**Viscous Fluid Flow**
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**Module - 09**
**Laminar Free Shear Flows**
**Lecture - 03**
# Free Shear Layer Between Two Different Streams

Hello everyone. So, we will continue with the Laminar Free Shear Flows. Today, we will consider Free Shear Layer Between Two Streams.

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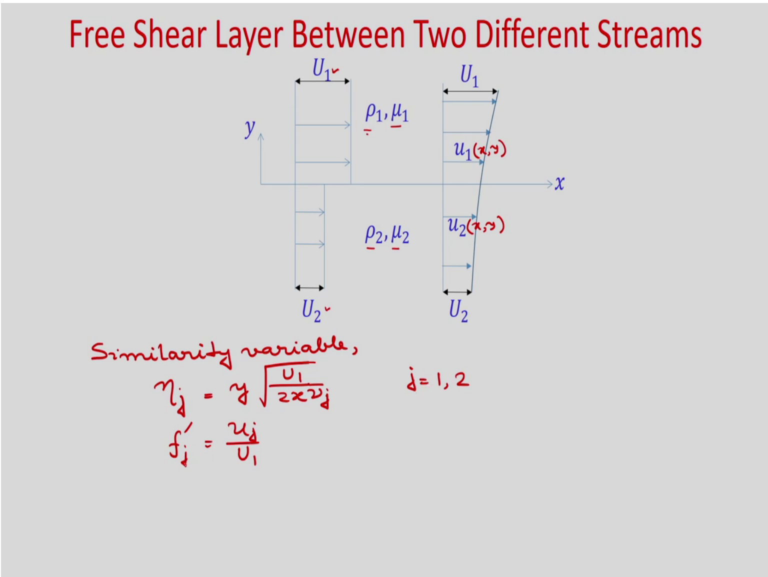


So, we can consider that two different streams are coming with different velocities; U 1 and U 2 and at x equal to 0, they are meeting and obviously, when it will go further in the axial direction, so there will be a velocity gradient. So, you can see that u 1 is the velocity which is function of x and y in the fluid stream 1, whose properties are density rho 1 and viscosity mu 1 and in fluid 2 domain, the velocity is u 2 which is function of x and y and the fluid properties are rho 2 and mu 2.

So, scientist Lock fist solved this problem using similar similarity variable used by Blasius. So, he used a different similarity variable for two different streams. So, let us write down the similarity variable this eta j. Similarity variable eta j is equal to y root U 1 divided by twice x nu j, so, where j is equal to 1 and 2. So, 1 is for fluid 1 in the upper layer and j is equal to 2 is the fluid layer 2 which is lower layer.

So, two different similarity variables were used for two streams and f prime j which is the representation of the velocity u, so that is u j by u 1 ok; where, u 1 is the scale of this velocity of first fluid layer or upper fluid layer. So, here you can see obviously for two different layers j is equal to 1 and 2, you will get different f j prime. So, this equation also we can write the boundary layer equation and we need to solve using these similarity variable.

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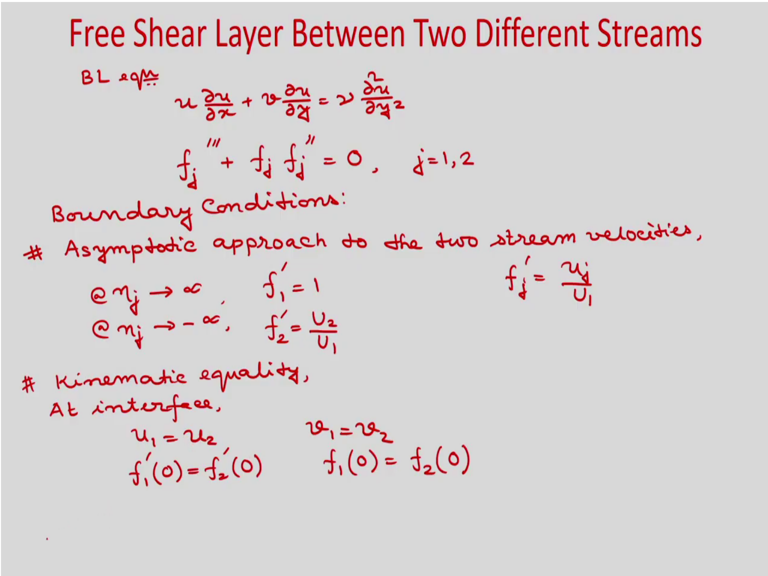


