Viscous Fluid Flow
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Module - 01
Introduction
Lecture – 04
Initial and Boundary Conditions

Hello, everyone. So, in the last class we derived that Navier-Stoke equations. Today, we will just write down the differential form of the momentum equations and the shear stresses acting on the fluid element in Cartesian coordinate, then we will write down these equations in cylindrical and spherical coordinates. Then we will discuss the initial conditions and boundary conditions for Viscous Fluid Flow.

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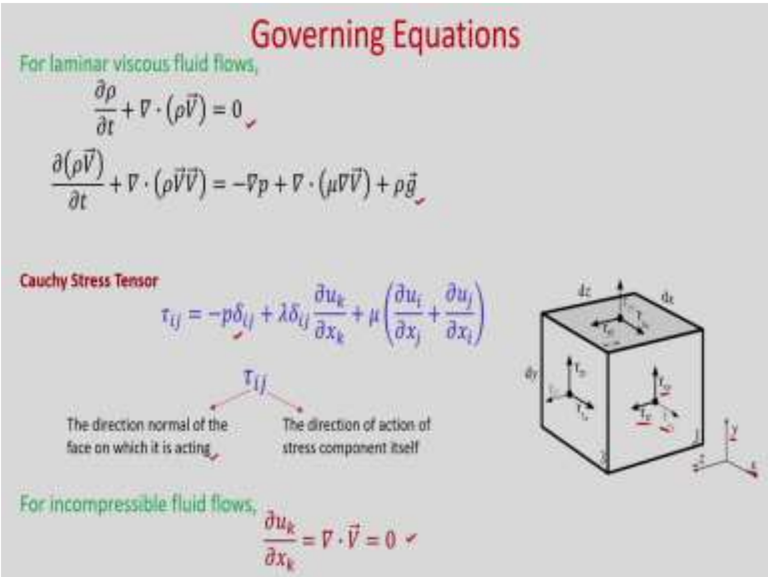

So, you can see that we have already derived these governing equations for laminar viscous fluid flows. So, this is the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla.\left(\rho \vec{V}\right) = 0$$

and this is the Navier-Stokes equations

$$\frac{\partial (\rho \vec{V})}{\partial t} + \nabla.\left(\rho \vec{V}\vec{V}\right) = -\nabla p + \nabla.\left(\mu \nabla \vec{V}\right) + \rho \vec{g}$$

So, these are written in general form and it is applicable for both incompressible and compressible fluid flows.

And, if you remember that we have also written the Cauchy stress tensor as a summation of these hydrostatic stress tensor and the deviatoric stress tensor. So, now if you remember that whatever we have written τ_{ij} this is the second ordered stress tensor and we can denote this τ_{ij} where this index i is the direction normal of the face on which it is acting. And, index j is the direction of action of the state component itself.

So, if you can see this is one fluid element the stresses acting on the surface x is in this direction, this is the y and this is the z-direction. If you consider this surface one where at this point you can see that sheer stress acting on the surface normal to the surface is τ_{xx}; that means, this the first x is for the direction normal of the face on which it is acting.

So, you can see on this for this surface 1 the normal direction is the x-direction right. So, τ_{xx} obviously, the fist x is the direction normal of the face on which it is acting and the second x is the direction of action of the stress component itself. So, it is acting in the x-direction. So, this is τ_{xx}.

Now, if you consider τ_{xy}, then x is the direction normal of the face on which it is acting. So, this is the normal direction you can see x, and in which direction it is acting? It is acting in the y-direction; that means, this y is the direction of x_n of stress component itself and similarly τ_{xz} you can see that z is the in the direction of action of stress component in the z-direction and x is normal to this surface.

So, in other surfaces two and three similarly, these stress components can be defined. If you consider incompressible fluid flows then obviously, for constant density incompressible fluid flows you can write $\frac{\partial u_k}{\partial x_k}$ which is nothing but $\nabla\vec{V}$ is equal to 0.

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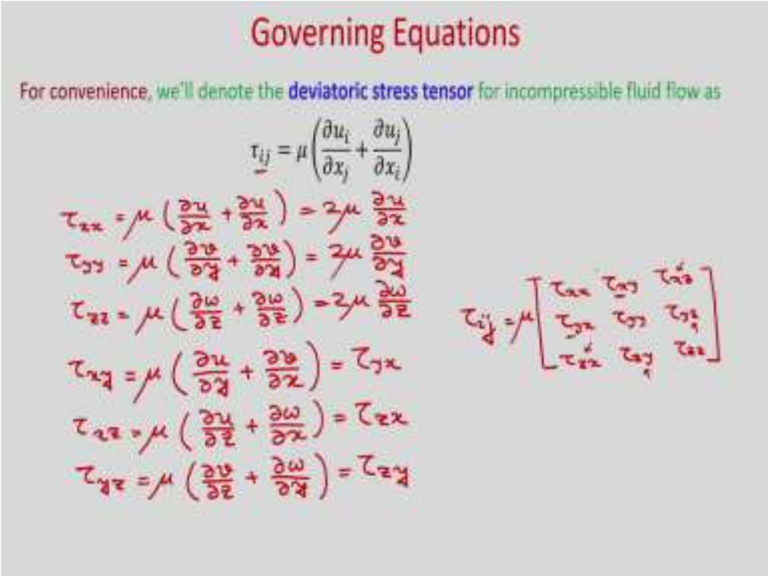

Now, onwards for convenience, we will denote the deviatoric stress tensor for incompressible fluid flows as

$$\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Earlier as we denoted as σ_{ij} and for incompressible fluid flows $\frac{\partial u_k}{\partial x_k}$ is equal to 0. So, these deviatoric stress tensor. So, this just in general will say that it is the shear stress acting on the fluid element. So, now, you can see that for 3-dimensional fluid flows we can have 9 components of this shear stress and out of that 6 will be unknown. So, let us write down the stress components in terms of the velocity gradient.

So, first, let us write that in the normal stress which is acting on the surface normal x and in the x-direction. So,

$$\tau_{xx} = \mu\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}\right) = 2\mu\frac{\partial u}{\partial x}$$

So, this is acting normal to the surface. Similarly,

$$\tau_{yy} = \mu\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial y}\right) = 2\mu\frac{\partial v}{\partial y}$$

Similarly,

$$\tau_{zz} = \mu\left(\frac{\partial w}{\partial z} + \frac{\partial w}{\partial z}\right) = 2\mu\frac{\partial w}{\partial z}$$

So, these are the stresses acting perpendicular to the surface.

