Viscous Fluid Flow

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Module - 10
Stability Theory
Lecture - 01
Derivation of Orr-Sommerfeld Equation

Hello everyone. So, in today's module, we will start a Stability Theory. You have already seen that for flow over flat plate, we often tell that critical Reynolds number for flow over flat plate is 5 into 10 to the power 5. So, what does it mean by critical Reynolds number? That means, the flow becomes turbulence, if you go above this critical Reynolds number. How did you determine that?

If you see that below critical Reynolds number, if you perturb the flow, then the disturbance will decay with time; that means, disturbances will damp out, but above the critical Reynolds number if you disturb the flow, then the disturbance will grow with time. So, this critical Reynolds number, we need to solve this ordinary differential equation which we will derive in today's class that is known as Orr-Sommerfeld Equation.

So, finding the Eigen values of this ordinary differential equation, you will be able to tell that when this flow becomes turbulence from laminar. So, in today's class, we will know how does turbulence come into being and what triggers it. So, you have seen that instability leads to turbulence.

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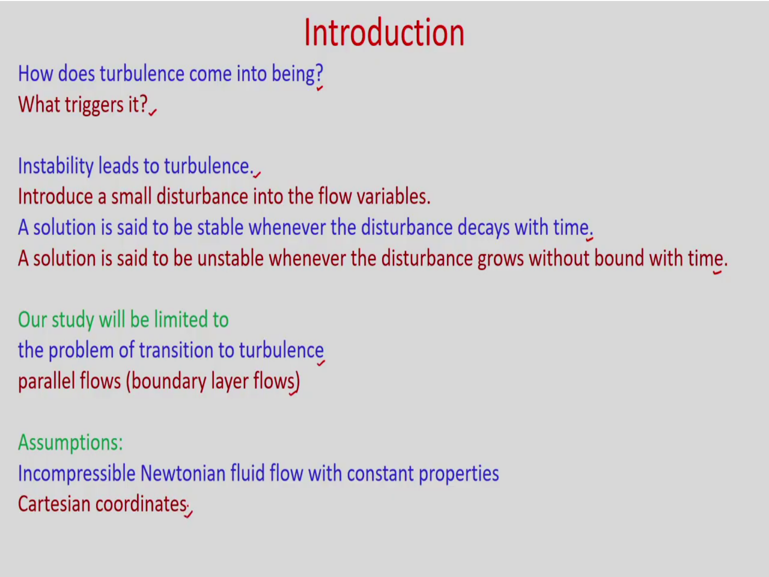

So, if we introduce a small disturbance into the flow variables, then a solution is said to be stable, whenever the disturbance decays with time. And a solution is said to be unstable, whenever the disturbance grows without bound with time. Although these study is very large, but our study will be limited to the problem of transition to turbulence and will consider parallel flows; mostly boundary layer flows.

So, the starting point of the stability theory is writing the governing equations. So, we will make these assumptions, incompressible Newtonian fluid flow with constant properties and will consider the geometries which fit in Cartesian coordinates.

So, these are the governing equations. So, this is the continuity equation; x component of momentum equation, y component of momentum equation and z component of momentum equation in Cartesian coordinate.

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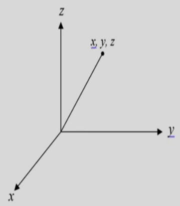
Governing Equations

Consider two-dimensional steady state, incompressible flow with constant properties.

In Cartesian coordinates (x, y, z)



Continuity equation:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

x – component momentum equation:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

y – component momentum equation:

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)$$

z – component momentum equation:

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)$$

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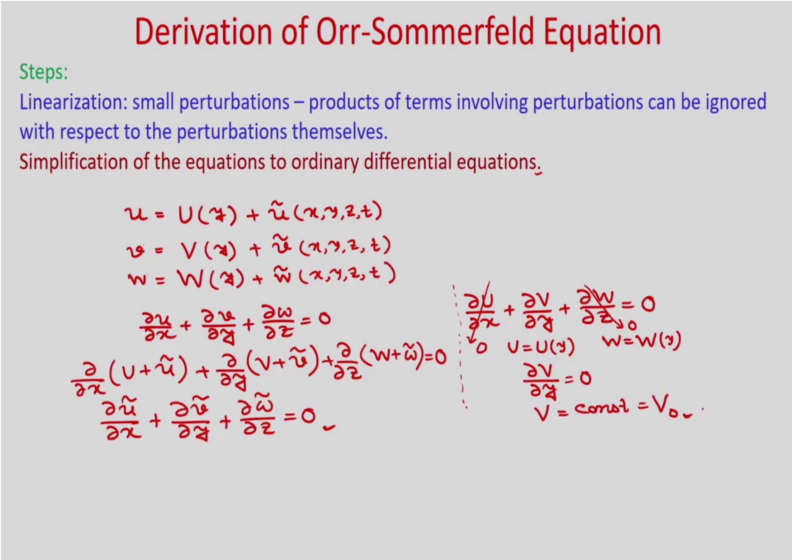

So, we will follow these two steps, first step is that linearization of the governing equation. So, we will consider small perturbation. So, why we are considering small perturbation? Because products of terms involving perturbations can be ignored with respect to the perturbation themselves. Then, if you are considering small perturbation, then we will tell it as linear stability analysis.