So, our governing equation that is your boundary layer equation is u del u by del x plus v del u by del y is equal to nu del 2 u y del y square ok. So, if we use those similarity variable and using similar approach made in the Blasius equation, we can write the Blasius type equation for each layer as f j triple prime plus f j f j double prime is equal to 0 ok; where, j is equal to 1 for fluid layer 1 and j is equal to 2 for fluid layer 2.

So, you can see that this is the ordinary differential equation; third order ordinary differential equation and this is non-linear. So, now, let us discuss about the boundary conditions. So, for each layer, we will have the boundary condition as well as we will have interface condition.

So, boundary conditions. So, to solve this ordinary differential equation, we will use these boundary conditions. First one is asymptotic approach to the two stream velocities ok. So, you can see that as eta j tends to infinity ok. So, that means, in the positive y direction. So, y tends to infinity. So, obviously, f 1 prime will be 1 because f j prime, we have defined at u j by U 1 right. So, at y tends to infinity, we have velocity e 1.

So, f 1 prime will become 1 and eta j tends to minus infinity. So, for j is equal to 2, you can see f 2 prime we can write as; so, u j will be U 2, so it will be U 2 by U 1 ok. So, there should be some kinematic equality. So, if you use kinematic equality, then we can write that velocity at the interface will be same for both the stream ok. So, at interface, we have ok. So, u 1 should be u 2 and v 1 should be v 2.

So, u 1 is equal to u 2 if you see, that you will get f 1 prime ok and the interface you can see that we have at y is equal to 0. So, f 1 at y is equal to 0 will be f 2 prime at y is equal to 0 ok. So, velocity continuity we have and also from v 1 is equal to v 2, you can see that it will be f 1 0 should be f 2 0 ok. So, these are the velocity continuity at the interface. Similarly, we will have the shear continuity at the interface.

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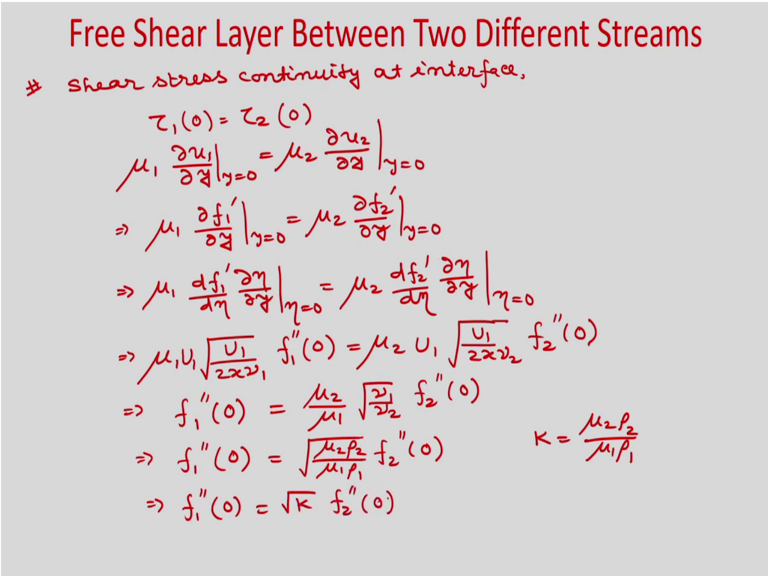


So, if you write that boundary condition, then we will get shear stress continuity at interface ok. So, at y is equal to 0 ok, so that means, eta is equal to 0. So, tau 1 at eta is equal to 0 should be tau 2 at eta is equal to 0. So, that means, mu 1 del u 1 by del y, so at y is equal to 0 ok; at y is equal to 0 should be mu 2 del u 2 by del y at y is equal to 0.