Now,

$$\tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \tau_{yx}$$

$$\tau_{xz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = \tau_{zx}$$

$$\tau_{yz} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = \tau_{zy}$$

So, you can see we have 6 components 1, 2, 3, 4, 5, 6 as τ_{xy} is equal to τ_{yx}, τ_{xz} is equal to τ_{zx} and τ_{yz} is equal to τ_{zy}. So, there are 9 components. So, we can write

$$\tau_{ij} = \mu\begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}$$

And if you can see that τ_{xy} is equal to τ_{yx}, τ_{xz} is equal to τ_{zx} and τ_{yz} is equal to τ_{zy}.

So, although there are 9 components, 6 are unknown. So, whatever Navier-Stokes equations we have derived. So, you have this non-linear term which is your convective term. So, let us express this convective term in differential form.

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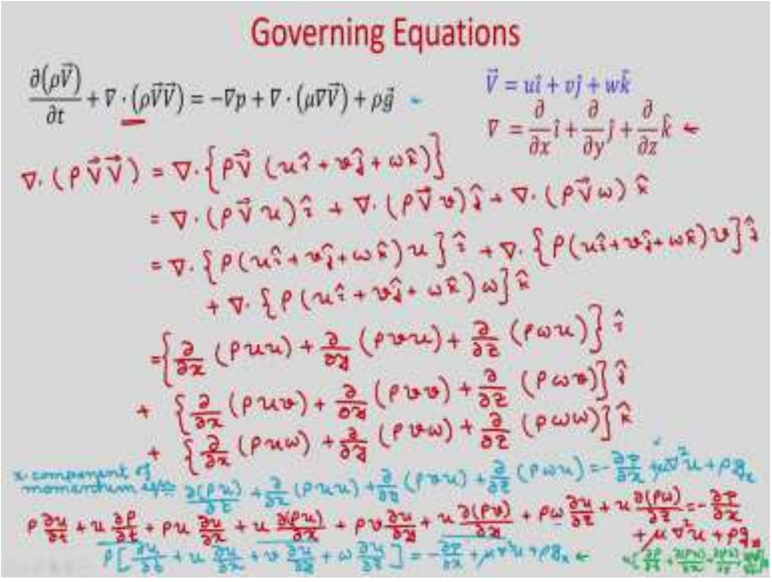

So, we have

$$\frac{\partial(\rho\vec{V})}{\partial t} + \nabla.\left(\rho\vec{V}\vec{V}\right) = -\nabla p + \nabla.\left(\mu\nabla\vec{V}\right) + \rho\vec{g}$$

where

$$\vec{V} = u\hat{\imath} + v\hat{\jmath} + w\hat{k}$$

$$\nabla = \frac{\partial}{\partial \mathrm{x}}\hat{\imath} + \frac{\partial}{\partial \mathrm{y}}\hat{\jmath} + \frac{\partial}{\partial \mathrm{z}}\hat{k}$$

So, now, if we write this convective term which is your non-linear term so, we can write

$$\nabla.\left(\rho\vec{V}\vec{V}\right) = \nabla.\left\{\rho\vec{V}\left(\ u\hat{\imath} + v\hat{\jmath} + w\hat{k}\right)\right\}$$

$$= \nabla.\left(\rho\vec{V}u\right)\hat{\imath} + \nabla.\left(\rho\vec{V}v\right)\hat{\jmath} + \nabla.\left(\rho\vec{V}w\right)\hat{k}$$

$$= \nabla.\{\left(\rho\left(\ u\hat{\imath} + v\hat{\jmath} + w\hat{k}\right)u\right)\hat{\imath} + \nabla.\left(\rho\left(\ u\hat{\imath} + v\hat{\jmath} + w\hat{k}\right)v\right)\hat{\jmath} + \nabla.\left(\rho\left(\ u\hat{\imath} + v\hat{\jmath} + w\hat{k}\right)w\right)\hat{k}$$

$$= \left\{\frac{\partial}{\partial \mathrm{x}}(\rho uu) + \frac{\partial}{\partial \mathrm{y}}(\rho vu) + \frac{\partial}{\partial \mathrm{z}}(\rho wu)\right\}\hat{\imath} + \left\{\frac{\partial}{\partial \mathrm{x}}(\rho uv) + \frac{\partial}{\partial \mathrm{y}}(\rho vv) + \frac{\partial}{\partial \mathrm{z}}(\rho wv)\right\}\hat{\jmath}$$
$$+ \left\{\frac{\partial}{\partial \mathrm{x}}(\rho uw) + \frac{\partial}{\partial \mathrm{y}}(\rho vw) + \frac{\partial}{\partial \mathrm{z}}(\rho ww)\right\}\hat{k}$$

So, now you can see that we have written this term now the temporal term whatever you have. So, if you write u component of momentum equations so, only x component if you write. So, you can see here you can write

$$\frac{\partial(\rho u)}{\partial \mathrm{t}}+\frac{\partial(\rho u u)}{\partial \mathrm{x}}+\frac{\partial(\rho v u)}{\partial \mathrm{y}}+\frac{\partial(\rho w u)}{\partial \mathrm{z}}=-\frac{\partial \mathrm{p}}{\partial \mathrm{x}}+\mu \nabla^2 u+\rho g z$$

$$\rho \frac{\partial \mathrm{u}}{\partial \mathrm{t}}+u \frac{\partial \rho}{\partial \mathrm{t}}+\rho u \frac{\partial u}{\partial \mathrm{x}}+u \frac{\partial(\rho u)}{\partial \mathrm{x}}+\rho v \frac{\partial u}{\partial \mathrm{y}}+u \frac{\partial(\rho v)}{\partial \mathrm{y}}+\rho w \frac{\partial u}{\partial \mathrm{z}}+u \frac{\partial(\rho w)}{\partial \mathrm{y}} = --\frac{\partial \mathrm{p}}{\partial \mathrm{x}}+\mu \nabla^2 u+\rho g z$$

Now, what is our continuity equation if you go back and see the continuity equation you can see $\frac{\partial\rho}{\partial t}+\nabla(\rho\,\vec{V})$ is equal to 0. So,

$$\rho\left[\frac{\partial \mathrm{u}}{\partial \mathrm{t}}+u \frac{\partial \mathrm{u}}{\partial \mathrm{x}}+v \frac{\partial \mathrm{u}}{\partial \mathrm{y}}+w \frac{\partial u}{\partial \mathrm{z}}\right]=-\frac{\partial \mathrm{p}}{\partial \mathrm{x}}+\mu \nabla^2 u+\rho g z$$

So, you can see this equation actually it is in the non-conservative form we have written ok. So, this is the equation in conservative form and this is the equation of x component momentum equation in non-conservative form and we have invoked the continuity equation and we have written it; that means, we have written you can see these terms.