And then we will simplify the equations, so, that we can convert it to ordinary differential equations. So, in the governing equation, you have seen that the variables u, v, w, p are the solutions of these governing equations. Now, we will write this u, v, w with some parallel flows, where the velocities are function of only one coordinates, let us say y and plus the perturbation.

So, let us consider this velocity u as parallel flow U which is function of y only and the disturbance; u tilde x, y, z and t. So, this is function of space as well as time, but the base flow is parallel flow and it is function of y only. Similarly, velocity v is also V which is function of y plus the perturbation, v tilde which is function of space and time. And w is function of W y, so, capital W is function of y only and the perturbation w tilde, which is function of space and time ok.

So, now, we can see that; obviously, this u, v, w will satisfy the continuity equation or continuity equation, we have del u by del x plus del v by del y plus del w by del z is equal to 0. Similarly, this U, V, W also will satisfy the continuity equation. So, we can write del U by del x plus del V by del y plus del W by del z is equal to 0, but U is function of y only right.

So, this term will become 0. W is function of y only right. So, this term also will become 0. So, we will get del V by del y is equal to 0; that means, V will be some constant ok. And let us say this is V naught ok. So, if you consider a parallel flow so, you can see that from this continuity equation, we can get V is equal to constant is equal to V naught.

And similarly, now, if you put the value of u v w here, what you will get? So, we will get del of del x U plus u tilde plus del of del y V plus v tilde plus del of del z W plus w tilde is equal to 0. So, here you can see U is function of y. So, we can write only del u tilde by del x. Here you can see V is becoming constant right. So, it is V naught.

So, it will be just del v tilde by del y and W is function of y. So, we can write del w tilde by del z is equal to 0. So, you can see these disturbances also satisfy the continuity equation. And we have seen that V is equal to constant V naught. Now, let us consider the x component of momentum equation and put these velocities with the parallel flow plus the disturbances..

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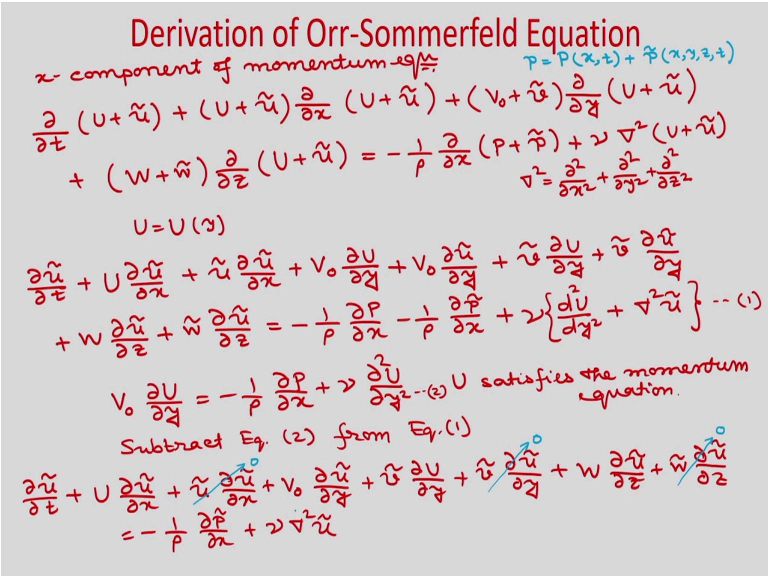

So, x component of momentum equation so, if you satisfy this then we will write just putting u is equal to U plus u tilde plus U plus u tilde del of del x U plus u tilde plus V naught plus v tilde del of del y U plus u tilde plus W plus w tilde del of del z U plus u tilde is equal to minus 1 by rho del of del x ok.

So, now, pressure p also will be capital P which is function of x and t. x is the axial direction plus the perturbation P tilde, which is function of x, y, z and t ok. So, if it is so, then we can write this term P plus p tilde. Remember P is function of x and t plus, we have the nu grad square U plus u tilde. So, this is the viscous term.

So, nabla square is nothing but del 2 by del x square plus del 2 by del y square plus del 2 by del z square. Now, let us simplify this equation and we will see that product of this disturbances ok will make it 0, because we are assuming that perturbations are very small and we are using the linear stability analysis and product of these two small components also will become very small.

So, we can neglect those terms. So, we will write you can see U is function of y only right U is function of y only. So, we can write it as del u tilde by del t plus, here we can write U; so, U is function of y only. So, it will be just del u tilde by del x plus u tilde del u tilde by delta x, V naught del U by del y, because U is function of y.

Se we have to write del U by del y plus V naught del u tilde by del y plus v tilde del U by del y plus v tilde del u tilde by del y plus W del u tilde by del z plus w tilde del u tilde by del z is equal to minus 1 by rho del P by del x plus minus 1 by rho del p tilde by del x and we will get plus nu.

So, we will get one term d 2 U by dy square ok, plus nabla square u tilde ok. And similarly, this u v w also will satisfy the momentum equation right. So, we can write V naught del U by del y is equal to minus 1 by rho, del P by del x plus nu, del 2 U by del y square ok. So, since U satisfies the momentum equation right..