So, if you write in terms of f, then you can write mu 1 del f 1 prime by del y at y is equal to 0 should be equal to mu 2 del f 2 prime by del y at y is equal to 0. So, now, this velocity gradient let us write this derivative with respect to eta. So, if you write that, then we will get mu 1 d f 1 prime by d eta del eta by del y at eta is equal to 0 is equal to mu 2 d f 2 prime by d eta del eta by del y at eta is equal to 0 ok.

So, you can see you can write this as mu 1 and del eta by del y which will be root U 1 divided by twice x nu 1. So, del f 1 prime by del eta, we will write just f 1 double prime at eta is equal to 0 is equal to mu 2 del eta by del y at eta is equal to 0. So, this will be. So, here 1 U 1 will be there and here, U 1 root U 1 by twice x nu 2 f 2 double prime at eta is equal to 0. So, if you rearrange it. So, you can write as f 1 double prime at eta is equal to 0 as.

So, you can see root U 1 by 2 x all this will get cancelled and we will have just mu 2 by mu 1 and we will have root nu 1 by nu 2 f 2 double prime 0. So, if you take it inside and do the simplification, you will get f double prime at eta is equal to 0 will be just root mu 2 rho 2 by mu 1 rho 1 into f 2 double prime 0 ok. Let us represent K is equal 2 mu 2 rho 2 divided by mu 1 rho 1. So, we can write it as f double prime 0 is equal to root K f 2 double prime 0.

So, now, let us consider few extreme cases. So, one case we will consider that lower layer velocity is very very small compared to the upper layer. So, in that case, we can consider that U 2 by U 1 is equal to 0 ok. So, U 2 is very very small compared to U 1.

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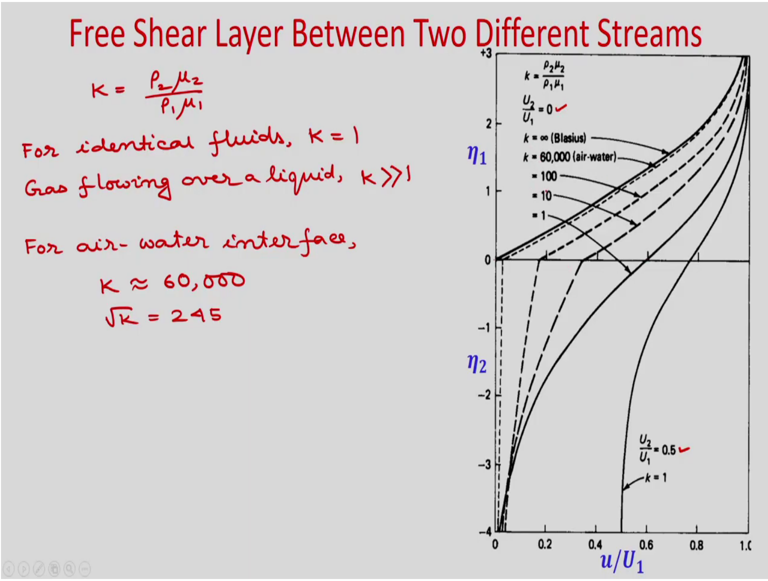

So, in that case, you can see that one case we can consider, where U 2 by U 1 is equal to 0 and another case we will consider U 2 by U 1 is equal to 0.5 ok and the value of K. So, K is equal to rho 2 mu 2 by rho 1 mu 1 ok. So, now, if you consider the identical fluid in both the streams ok same fluid; then obviously, properties will be same. So, K will become 1 ok.

So, we can write for identical fluids, so these fluid properties will be same; so, K is equal to 1. And if you consider gas flowing over a liquid, so obviously, liquid this mu 2 rho 2 will be very high right and compared to rho 1 mu 1. So, you can write that K is much much greater than 1.

So, for example, you can see for air-water interface K value is 60,000 ok. So, that means, root K will be almost 245. So, for this U 2 by U 1 is equal to 0 we will consider different K value. So, if K is equal to infinity that means, it is a Blasius equation which we have already solved and K 60,000, then air-water and if K is equal to 1, then identical fluids. So, you can see these are the velocity distribution u by U 1 and this is eta 1 and eta 2 ok. So, this is eta is equal to 0.

So, you can see that when you consider K is equal to infinity; that means, it is flow over a flat plate case and this is the case ok. So, this is the Blasius profile and if you consider K is equal to 1; that means, it is identical fluids, so you can see this is the velocity profile ok. So, you can see that in this case obviously, the interface velocity is greater than 0.5 ok.