So, if you take u common then you will get

$$\mathrm{u}\left[\frac{\partial \rho}{\partial \mathrm{t}}+u \frac{\partial(\rho u)}{\partial \mathrm{x}}+v \frac{\partial(\rho v)}{\partial \mathrm{y}}+w \frac{\partial(\rho w)}{\partial \mathrm{z}}\right]$$

So, this is the continuity equation. So, we have invoked this continuity equation has 0 and we have written this continuity this x-component of momentum equation in non-conservative form.

So, if you see in the Navier-Stokes equation we have the pressure gradient term as well as the body force term. So, can we club this together to get some quantity which is having some physical significance?

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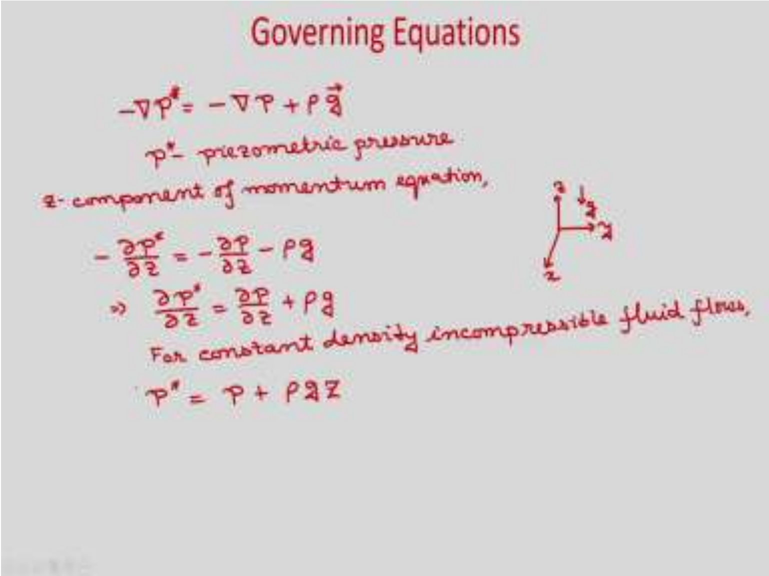

So, whatever pressure get in term is there you can see that we have written

$$-\nabla p^* = -\nabla p + \rho \vec{g}$$

So, if we are considering body forced term as the gravitational acceleration, then obviously, this p^* will denote as the piezometric pressure. So, let us just write the z-component of the momentum equation and we considered this pressure gradient term and the body force term and let us say that this is the z-direction, this is the y and this is the x-direction and gravity is acting in a negative z-direction.

So, you can see that in x and y components of momentum equation obviously, we do not have any component of this gravity. However, in the negative z-direction we have the gravity term as g. So, this is the g so, obviously, for z component of momentum equation we can write

$$-\frac{\partial p^*}{\partial z} = \frac{\partial \mathrm{p}}{\partial z} - \rho g$$

$$\Rightarrow \frac{\partial p^*}{\partial z} = \frac{\partial \mathrm{p}}{\partial z} + \rho g$$

So, now, we will integrate it and for constant density incompressible flow. So, rho we can keep it as constant. So, we can write

$$p^* = p + \rho g z$$

So, you can see with some constant, but if you put that at z is equal to 0, p is equal to 0, then obviously, you will get that constant as 0. So, you can see that in general Navier – Stokes equation instead of this pressure gradient and the body force term, we can write in terms of some modified pressure which is your piezometric pressure.

As you are considering the gradient of the pressure in the governing equation, so you can see that it does not matter whether it is a modified pressure or it is a thermodynamic pressure or mechanical pressure. So, we can actually club together this pressure gradient term and body force term and we can write one modified pressure when we considered gravity as the body force term then; obviously, it comes down to the piezometric pressure.

So, we can drop the gravity term and we can deal with the piezometric pressure and as we are dealing with the pressure gradient so, it does not matter whether it is piezometric pressure or it is thermodynamic pressure.

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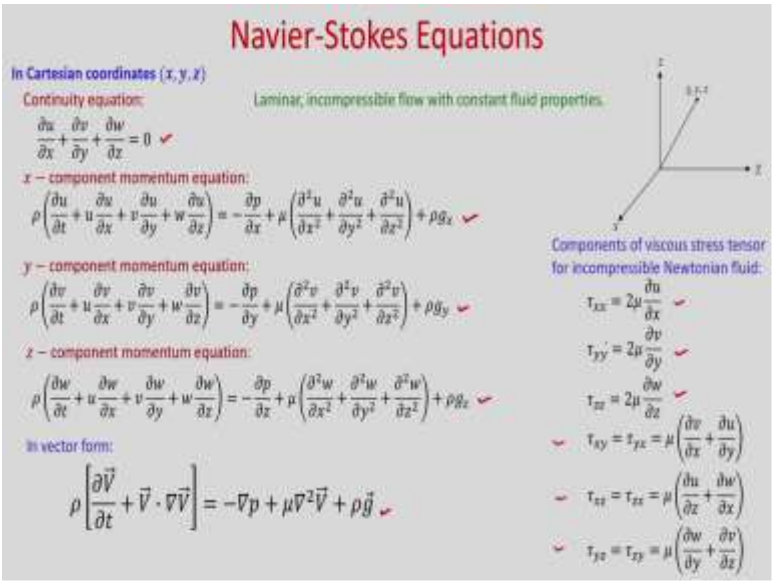

So, you can see that in Cartesian coordinate we can write the governing equation for laminar incompressible flow with constant fluid properties. So, you can see this is the continuity equation. So, this we have already derived.