So, the you see from here; so, if this is equation 1 and this is equation 2, then you subtract this equation 2 from equation 1 ok. So, subtract equation 2 from equation 1 ok. So, what you will get? So, you see here V naught del U by del y is. So, this term will not be there and this also will not be there and this term will not be there.

So, we will get this equation of perturbation del u tilde by del t plus U del u tilde by del x plus u tilde del u tilde by del x plus. So, this term will not be there. V naught del u tilde by del y plus v tilde del U by del y plus v tilde del u tilde by del y plus W del u tilde by del z plus w tilde del u tilde by del z and in the right hand side this first term will not be there, we will have only minus 1 by rho del p tilde by del x plus these term also will not be there only nu grad square u tilde.

So, now, you see all the terms carefully and drop the terms, where we have product of these disturbances ok, because we have assumed disturbances very small. So, you can see that this term u tilde del u tilde by del x. So, this we can drop ok. Then we have v tilde del u tilde by del y so, these term also we can drop. w tilde del u tilde by del z so, this term also we can drop ok.

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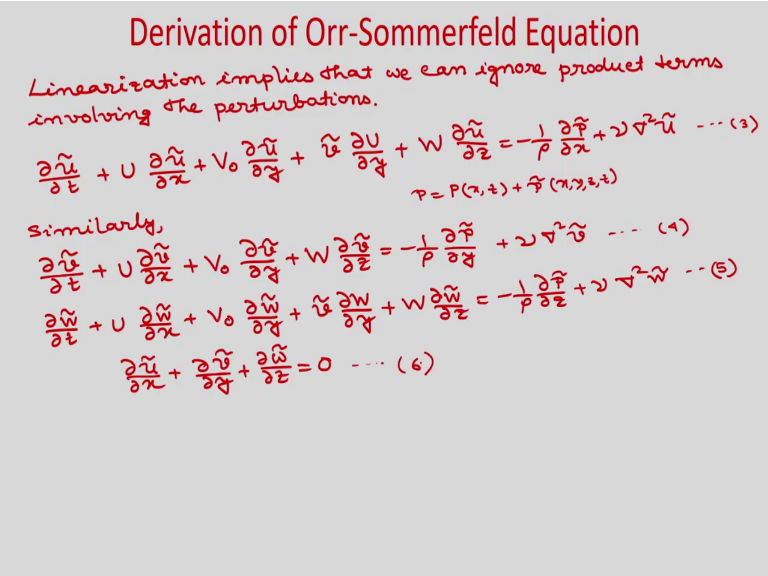

So, now if you simplify this. So, now, you will get a linear equation. So, linearization implies that, we can ignore product terms involving the perturbations ok. So, we can now, write del u tilde by del t plus U del u tilde by del x plus V naught del u tilde by del y plus v tilde del U by del y plus W del u tilde by del z is equal to minus 1 by rho del p tilde by del x plus nu nabla square u tilde.

Let us say this equation number is 3. Similar way from y and z momentum equations, you can derive the equation for perturbations. So, similarly we can write. So, we can derive these perturbation equations for v tilde del v tilde by del t plus u del v tilde del x plus V naught del v tilde del y plus W del v tilde del z is equal to minus 1 by rho del p tilde by del y plus nu.

So, you can see that P is P is function of x t and p tilde x, y, z, t. So, it will become only del p tilde by del y plus nu grad square v tilde. So, similarly we can derive del w tilde by del t plus U del w tilde del x plus V naught del w tilde del y plus v tilde del w by del y plus W del w tilde del z is equal to minus 1 by rho del p tilde del z plus nu grad square w tilde ok.

So, we have already derived the continuity equation, del u tilde by del x plus del v tilde by del y plus del w tilde by del z is equal to 0 ok. So, now, let us give the equation number. So, this is 4, this is 5 and this continuity equation, as equation number 6. Whatever disturbances we have considered, so, these disturbances are assumed to be travelling periodically, in space as well as in time.

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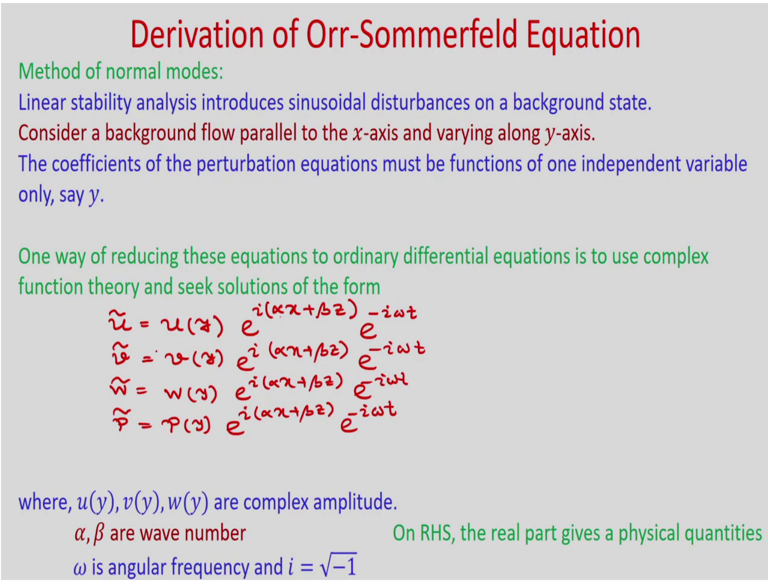

So, for that reason it is known as Fourier analysis or normal mode analysis. So, you can see, if you use method of normal modes linear stability analysis introduces sinusoidal disturbances on a background state. So, and these disturbances are assumed to be travelling waves and hence periodic in the flow direction and time.