So, u by U 1 this interface velocity is greater than 0.5 ok. It is due to that we have different convective deceleration in two streams. So, we will get this interface velocity u by U 1 greater than 0.5. When you consider K is equal to 1 and U 2 by U 1 is equal to 0.5, so this is the velocity profile.

So, this actually the scientist Lock computed and plotted for different values of K. Till now, we have used analytical method to solve the boundary layer equations. Another way to solve this boundary layer equation is using numerical method. So, you can use different discretization scheme like finite difference method, finite volume method or finite element method; but for flow over a flat plate, we will use finite difference method.

When we use finite difference discretization method, we need to divide the domain into grid so that we can solve this discretize equation at each grid point. So, first let us write down the boundary layer equation in general and we will use suitable discretization scheme to write the final algebraic equation and then, we will discuss about the boundary conditions. If you look into this boundary layer equation, it is parabolic in nature.

So, it is actually marching in x direction. So, if you know the inlet velocity at x equal to 0 which is your free stream velocity U infinity, then you can march in x direction and next level where say next downstream level, you can compute the velocities from the known velocity in the at inlet.

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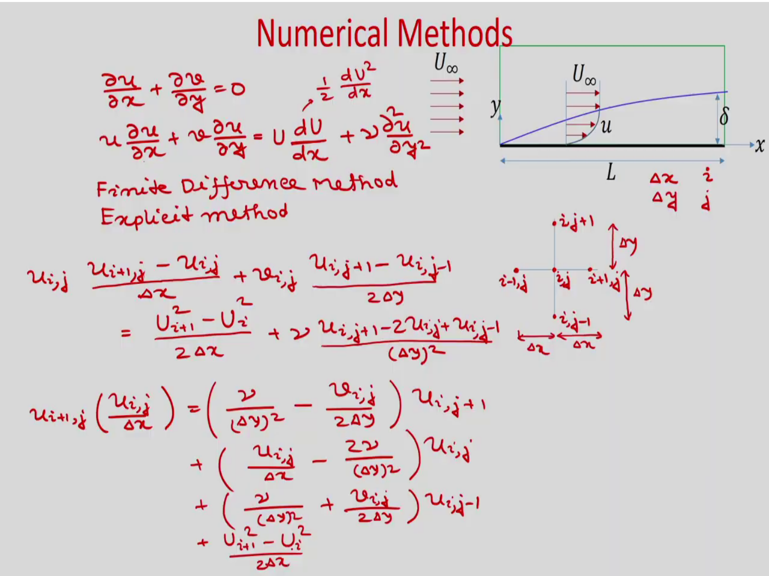


So, let us write the governing equation first. So, our governing equation is continuity equation; del u by del x plus del v by del y is equal to 0 and we have boundary layer equation, u del u by del x plus v del u by del y is equal to this. We will also discretize this term which is your pressure gradient term although for flow over flat plate it is 0; but you can use it for any cylinder body plus nu del 2 u by del y square.

So, when we need to use this numerical method, so this is the flat plate and you know that boundary layer will grow like this. So, you take a domain which is larger than this boundary layer thickness and you can see we have represented the computational domain with this green colour boundary.

Now, you need to divide the domain ok into grid. So, we will have in x direction, the grid size delta x and in y direction, we will have the grid size delta y and indices will use in the x direction i and in y direction j. So, if you use that at these point, we need to discretize this equation, then we will give the indices at i j and we will have the neighbour points as this will be i plus 1 j; this is i minus 1 j and this is i j plus 1 and this is i j minus 1.

So, we will discretize this momentum equation in this grid point I, j and we will use explicit method ok. So, we will find the value of i plus 1 from the known value of i j ok. So, this equation now we will use forward difference for this convective term. So, you can write.

So, if you discretize using finite difference method ok; so using Taylor series expansion, you can express this gradient in terms of the discreet point values. So, using finite difference method and we will use explicit method. So, we can discretize this equation as u i, j. So, at this point we are discretizing.