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

$$\rho\left[\frac{\partial \mathrm{u}}{\partial \mathrm{t}}+u\frac{\partial \mathrm{u}}{\partial \mathrm{x}}+v\frac{\partial \mathrm{u}}{\partial \mathrm{y}}+w\frac{\partial u}{\partial \mathrm{z}}\right]=-\frac{\partial \mathrm{p}}{\partial \mathrm{x}}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+\rho g_x$$

So, this is the x component momentum equation and similarly y and z component of momentum equation you can write like this.

$$\rho\left[\frac{\partial \mathrm{v}}{\partial \mathrm{t}}+u\frac{\partial \mathrm{v}}{\partial \mathrm{x}}+v\frac{\partial \mathrm{v}}{\partial \mathrm{y}}+w\frac{\partial v}{\partial \mathrm{z}}\right]=-\frac{\partial \mathrm{p}}{\partial \mathrm{y}}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)+\rho g_y$$

$$\rho\left[\frac{\partial \mathrm{w}}{\partial \mathrm{t}}+u\frac{\partial \mathrm{w}}{\partial \mathrm{x}}+v\frac{\partial \mathrm{w}}{\partial \mathrm{y}}+w\frac{\partial w}{\partial \mathrm{z}}\right]=-\frac{\partial \mathrm{p}}{\partial \mathrm{z}}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)+\rho g_z$$

So, you can see the this we have written in non-conservative form and in vector form in general we can write

$$\rho\left[\frac{\partial(\vec{V})}{\partial t}+\vec{V}.\nabla\vec{V}\right]==-\nabla p+\mu\nabla^2\vec{V}+\rho\vec{g}$$

And, a component of viscous stress tensor for the incompressible Newtonian fluid you can see that we have already written these expressions. So, these are the normal stresses and these are the sheer stresses.

So, this τ represents obviously, you can see for incompressible fluid flow these the deviatoric stresses that we have represented here.

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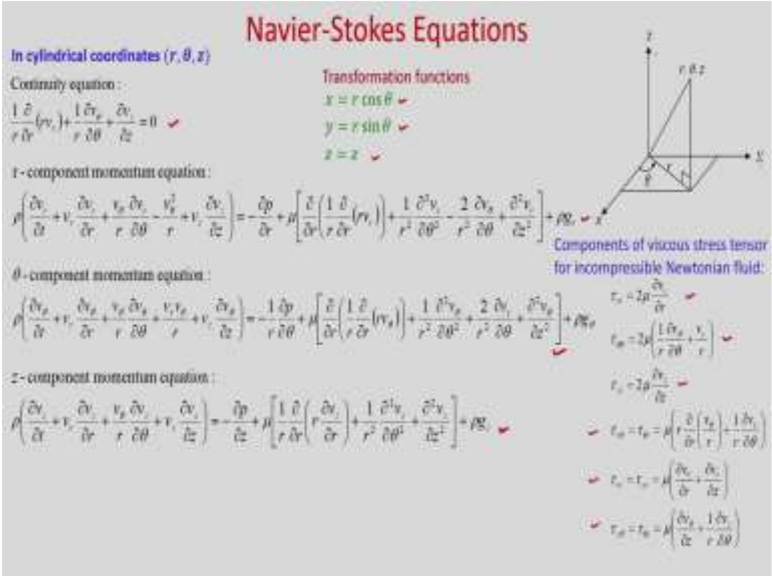

Similarly, if you consider a cylindrical coordinate this is the x, y and z. So, at this point it is r, θ, z; r is the radius, and θ is measured from here. So, whatever governing equation we have derived in Cartesian coordinate and if you use this transformation function because from here you can see that x equal to $r\cos\theta$, y is equal to $r\sin\theta$ and z is equal to z.

So, using this transformation function you can convert these governing equations from Cartesian coordinate to cylindrical coordinate. So, you can see this is the continuity equation, this is the r component of momentum equation, this is the θ component of momentum equation and this is the z component of the momentum equation and corresponding viscous stresses are written here. So, these are the normal stresses τ_{rr}, $\tau_{\theta\theta}$ and τ_{zz} and these are the shear stresses.

So, in this case $\tau_{r\theta}$ will be $\tau_{\theta r}$, τ_{rz} is equal to τ_{zr} and $\tau_{r\theta}$ is equal to $\tau_{\theta z}$ where v_r is the velocity in the radial direction, v_θ is the velocity in the tangential direction and v_z is the velocity in the z-direction.

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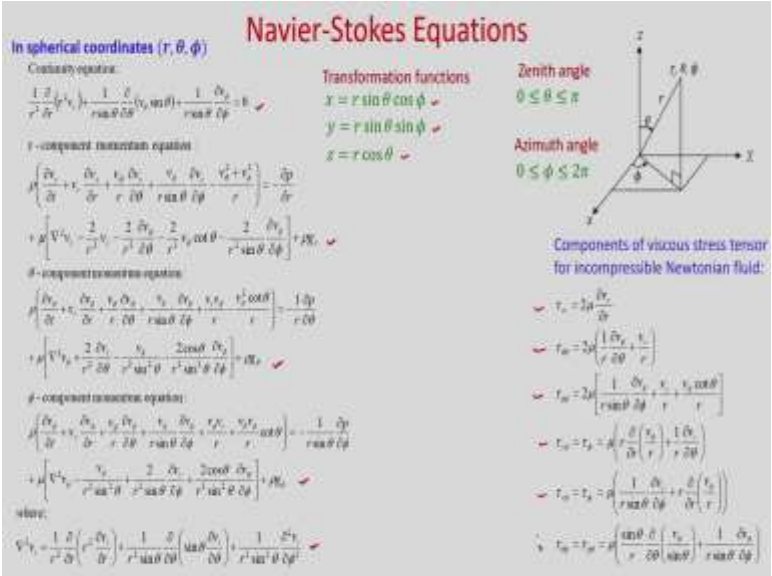

Similarly, in spherical coordinate, if you consider this as r, θ, ϕ and θ is the zenith angle it varies from 0 to ϕ and ϕ is the azimuth angle. So, it varies from 0 to 2ϕ then if you use this transformation function x equal to rsinθ cosϕ and y is equal to rsinθsinϕ and z is equal to rcosθ. So, from the Cartesian coordinate governing equations to cylindrical to spherical coordinates governing equation you can convert.