So, we are considering a background flow parallel to the x axis and varying along y axis, the coefficients of the perturbation equations must be functions of one independent variable only and let us say that is y. So, one way of reducing these equations to ordinary differential equation is to use complex function theory and solutions of the form.

So, these disturbances will be u tilde is equal to u y e to the power i alpha x plus beta z e to the power minus i omega t ok. So, this method also is known as Fourier analysis, v tilde is equal to v y e to the power i alpha x plus beta z e to the power minus i omega t. w tilde is equal to w function of y e to the power i alpha x plus beta z e to the power minus i omega t. And p tilde is equal to p, which is function of y e to the power i alpha x plus beta z e to the power minus i omega t ok.

So, where this u, v, w are complex amplitude and alpha and beta are wave number and this omega is angular frequency ok. And i is just root minus 1. So, you can see on the right hand side the real part gives a physical quantities. Now, these perturbation now, we will substitute in the perturbation equations which we have already derived.

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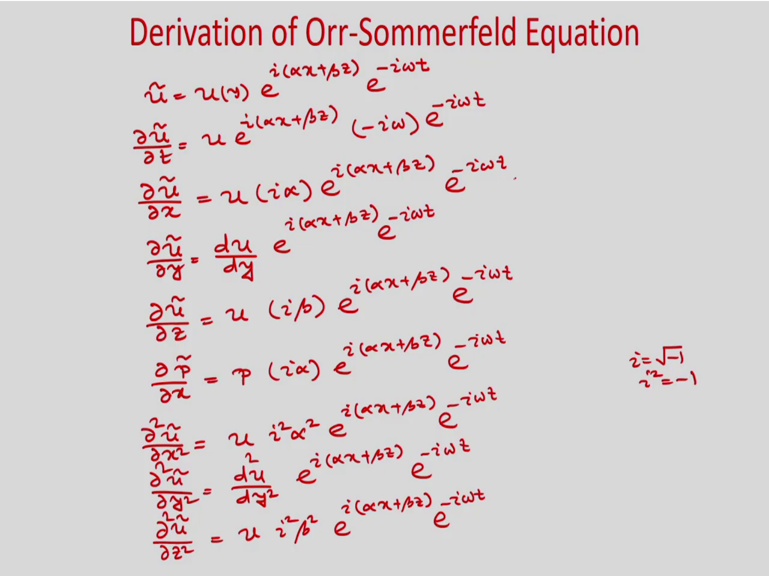

We have u tilde is equal to u which is function of y e to the power i alpha x plus beta z e to the power minus i omega t ok. Now, let us consider this x component of this disturbance equation and let us find the gradient ok. So, we have del u tilde by del t right.

So, this you will get u e to the power i alpha x plus beta z minus i omega e to the power minus i omega t. Similarly, you can write del u tilde by del x is equal to u i alpha e to the power i alpha x plus beta z e to the power minus i omega t. So, now, we can see del u tilde by del y. So, u is function of y right.

So, we can write du by dy ok. Then, e to the power i alpha x plus beta z e to the power minus i omega t. del u tilde del z. So, we can write u i beta e to the power i alpha x plus beta z e to the power minus i omega t. And del p tilde by del x will be p i alpha e to the power i alpha x plus beta z e to the power minus i omega t. And similarly, if you find del 2 u tilde by del x square ok.

So, from here you can see, it will be just u i square, alpha square e to the power i alpha x plus beta z e to the power minus i omega t and i is equal to root minus 1. So, i square is minus 1 ok. And del 2 u tilde by del y square. So, we will get d 2 u by dy square e to the power i alpha x plus beta z e to the power minus i omega t and del 2 u tilde by del z square.

So, it will be u i square beta square e to the power i alpha x plus beta z e to the power minus i omega t. So, now, all these terms you please, substitute in the x component of the disturbance equation and let us simplify that equation. So, from equation 3, now, we can write in the left hand side it will be e to the power i alpha x plus beta z e to the power i omega t with a minus sign.