So, this gradient del u by del x if you use forward difference, then we will can write as u i plus 1 j minus u i j divided by delta x and this we can write plus v i j u i j plus 1. So, this convective term del u by del y, we will use central difference ok. So, in the y direction, we will use central difference. This we are using forward difference because you are using explicit method so that u i plus 1 j, we can find from the known values of u i j and other neighbours.

So, this we will use del u by del y central difference, so we can write u i j plus 1 minus u i j minus 1 and you can see the difference between these two points is 2 delta y. So, you can see that this is delta y, this is also delta y ok and the distance between these two grid points is delta x and also this is delta x. So, these are the grid size.

So, obviously, you can see when you are writing u i j plus 1 minus u i j minus 1, so it will be divided by 2 delta y. This we can write as half d u square by d x ok. So, if you discretize that, then we can write u i plus 1 square minus u i square divided by 2 delta x because u may be function of x, so we are using forward difference plus nu. So, this we will use central difference.

So, we will use from the known value at point i ok. So, u i j plus 1 minus 2 u i j plus u i j minus 1 divided by delta y square. So, you can see that as this boundary layer equation is parabolic, so we are marching in x and explicit method, we are telling as we are finding the value of velocity that i plus 1 location from the values of i location ok.

So, now, if you simplify this, then we can write u i plus 1 j. So, that is unknown. So, you can take it left hand side. We have this coefficient u i j by delta x. So, this is an unknown parameter and all are known from the ith point ok. So, you can see i plus 1 j we are finding from the these values, known values ok. That way we were marching in x direction.

So, in right and side now u i j plus 1 coefficient if you see, so it will be nu by delta y square minus v i j divided by 2 delta y into u i j plus 1; then u i j coefficient if you see, so it will be u i j divided by delta x minus twice nu divided by delta y square into u i j. And we have nu by delta y square plus v i j divided by 2 delta y and coefficient this is the coefficient of u i j minus 1 and we have this term u i plus 1 square minus u i square divided by 2 delta x.

So, now, we can find the value of u at i plus 1 j, you can find from the known values of i j i j minus 1 i j plus 1 and i minus 1 j.

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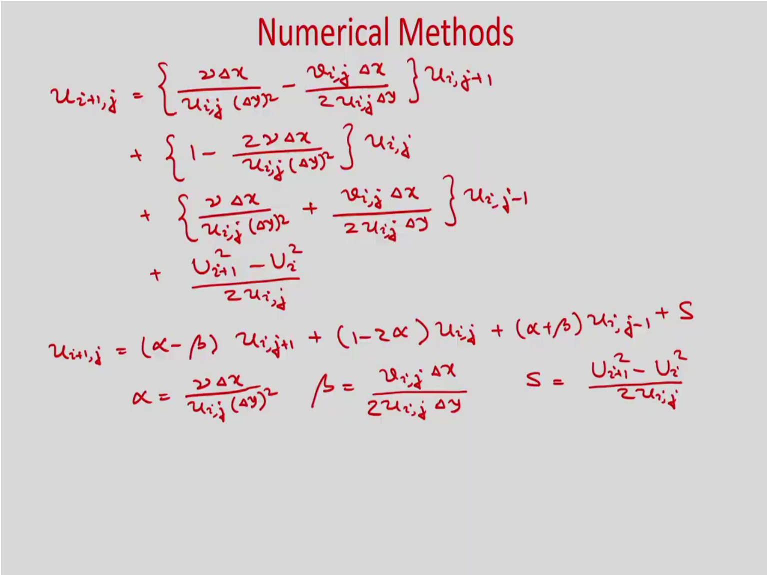


So, we can write now the velocity u i plus 1 j is equal to nu delta x by u i j delta y square minus v i j delta x divided by 2 u i j delta y u i j plus 1 plus 1 minus twice nu delta x divided by u i j delta y square u i j plus nu delta x divided by u i j delta y square plus v i j delta x by 2 u i j delta y u i j minus 1 plus u i plus 1 square minus u i square divided by twice u i j.

So, we can write so, this is the discretized equation right. So, we can write in this form; u i plus 1 j is equal to alpha minus beta u i j plus 1 plus 1 minus 2 alpha u i j plus alpha plus beta u i j minus 1 plus source term S. So, where, alpha is nu delta x by u i j delta y square; beta is v i j delta x divided by u i j delta y and there will be 2 and the source term S is equal to u i plus 1 square minus u i square divide by 2 u i j.