So, these are the continuity equations where v_r is the velocity in the r direction, v_θ is the velocity in θ direction and v_ϕ is the velocity in ϕ direction. So, similarly, we can write the r component of momentum equation, θ component momentum equation and ϕ component of momentum equation where this nabla square v_i in spherical coordinate is denoted like this. And, the viscous stress tensor in spherical coordinates you can write τ_{rr}, $\tau_{\theta\theta}$, $\tau_{\phi\phi}$ these are the normal stresses and these are the shear stresses.

So, you can see that the governing equation whatever we have derived the Navier-Stokes equation we have four independent variables that is x, y, z, and t and four dependent variables u, v, w and p. And, we have four scalar equations three component of momentum equations and one continuity equation. In general, you can see these Navier-Stokes equations are non-linear and couple.

So, the general solution after integrating this equation is very difficult. However, for simplified geometry and simplified assumptions, we can have the integral solution of these equations.

Now, when we solve these equations by integrating and invoking the assumptions we must specify the physical conditions to constant the flow at the boundaries.

So, those are known as boundary conditions and for time-dependent flow we must specify the state of the flow at an initial condition and that is known as initial conditions. So, you can see that the governing equations whatever we have written so, we have a time derivative. So, these are time-dependent governing equations.

So, if you are solving any time-dependent problem or unsteady problem, then at time t is equal to 0 you need to specify the initial condition. So, how do you determine that how many initial conditions and how many boundary conditions are required for a particular problem? So, it depends on the highest order of the partial differential equation.

So, we can see in the Navier-Stoke equation we have the first derivative with respect to time. So, obviously, one initial condition is required and if you see in the special variable we have highest order is second order. So, we need two boundary conditions in each direction ok. So, if it is a 3-dimensional flow then in three directions in each direction we need two boundary conditions. So, total 6 boundary conditions are required.

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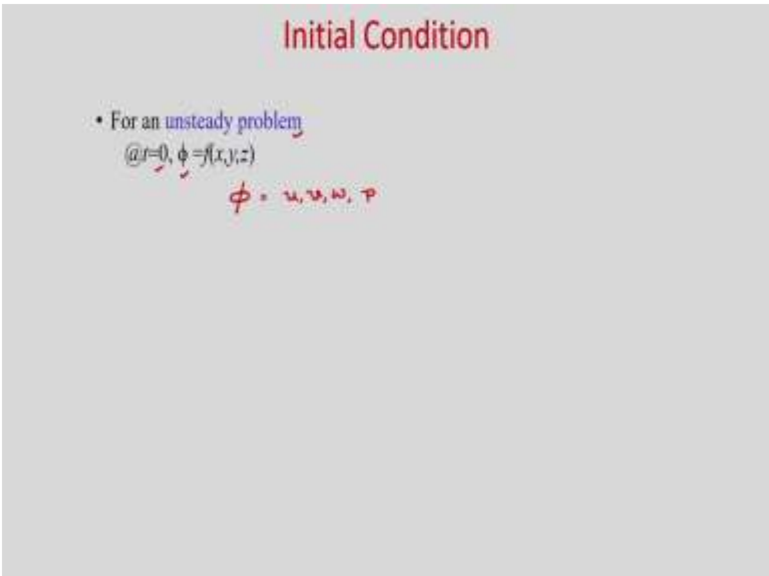

So, we can see that for an unsteady problem at time t is equal to 0, we need to specify the value of any variable at interior domain. So, it may be a function of x, y, z or it may be 0 or it may be constant where ϕ represents any variable u, v, w or p ok.

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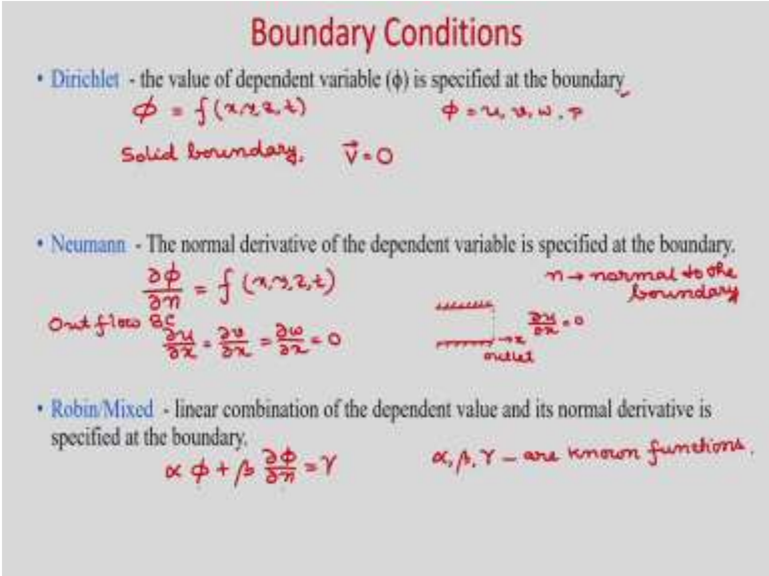

Now, in general, we can have three different types of boundary conditions; first is the Dirichlet type boundary condition where the value of a dependent variable is specified at the boundary ok. So, you can see that we can have this value of ϕ as a function of space and time, ok.

So, we can see that this ϕ may represent u, v, w or p. So, if you are specifying the values of this dependent variable at the boundary then that is known as Dirichlet type boundary condition and it may be a function of space only or function of time only or it may be constant or it may be 0.

So, if we can see that if we consider a solid boundary ok; so, in the solid boundary; obviously, we invoke the no-slip condition; that means, there will be no relative motion between the boundary and the fluid. So, that means, your velocity will be 0 at the solid boundary. So, you can see that this is the example of a Dirichlet boundary condition even at the inlet you can specify the velocity or at the outlet, you can specify the velocity. So, those will be Dirichlet type boundary conditions.