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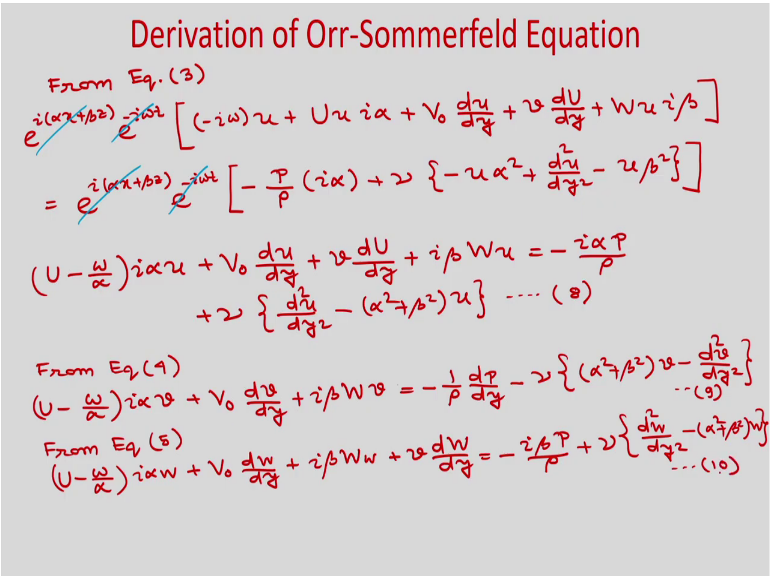

If you take it common then, if you put all these derivatives, we will write minus i omega u plus U u i alpha plus V naught du by dy plus v du by dy plus W u i beta and in the right hand side, you will get e to the power i alpha x plus beta z e to the power minus i omega t and you will get minus p by rho i alpha plus nu minus u alpha square plus d 2 u by dy square minus u beta square ok.

So, we can see these term will get cancelled. So, you can simplify it and write as U minus omega by alpha i alpha u plus V naught du by dy plus v d U by dy plus i beta W u is equal to minus i alpha P by rho plus nu d 2 u by dy square minus alpha square plus beta square u ok, and let us say this is equation number 8.

Similar analysis, if you do for the other perturbation equation then, we can derive from equation 4, we can derive U minus omega by alpha i alpha v plus V naught dv by dy plus i beta w v is equal to minus 1 by rho dp by dy minus nu alpha square plus beta square v minus d two v by dy square ok.

And from equation 5, we can derive U minus omega by alpha i alpha w plus V naught dw by dy plus i beta W w plus v dw by dy is equal to minus i beta P by rho plus nu d 2 w by dy square minus alpha square plus beta square w. Now, let us consider the continuity equation and simplify it.

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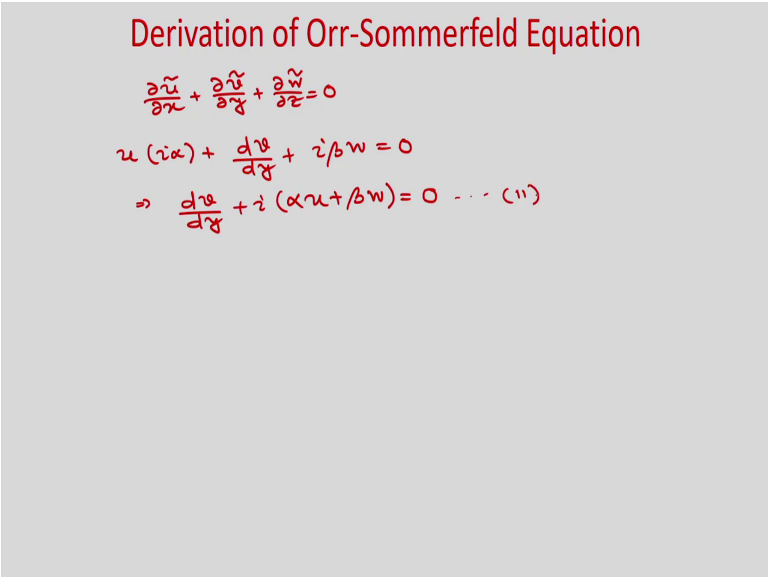

So, our continuity equation is del u tilde by del x plus del v tilde by del y plus del w tilde by del z is equal to 0. So, now, if you put it here, you will get u i alpha plus dv by dy plus i beta w is equal to 0. So, from here you can write that, dv by dy plus i alpha u plus beta w is equal to 0. Let us say that; this equation number as 9 this is 10 and this is 11.

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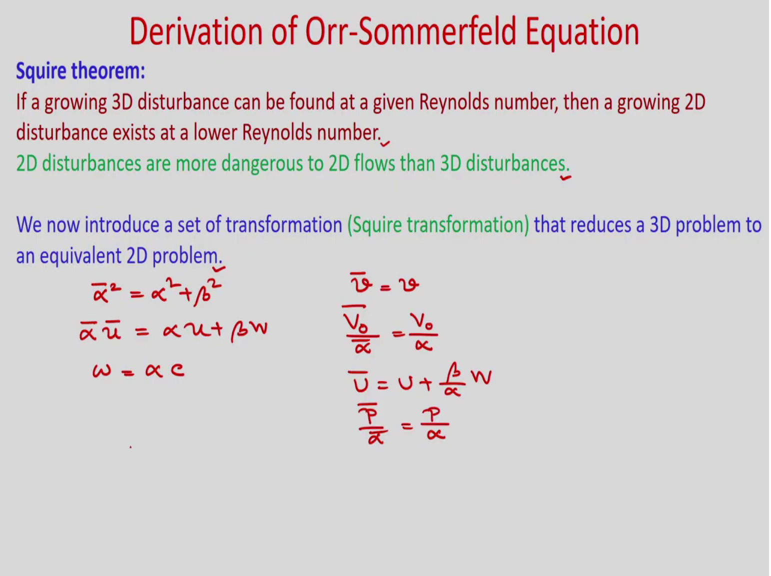

So, now, we will discuss about one important theorem that is known as squire theorem. So, squire theorem states that if a growing 3D disturbance can be found at a given Reynolds number, then a growing 2D disturbances exists at a lower Reynolds number. That means, 2D disturbances are more dangerous to 2D flows than the 3D disturbances.