So, you can see that this equation for every i j if you right, then you will get system of algebraic equations and that you need to solve using suitable iterative method. So, as we used explicit method. So, if you do the Von Neumann stability analysis. So, there will be restriction to choose the value of delta x ok.

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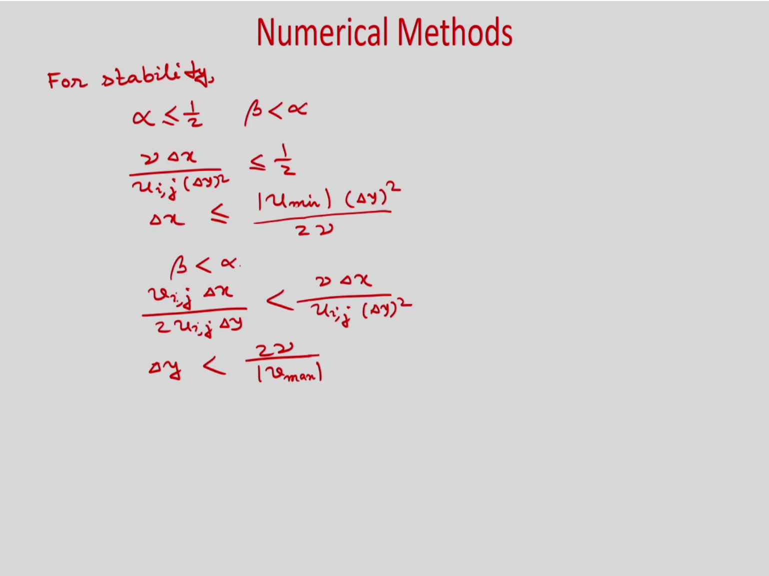


So, if you do the Von Neumann stability analysis, then based on the stability criteria, we can write that for stability, we can write that alpha should be less than equal to half and beta should be less than alpha ok. So, alpha is nu delta x by u i j delta y square should be less than equal to half and delta x, you can choose as you can see that it will be u minimum in the domain delta y square divided by twice nu ok.

So, you have to choose delta x based on the minimum velocity in the domain, then the y gets a delta y and the fluid property nu ok and beta is less than alpha. So, you can see that v i j delta x divided by twice u i j delta y should be less than nu delta x divided by u i j delta y square.

So, this you can find that delta y should be less than twice nu divided by maximum velocity v max ok because here v is there, so it should be v max so that delta will be the minimum value you can take. So, we use the explicit method to discretize the boundary layer equation and we have seen that there is a restriction to choose the value of delta x.

But if we use some suitable implicit method, then there will be no restriction to choose the value of delta x. Because we can see using the Von Neumann stability analysis that that the scheme will be unconditionally stable.

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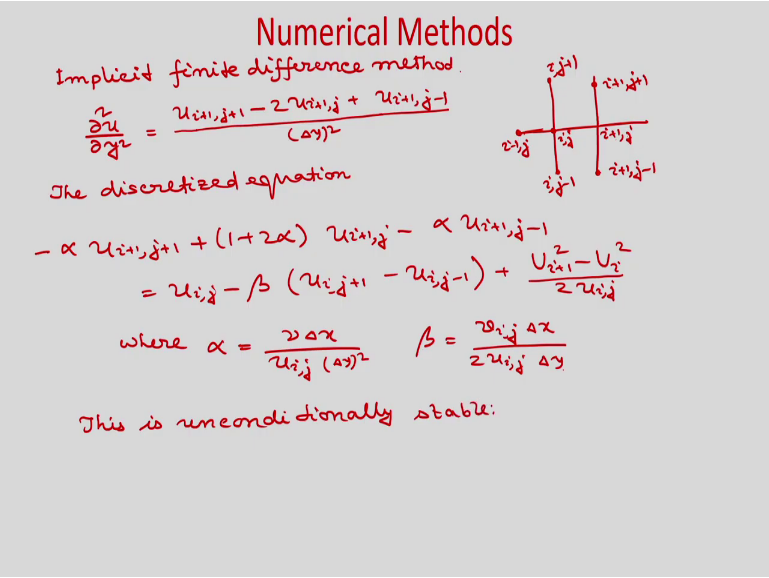