The next one is Neumann the normal derivative of the dependent variable is specified at the boundary. So, for any surface, if that normal derivative let us say if n is normal. So, $\frac{\partial \phi}{\partial n}$ is specified. So, it may be spatially varying or with time it may vary. So, you can see this is the Neumann type boundary condition. So, you can see ϕ maybe u, v, w or p and this may be equal to 0 or it may be a function of only space or it may be a function of only time or it may be constant ok.

So, generally at outflow boundary condition for the velocities, we specify that $\frac{\partial u}{\partial x}$ is equal to 0, where say if you consider any channel flow and if it is fully developed flow at the outlet and if this is the x-direction then normal to these boundary you can see $\frac{\partial u}{\partial x}$ will be 0 and similarly, $\frac{\partial v}{\partial x}$ will be 0 and $\frac{\partial w}{\partial x}$ will be 0.

So, this is an outflow boundary condition. So, this is one example of Neumann boundary condition. Another type of boundary condition we can have Robin or mixed type boundary condition which we are having the linear combination of the dependent value and its normal derivative is specified at the boundary.

So, we can write this as

$$\alpha\phi + \beta\frac{\partial \phi}{\partial n} = \gamma$$

So, n is normal to the boundary ok. So, where α, β and γ are the known functions; α. So, you can see that we have a specified value at ϕ and the normal gradient we have $\frac{\partial \phi}{\partial n}$ is equal to γ.

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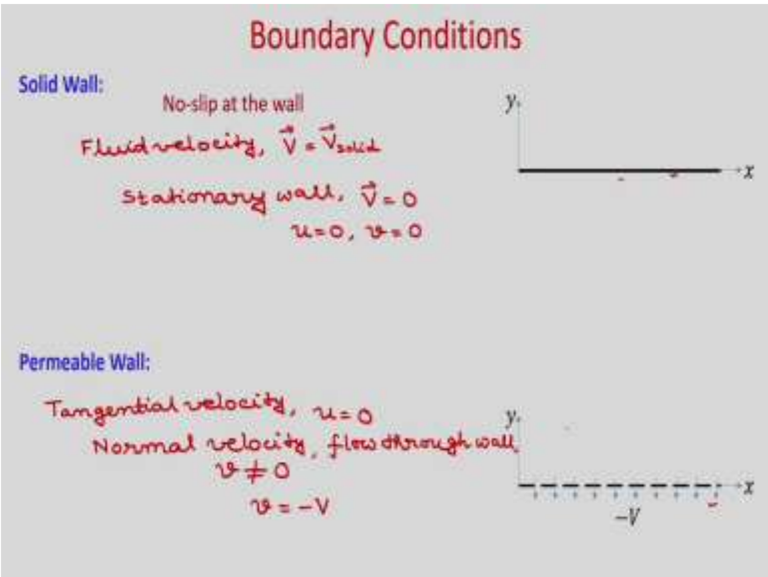

So, now we will discuss about different boundary conditions which we encounter in fluid flow problems. First we will discuss about solid wall boundary condition which we actually apply no slip boundary condition at the wall; that means, the fluid sticks to the boundary this boundary condition says that the fluid in contact with the wall will have the same velocity of the wall.

So, you can see that if this is the solid wall and the fluid velocity V will have the same velocity at the solid. So, obviously, for this stationary wall this solid wall velocity will be 0, so V will be 0. So, in this particular case if it is 2-dimensional flow then u will be 0 and v will be 0. So, this boundary condition is commonly known as no slip boundary condition.

Then we will discuss about the permeable wall, if the boundary is permeable then fluid can cross the boundary. So, you can see that in this figure we have shown permeable wall where suction is taking place; that means, the fluid velocity here normal to this boundary will have the velocity minus v; that means, here in this case tangential velocity u is equal to 0 and normal velocity.

So, this is flowing through the wall v not equal to 0 and for this particular case v will be just minus v for suction and if it is a blowing then obviously, this v will take place in the positive y direction. So, v will be positive V.

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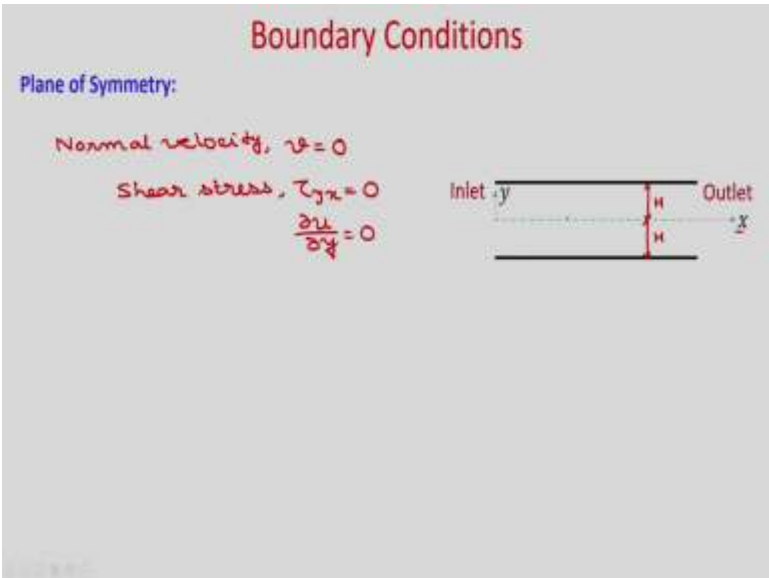

In some flows there is a plane of symmetry since the velocity field is the same on either side of the plane of symmetry the velocity must go through a minimum or maximum at the plane of symmetry.

So, if we consider this channel flow let us say flow inside two infinite parallel plates and x is the axial direction in this particular case the flow will be symmetric about this central axis the distance between two parallel plates. So, this will be from the centerline if we measure so, it will be h and it will be h, then at y is equal to 0, there will be a symmetry plane.

So, in this case, normal velocity so, v will be 0 and shear stress τ_{yx} will be 0 and in this particular case obviously, y is measured from the centerline. So, $\frac{\partial u}{\partial y}$ will be 0; that means, the flow will be maximum or minimum at this plane of symmetry.