So, now, squires used some transformation and simplified the equations which we just derived. So, we now, introduce a set of transformation, which is known as squire transformation that reduces a 3D problem to a an equivalent 2D problem.

So, you can see that now, we will introduce these transformation; one is alpha bar square is equal to alpha square plus beta square, alpha bar u bar is equal to alpha u plus beta w, omega is equal to alpha into c ok.

And v bar is equal to v, V naught bar by alpha bar is equal to V naught by alpha, U bar is equal to U plus beta by alpha W and P bar by alpha bar is equal to P by alpha ok. So, we will use this transformation and we will simplify the equations. So, as we proceed towards the free stream U tends to U infinity, del u by del y decreases at a continuous lesser rate in the y direction. So, del 2 u by del y square remains less than 0 near the edge of boundary layer ok.

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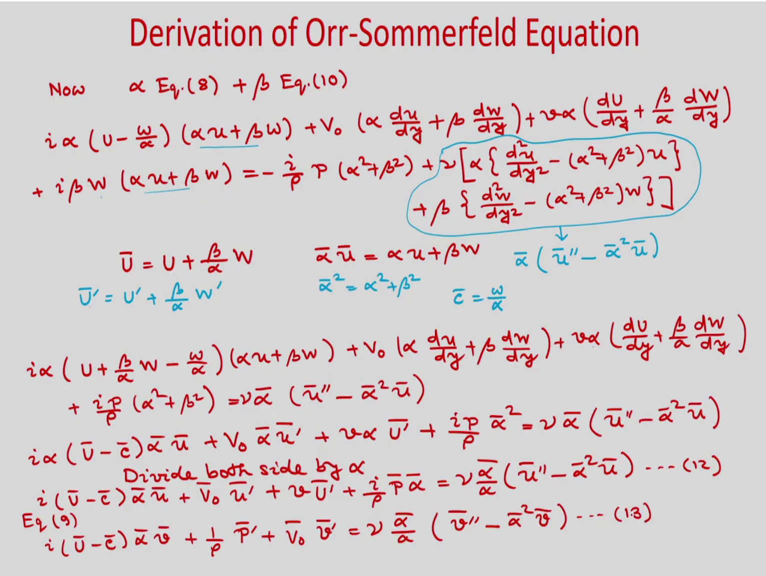

So, now, let us make this operation, alpha times equation 8 plus beta times equation 10. Then what we will get? So, we can see you can get i alpha U minus omega by alpha, alpha u plus beta w plus V naught alpha du by dy plus beta, dw by dy plus v alpha dU by dy plus beta by alpha, dW by dy plus i beta W alpha u plus beta w is equal to minus i by rho P alpha square plus beta square, plus nu alpha d 2 u by dy square minus alpha square plus beta square, u plus beta d 2 w by dy square minus alpha square plus beta square w ok.

So, now, let us use the squares transformation. So, we will use U bar is equal to u plus beta by alpha w, we will use alpha bar u bar is equal to alpha u plus beta w ok. So, you can see from here that, we have these terms alpha u plus beta w and this is also alpha u plus beta w. So, we can use alpha bar u bar here and also you can see here that we have this U and W and we have here also d U by dy plus beta by alpha dW by dy.

So, we can now write the U prime; U prime we are representing du by dy ok. So, plus beta by alpha dw by dy so, we will write w prime. So, this we can represent as U prime bar ok. So, and here you can see that we have del two u by del y square minus alpha square plus beta square u and alpha square beta square we are just writing; alpha bar square is equal to alpha square plus beta square and if you see this.

So, we will write these terms. So, you can see from here, we will write these terms as alpha bar. So, we will write just d 2 u by dy square, we will write u double prime ok. So, these derivative we are writing u double prime, but it will become bar minus alpha bar square u bar ok. So, this inside nu, whatever we have we will represent this term as this ok.

So, these terms whole term will represent as this. So, we will write c bar is equal to omega by alpha. So, from here you can see that we can write. So, here we will just write w is there beta w. So, if you take alpha outside. So, you will get i alpha U plus; so, it will be beta by alpha w ok and minus omega by alpha and this will be alpha u plus beta w ok. And this one just will keep as it is, V naught alpha du by dy plus beta dw by dy plus v alpha.

So, here you can see that this term we will write dU by dy plus beta by alpha dW by dy ok. And this term already we have added here, this term we will take in the left hand side. So, it will be i P by rho alpha square plus beta square and these term will be as it is and we are writing alpha bar, u bar double prime minus alpha bar square u bar ok.

So, now, you simplified these term. So, you can see that it will be i alpha so, U plus beta by alpha W. So, these we can write U bar w by alpha so, you can write c bar alpha u plus beta w so, this will write alpha bar u bar plus V naught. Now, you can see this will be just alpha bar u prime bar right, because then it will become u prime and w prime.