So, let us use this implicit finite difference model. So, here we will use del 2 u by del y square will discretize like u i plus 1 j plus 1 minus 2 u i plus 1 j plus 2 u i plus 1 j minus 1 divided by delta y square. So, if you can see that we have used. So, this is i j; this i plus 1 j; this point is i plus 1 j plus 1 and this is i plus 1 j minus 1 and this is i j minus 1 and this is i j plus 1 and this is i minus 1 j.

So, we have discretized this gradient del 2 u by del y square at this point i plus 1 j so that we will write u i plus 1 j plus 1 minus 2 u i plus 1 j plus u i plus 1 j minus 1 divided by delta y square. So, you can see that we have more than 1 unknown of u at point i plus 1.

So, if you discretize this, then write the final discretize equation as minus alpha u i plus 1 j plus 1 plus 1 plus 2 alpha u i plus 1 j minus alpha u i plus 1 j minus 1 is equal to u i j minus beta u i j plus 1 minus u i j minus 1 plus u i plus 1 square minus u i square divided by twice u i j; where alpha is equal to nu delta x by u i j delta y square and beta is v i j delta x divided by twice u i j delta y.

So, if you do the Von Neumann stability analysis, this is unconditionally stable. So, if you see the discretize equation, so in the left hand side, we have more unknown terms ok. So, we have u i plus 1 j plus 1, u i plus 1 j, u i plus 1 j minus 1. So, for each grid point, if you write this equation, so you will get a system of linear equations and that you need to solve using some suitable iterative method.

So, you can use like Gauss iterative method or some other advance solver like Conjugate gradient method or Bi-conjugate gradient method, you can use to solve this implicit equation. So, regarding the boundary condition, obviously, it is easy because you know that at the valve we have velocity 0. So, you can make velocities at the boundaries 0. At inlet, obviously, we have the pristine velocity ok. So, you need to give that u is equal to the free stream velocity U infinity and v should be 0.

At the upper boundary obviously, it will be free stream velocity. So, it will be U infinity and v is equal to 0 and obviously, at the right boundary or exit boundary, you do not need in the boundary condition because you are marching in the x direction. Obviously, when you will go to the outer boundary, right most boundary, then you will be able to calculate the value of u from the previous grid point.

So, in today’s class, first we considered the free shear flows between two different streams. So, initially it was having two different velocities U 1 and U 2 in upper and lower level. Then it will have the velocity gradient u 1 in the upper level and u 2 in the lower level.

So, scientist Lock actually solved this problem similar to Blasius equation. So, he used the same similarity variable eta for two different streams. Then, he wrote the ordinary differential equation which is third order non-linear ordinary differential equation. Then, we discussed about the three different boundary conditions; one is that at the valve, the velocity u and v are.

So, we discuss about three different boundary conditions. So, you can see that upper level obviously at eta tends to infinity, the velocity will be equal to U 1; that means, your f 1 prime will be 1 and the at eta tends to minus infinity, that means, at the lower level, we have the velocity U 2. So, using this similarity variable, you can see that f 2 prime will be U 2 by U 1.

Then, we have used the velocity continuity as well as the shear stress continuity at the interface and from there, we have seen the velocity profile for two different cases, where U 2 by U 1 is equal to 0, where lower stream velocity is very very small compared to the upper-level velocity..

And next, we considered U 2 by U 1 as 0.5. Then, we discussed our different values of K which is the root square ratio of this mu 2 rho 2 divided by mu 1 rho 1 and in that case, K is equal to if you consider 1, then obviously, two stream properties are same; so, that means, we will have the same fluid. And if K tends to infinity, obviously, the lower-level velocity will be 0 and for K tends to infinity, we considered that flow over a flat plate which is the Blasius solution.

Then, we used finite difference method to solve the boundary layer equation. As boundary layer equation is parabolic in nature, so we are marching in direction x. So, we used both explicit and implicit method to discretize the boundary layer equation and we have written the final algebraic equation and finally, we discuss about the boundary conditions.

Thank you.