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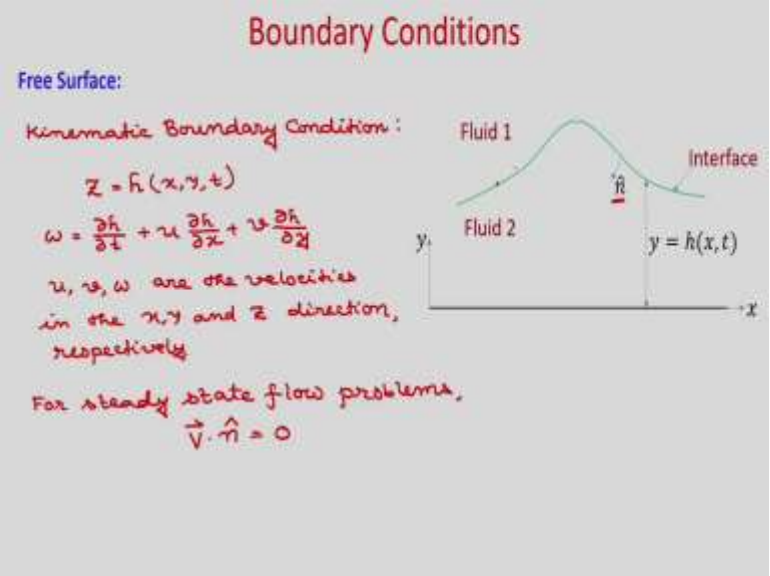

Now, another boundary condition we will discuss is free surface. So, you can see there are two fluids; fluid 1 and fluid 2 immiscible fluids and there is an interface. So, free surface occurs at the interface between two fluids such interfaces require two boundary conditions to be applied. So, one is the kinematic boundary condition.

So, kinematic boundary condition which relates the motion of the free interface to the fluid velocities at the free surface. So, if there is any fluid particle sitting on this interface; obviously, it will always remain part of this free surface ok. So, if we say that Z is this h, the height from the bas,e and it is function of x, y and t then the velocity in z-direction w will be

$$w = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y}$$

So, in this particular case obviously, you can see u, v, w are the velocities in the x, y and z direction respectively. And, if we consider the flow to be steady, then for steady-state flow problems obviously, it will not be a function of t. So, $\vec{V}.\hat{n}$ is equal to 0 where n is normal to the free surface. So, you can see there is no flow through this interface. So, $\vec{V}.\hat{n}$ is equal to 0; that means, there will be no flow through this free surface.

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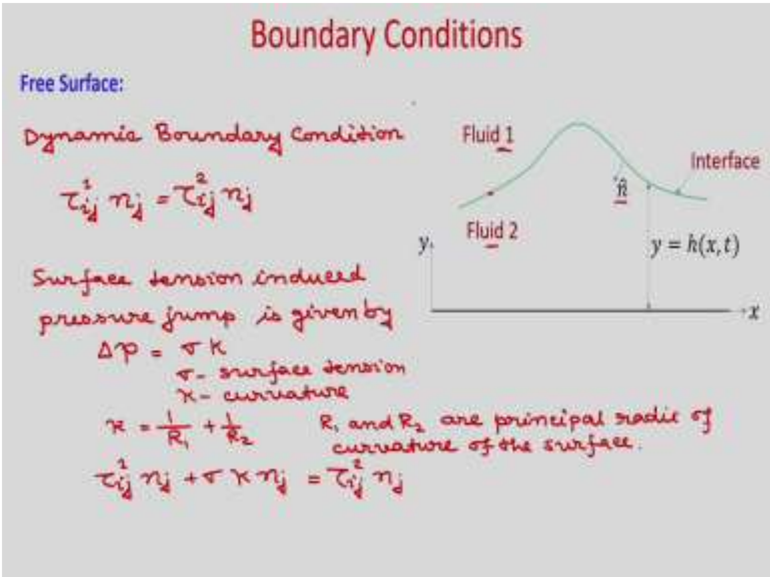

So, now, we will discuss about the other boundary condition in case of free surface so that is a dynamic boundary condition. Dynamic boundary condition requires the stress to be continuous across the free surface which separates the two fluids. So, in this case the traction exerted by fluid 1 onto the fluid 2 is equal and opposite to the traction exerted by fluid 2 on fluid 1.

So, that means, in this particular case if we can write the shear stress tensor

$$\tau_{ij}^{1} n_j = \tau_{ij}^{2} n_j$$

So, n is the normal to this interface. If you have this curved surface so, in this case surface tension can create a pressure jump across the free surface.

So, the surface tension induced pressure jump is given by

$$\Delta p = \sigma \kappa$$

where σ is the surface tension and κ is the curvature, ok.

And, we know that the curvature you can calculate as

$$\kappa = \frac{1}{R_1} + \frac{1}{R_2}$$

Where R_1 and R_2 are principal radii of curvature of the surface. So, if you consider the surface tension then this dynamic boundary condition you can write as

$$\tau_{ij}^1 n_j + \sigma \kappa n_j = \tau_{ij}^2 n_j$$

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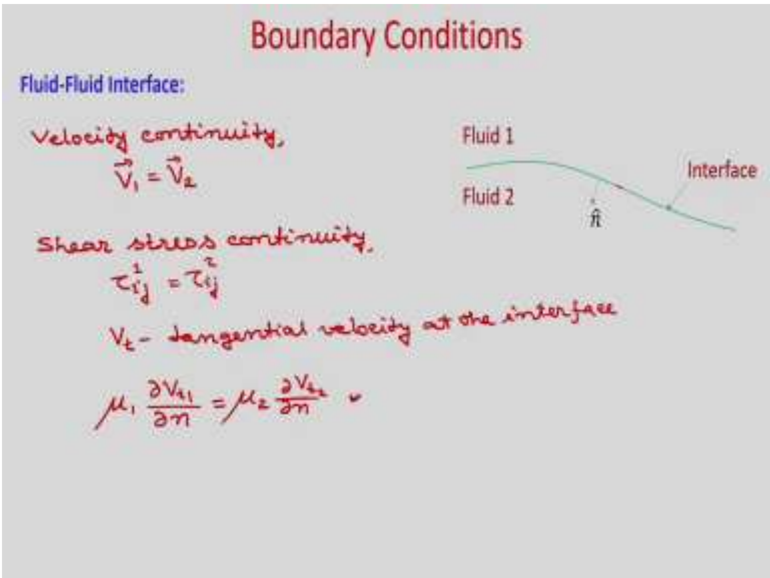

Next, we will have another boundary condition fluid-fluid interface. So, similar to whatever we have discussed in the last slide, but we neglect the surface tension effect then; obviously, you can see at this fluid-fluid interface there will be shear stress continuity and the velocity continuity neglecting the effect of the surface tension.