So, from here we are writing alpha bar u prime bar. So, u prime is the derivative ok, du bar by dy and then this term. So, it will be v alpha and this you can see from here. So, this we can write U prime bar. So, U prime bar plus this will be i P by rho and this will be just alpha bar square alpha bar square is equal to in right hand side ok.

So, there will be one nu here. So, it will be nu alpha bar; so, it will be u double prime bar minus alpha bar square u bar ok. So, this we have written using the squares approximation, now, divide both side by alpha. So, now, divide both side by alpha ok. So, we will get i U bar minus c bar alpha bar u bar and here you will get, V naught by alpha and V naught by alpha, you can write V naught bar by alpha bar and this alpha bar alpha bar will get cancelled.

So, we will get V naught bar u prime bar plus here, you will get v U prime bar plus i by rho P; so, it will be alpha bar P bar, because P by alpha will be P bar by alpha bar and one alpha bar will get cancelled is equal to nu alpha bar by alpha u double prime bar minus alpha bar square u bar and this equation let us say that equation 12 ok.

And, similarly from equation 9, we can write i U bar minus c bar alpha bar v bar plus 1 by rho P prime bar plus V naught bar, v prime bar is equal to nu alpha bar by alpha v double prime bar minus alpha bar square v bar ok. Let us say that this is equation number 13 ok.

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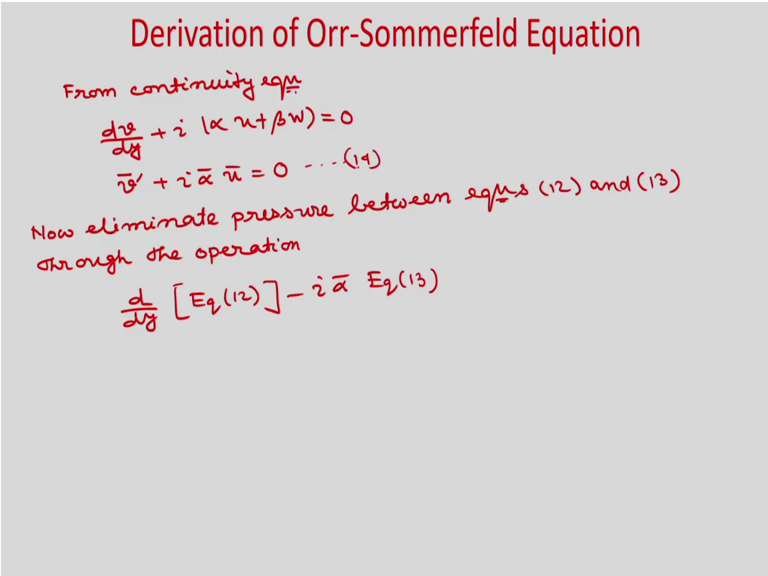

So, now, from continuity equation, we will get dv by dy plus i alpha u plus beta w is equal to 0. So, this we can write ok, v prime bar plus i alpha bar u bar is equal to 0 using squares approximation ok. So, now, eliminate pressure between equations 12 and 13 through the operation, d of dy of equation 12 minus i alpha bar equation 13 ok and this equation let us say it is 14.

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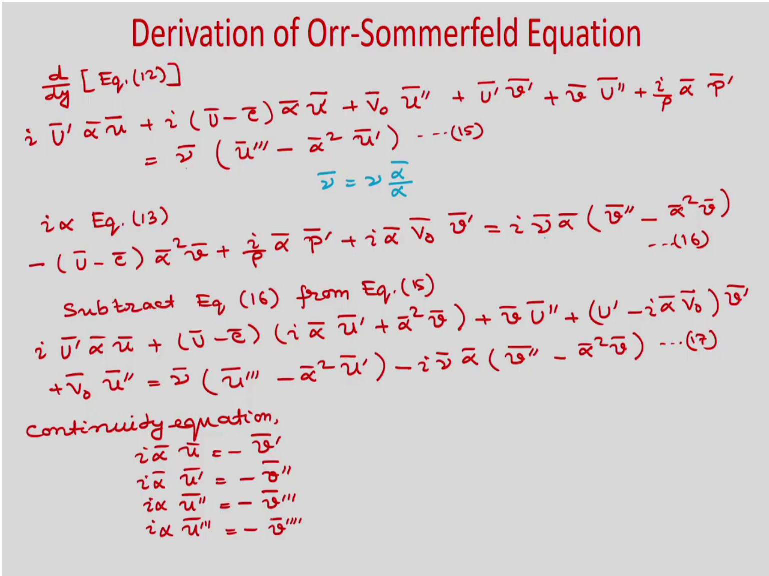

So, now, if we do d of dy of equation 12, then we will get U prime bar alpha bar u bar plus i U bar minus c bar alpha bar u prime bar plus V naught bar u double prime bar plus U prime bar v prime bar plus v bar u double prime bar plus i by rho alpha bar P prime bar is equal to in the right hand side we will get nu bar.

So, it will be u triple prime; that means, d cube u bar by dy cube minus alpha bar square and u prime bar ok. Say let us say, this is equation 15. And if we make this operation, i alpha into equation 13 ok, then we will get minus i square we are writing minus.

