del x i, where mu t is the eddy viscosity or turbulent viscosity and k is the turbulent kinetic energy represented as half of u prime square bar plus v prime square bar plus w prime square bar.

So, this is the turbulent kinetic energy and delta i j is the Kronecker delta. So, now, we can write a, the shear stress tau i j as minus p bar delta i j this is the normal stress. So, this is the deviatoric stress. So, this is a normal stress and this is the deviatoric stress minus rho u i prime u j prime bar.

So, which is your Reynolds stress ok. So, these Reynolds we can actually use these Boussinesq eddy viscosity approximation and if you substitute it here. So, tau i j will become this ok. So, now, these if you take together.

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Reynolds Stress

$$\tau_{ij} = -\left(\bar{p} + \frac{2}{3}\rho k\right)\delta_{ij} + (\mu + \mu_t)\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)$$

$$\tau_{ij} = -\bar{p}_{eff}\delta_{ij} + \mu_{eff}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)$$

$$\bar{p}_{eff} = \bar{p} + \frac{2}{3}\rho k$$

$$\mu_{eff} = \mu + \mu_t$$

$$\rho\left(\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial \overline{u_i u_j}}{\partial x_j}\right) = -\frac{\partial \bar{p}_{eff}}{\partial x_i} + \mu_{eff}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)$$

And these if you take together, then we can write tau i j as minus p bar plus 2 by 3 rho k delta i j plus mu plus mu t del u i bar by del x j plus del u j by del u j bar by del x i. So, you can see these together we can write as p bar effective and mu plus mu t we can write as mu effective ok. So, you can see mu is the molecular viscosity right, mu is molecular viscosity and this is your mu t is coming due to these fluctuating components and this is known as eddy viscosity.

And this we can write p bar effective as p bar plus 3 by 3 rho k and mu effective we can write mu plus mu t, where mu is the molecular viscosity and mu t is the eddy viscosity. So, now, if you put it in the Reynolds average Navier Stroke equation. So, this we can write you can see that this is your a left hand side terms are the tangent terms and the convective terms, this is the pressure gradient term, but we have written in terms of the effective pressure plus mu effective and the this viscous term.

So, now, we can see that this mu effective contents mu t, which is unknown ok. So, now, to find this unknown eddy viscosity, there are many proposed methods are there, where you can use different turbulence modeling one equation modeling or zero equation modeling or two equation modeling, ok. So, all these turbulence modeling, you can use to find this unknown eddy viscosity mu t.

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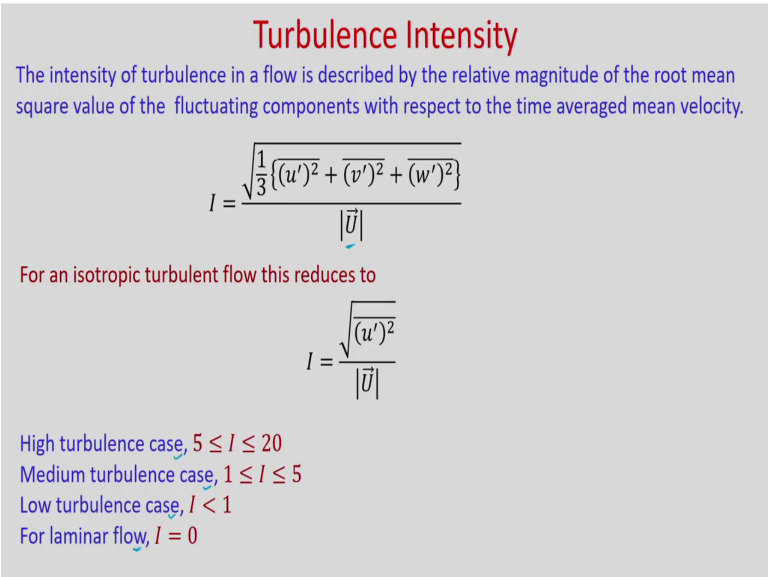

Now, we will define the turbulence intensity. The intensity of turbulence in a flow is described by the relative magnitude of the root mean square value of the fluctuating components with respect to the time average mean velocity.

So, you can see this is the time average mean velocity and this intensity of turbulence is defined is equal to I is equal to this root of 1 third of u prime square bar plus v prime square by plus w prime square bar divided by these magnitude of this velocity vector. So, you can see this is known as turbulence intensity and if it if you use the isotropic turbulent flow, then u prime is equal to v prime is equal to w prime.

So, it will become root of u prime square bar divided by magnitude of this velocity vector. So, now generally if this turbulence intensity is between 5 and 20, then it is known as high turbulence case. If this turbulence intensity varies between 1 and 5, then it is known as medium turbulence case. And if this turbulence intensity is less than 1 than it is low turbulence case and; obviously, if I is equal to 0, then it is a laminar flow.

So, when you are solving some problem of turbulent flows, then you need to give the turbulent intensity at the inlet ok. So, if you are numerically solving these equations, then you need to define the turbulent intensity at inlet. So, in today's class first we discussed about some rules of time averaging, then we considered the Navier Stoke equations and we put the Reynolds decomposition in this equation and we derived the Reynolds average Navier stock equations.

So, in Reynolds average Navier Stroke equations you can see that due to this velocity fluctuation there are some additional terms coming. So, if you see the Reynolds stress there are total 9 components out of that 6 are unknowns. So, we have total 4 governing equations and total 10 unknowns.

So, we have a problem of the closer and so this is known as closer problem. So, now, Boussinesq proposed that this stresses due to these fluctuating components, we can write in terms of the velocity gradient and turbulence kinetic energy and using that we have written the Reynolds average Navier Stroke equation, where one unknown is there that is your eddy viscosity mu t.

So, we have written the total viscosity as molecular viscosity plus the eddy viscosity and we have written the effective pressure as the p bar plus the terms which is coming from the kinetic energy.

Thank you.