So, if you have two immiscible fluids then at the interface there will be velocity continuity; that means, so, from fluid 1 side velocity $\vec{V}_1$ will be equal to the fluid velocity fluid 2 velocity $\vec{V}_2$. So, in this particular case obviously, the velocity at the interface from fluid 1 side and fluid 2 sides it will be the same and there will be also shear stress continuity at the interface.

So, in this case, you can see obviously, from the fluid side τ_{ij}^1 will be τ_{ij}^2. So, in this particular case if we say that tangential velocity is V_t at the interface and n is normal to the boundary, then we can say that

$$\mu_1 \frac{\partial V_{t_1}}{\partial n} = \mu_2 \frac{\partial V_{t_2}}{\partial n}$$

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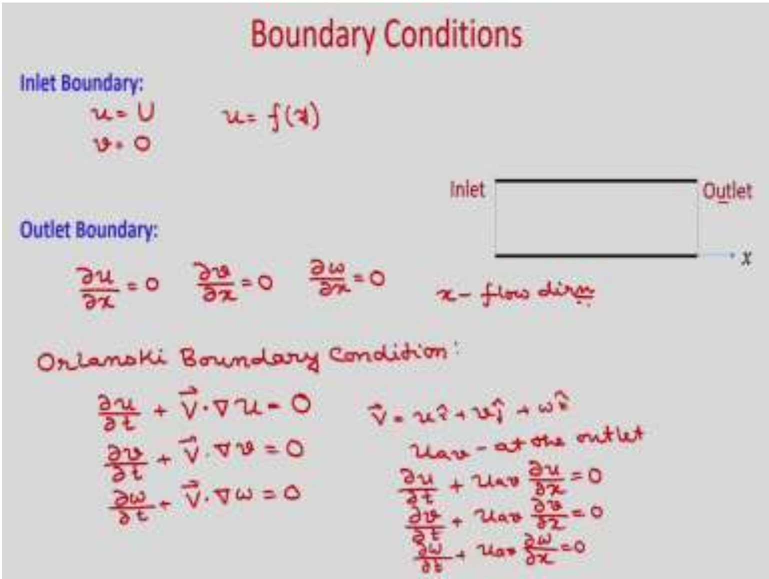

So, this is the shear stress continuity and we can have the inlet boundary condition. So, at the inlet, we can specify the value of the variable. So, obviously, in this case, if you consider that flow between two infinite parallel plates so, at the inlet you can either specify a constant velocity or you can have a fully developed velocity profile you can specify at the inlet.

So, obviously, the flow is assumed to be constant or fully developed. So, in this case, if x is the axial direction, then in this case you can see that velocity u will be either U which is a constant velocity and v you can make 0 or you can also give a parabolic profile u which will be a function of y.

So, you can specify at the inlet the condition at the outflow plane now we will discuss. So, you can see in this particular case it is the outlet. So, at the outlet generally, we specify pressure is equal to 0 and for other variables, we specify that in the flow direction the gradient is 0 of all the velocities.

So, in this particular case, you can see the conditions of the outflow plane are extrapolated from within the domain and have no impact on the upstream flow. So, in this particular case, we can write if x is the axial direction which is the flow direction. So, we can write

$$\frac{\partial u}{\partial x} = 0 \qquad \frac{\partial v}{\partial x} = 0 \qquad \frac{\partial w}{\partial x} = 0$$

where x is the flow direction in this particular case ok. So, that means, the gradient of velocities in the flow direction is 0.

For unsteady flow generally, one more appropriate boundary condition is OrLanski boundary condition; which you can apply at the outlet for unsteady flow problem. So, this we can write

$$\frac{\partial u}{\partial t} + \vec{V}.\nabla u = 0$$

Similarly, you can write

$$\frac{\partial v}{\partial t} + \vec{V}.\nabla v = 0$$

and

$$\frac{\partial w}{\partial t} + \vec{V}.\nabla w = 0$$

Where $\vec{V}$ we can take as

$$\vec{V} = u\hat{\imath} + v\hat{\jmath} + w\hat{k}$$

So, in this particular case we write the average velocity at the outlet. So, we need to calculate the average velocity then if the average velocity let us say if it is u_{av} at the outlet, then we can write

$$\frac{\partial u}{\partial t} + u_{av}\frac{\partial u}{\partial x} = 0$$

$$\frac{\partial v}{\partial t} + u_{av}\frac{\partial v}{\partial x} = 0$$

$$\frac{\partial w}{\partial t} + u_{av}\frac{\partial w}{\partial x} = 0$$

So, this is known as the OrLanski boundary condition and this is more appropriate for unsteady flow problem.

So, in today's class, we started with the Navier-Stoke equation in general form and then we have invoked the incompressible flow assumptions and we have written the momentum equations in differential form and in Cartesian coordinate we have written the continuity equation and three components of momentum equations in differential form and in non-conservative form.

Also, we have written the shear stresses three components of normal stresses and three components of six components of shear stresses and out of these you can see out of nine components six are unknown. Next, we have written the governing equations in the cylindrical coordinate and spherical coordinate and corresponding viscous stresses also we have written.

Then we discussed about piezometric pressure. So, we combined the thermodynamic pressure or the mechanical pressure plus the body force term in the piezometric pressure. Next, we discussed about the initial condition where at time t is equal to 0, we need to specify the value of the dependent variable.

Next we considered different types of boundary conditions Dirichlet, Neumann, and Robin boundary conditions and for the dependent variable if the value is specified at the boundary, then it is known as Dirichlet type boundary condition; if the normal gradient is specified, then it is known as Neumann type boundary condition and if the value of the dependent variable and it is normal gradient is specified then it is Robin boundary condition.

In addition, we have discussed about other boundary conditions like no-slip boundary condition, then free slip boundary condition or symmetry boundary condition and also if we have two different immiscible fluids at the interface we need to consider the boundary condition. So, that also we have discussed.

Thank you.