So, U bar minus c bar alpha bar square v bar plus i by rho alpha bar P prime bar plus i alpha bar V naught bar v prime bar is equal to i nu bar alpha bar v double prime bar minus alpha bar square v bar and this equation 16. Here, we have made this nu bar is equal to nu alpha bar by alpha ok.

So, these we have represented nu bar here and nu bar here ok. So, now, subtract equation 16 from equation 15 ok. Then, we can eliminate the pressure. You can see here, we have i by rho alpha bar P prime bar and here also we have i by rho alpha bar P prime bar. So, this will get cancelled. So, we can eliminate the pressure from these two equations.

So, finally, we will get i U prime bar alpha bar u bar plus, u bar minus c bar i alpha bar u prime bar plus, alpha bar square v bar plus v bar u double prime bar plus, U prime minus i alpha bar V naught bar v prime bar plus V naught bar u double prime bar is equal to nu u triple prime bar minus alpha bar square u prime bar. Minus i nu bar alpha bar it will be nu bar and we will get v double prime bar minus alpha bar square v bar ok.

So, this is equation number 17. And from continuity equation you can see, we will get i alpha bar u bar is equal to minus v prime bar ok. So, we can write. So, whatever we have this u triple prime, u prime; so, all these we can U double prime all these we can actually substitute in terms of v ok.

So, you can see that we can write, i alpha bar u prime bar is equal to minus v double prime bar then i alpha u double prime bar is equal to minus v triple prime bar and i alpha u triple prime bar is equal to minus v 4 prime bar ok. So, here let us substitute all these u in terms of v ok.

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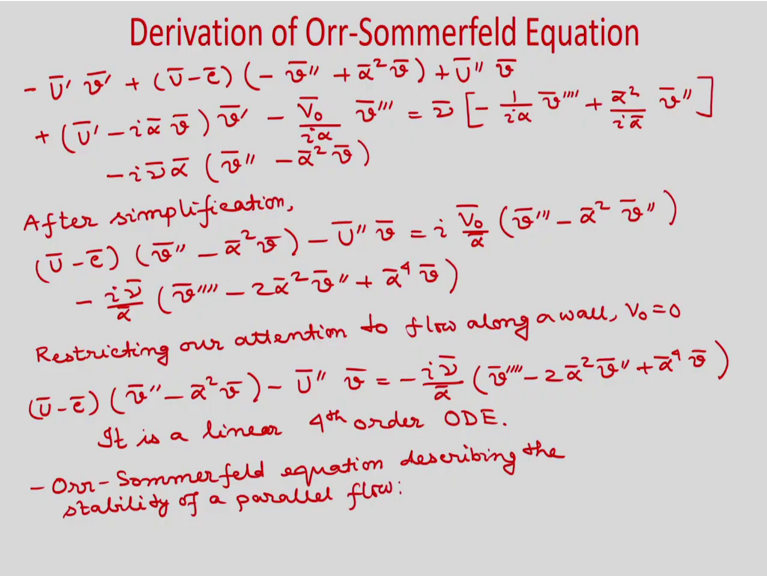

Then finally, you we will get if you substitute, minus U prime bar, v prime bar plus U bar minus c bar minus v double prime bar plus alpha square, v bar plus U double prime bar v bar plus U prime bar minus i alpha bar, v bar v prime bar minus V naught bar by i alpha v triple prime bar is equal to nu bar minus 1 by i alpha v 4 prime bar plus alpha bar square divided by i alpha bar v double prime bar minus i nu bar alpha bar v double prime bar minus alpha bar square v bar ok.

So, after simplification, you will get U bar minus c bar v double prime bar minus alpha bar square v bar minus U double prime bar v bar is equal to i V naught bar by alpha bar, v triple prime bar minus alpha bar square v double prime bar, minus i nu bar by alpha bar v 4 prime bar minus 2 alpha bar square v double prime bar plus alpha to the power four v bar ok.

So, now, if we restrict our study just to flow along a wall, then V naught the normal velocity through this wall; obviously, will be 0. So, we can write then this equation simplifying just after putting V naught is equal to 0. So, you are putting V naught is equal to 0.

So, you will get U bar minus c bar, v bar double prime minus alpha bar square v bar, minus u double prime bar v bar is equal to minus i nu bar by alpha bar, v bar 4 prime minus 2 alpha bar square, v double prime bar minus alpha to the power 4 v bar. So, these equation is known as Orr-Sommerfeld equation ok.

So, we can see it is a linear and 4th order ordinary differential equation. So, it is a linear 4th order ordinary differential equation and this equation is known as Orr-Sommerfeld equation describing the stability of a parallel flow. In today's class, we studied the stability theory, we started with the flow equation, then we introduced this velocity as summation of parallel flows and we introduce some perturbation in the parallel flow.

And these perturbations; if those are very small, then we can say that it is a linear stability analysis. So, using this parallel flow, we converted this momentum equation into the perturbation equation and then we use the squares theorem.

We use the Fourier analysis, then using that analysis we derived this ordinary differential equation, which is linear and 4th order differential equation. So, this equation, if we solve and find the Eigen values, then we will be able to comment on the critical Reynolds number for the transition from laminar to turbulent flows.

Thank you.