Viscous Fluid Flow
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Module - 9
Laminar Free Shear Flows
Lecture - 02
Flows in the Wake of a Flat Plate

Hello everyone. So, today, we will consider another example of Free Shear Flow. We will consider today, the Flow near a wake behind the Flat Plate of finite length. So, we can see that immediately downstream of the flat plate, the wake is developing and non-similar; but about 3 lengths downstream, the wake is developed and velocities are self similar.

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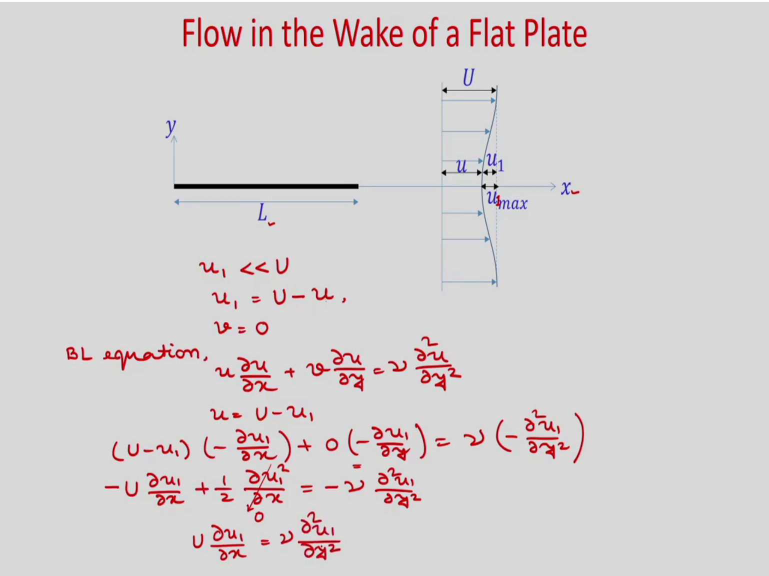

So, let us consider that this is the flat plate of finite length L; y is measured from the flat plate and this is the axial direction x.

So, you can see just after the flat plate, the wake will be developing and wake is non-similar; but far away from here, about 3 times the downstream, this wake is developed and velocity profiles are self-similar. So, here you can see that we have U as free stream velocity and in the wake, obviously, there will be a momentum deficit and you can see that velocity profile u is function of x and y and this u 1, u 1 is the velocity defect ok.

So, you can see we can write u 1 as U minus small u and obviously, velocity defect will be maximum at the central line. That means, at y is equal to 0, u 1 will be maximum. Here, we will assume that this velocity reflect u 1 is very very small compared to the free stream velocity U.

So, in this case u 1 is very very small compared to the free stream velocity U. And from here, you can see that we can write u 1 is capital U minus small u and v is also 0. So, we have the boundary layer equation. So, it will be valid for these free shear flows.

So, that is u del u by del x plus v del u by del y is equal to nu del 2 u by del y square. So, this boundary layer equation, now we will write in terms of the velocity defect. So, substitute u 1 is equal to capital U minus small u.

So, you can see that from here, we can write u is equal to U minus u 1 and if you substitute it here, so you can write U minus u 1 and you can see del u by del x, if you see, u is constant. So, it will be minus del u 1 by del x plus v will be 0. So, del u by del y, you can write minus del u 1 by del y is equal to nu del 2 u by del y square, so that we can write minus del 2 u 1 by del y square ok.

So, you can see this term is 0 and this if you multiply, then you will get minus U del u 1 by del x and it will be plus u 1 into del u 1 by del x. So, that we can write as half del u 1 square by del x. This will be 0 and it will be minus nu del 2 u 1 by del y square. So, you can see that u 1 is very very small compared to U and u 1 square will be much much smaller than the U square.

So, obviously, this term you can neglect ok. So, if you neglect this term, you can write the equation U del u 1 by del x is equal to nu del 2 u 1 by del y square. So, you can see that this boundary layer equation after simplification, we have written in terms of velocity defect. Now, let us write the boundary conditions.

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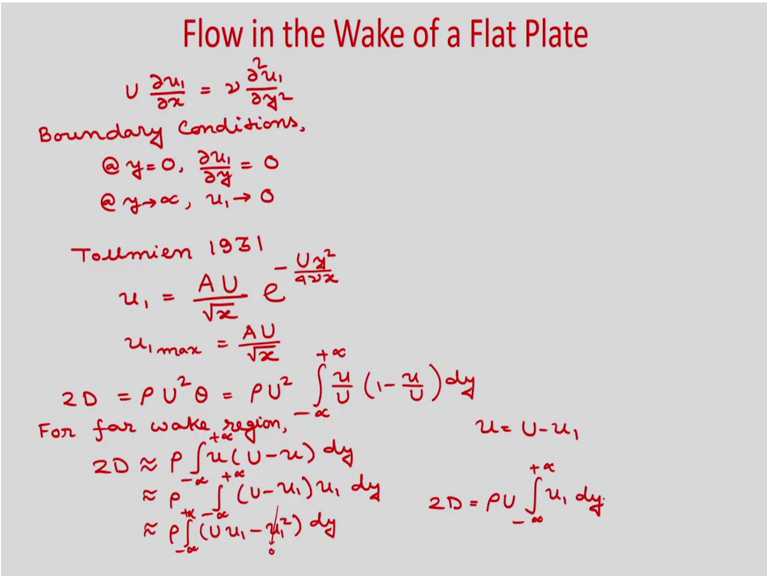

So, we have the equation U del u 1 by del x is equal to nu del 2 u 1 by del y square and boundary conditions, we have at y is equal to 0 ok. At y is equal to 0, it is symmetry right. We will have this maximum velocity of u 1.

So, that means, you will get del u 1 by del y will be 0 and at y tends to infinity, obviously, velocity effect will become 0. So, u 1 tends to 0. So, this equation was solved by a Tollmien, he expanded the stream function and invoking the boundary conditions, he solved this velocity defect and the expression of this velocity defect solution is.

So, Scientists Tollmien in 1931, he solved this problem and velocity defect expression, he found as A is constant, U is the free stream velocity divided by root x e to the power minus U y square divided by 4 nu x ok.

So, from here, we can write that u 1 maximum is equal to AU by root x ok. Now, we need to find this constant A. So, for that, we will use the Drag coefficient ok. So, we know that drag coefficient for flow over flat plate. Now, that we will use to find this constant A. So, you can see that decrease of flow of momentum in the wake is equal to the friction drag of the plate.

So, you can write 2 D is equal to rho U square theta, where theta is the momentum thickness and that you can write rho U square minus infinity to plus infinity u by U 1 minus u by U dy ok. So, this is per unit width of the plate we are writing. So, for the far wake region, we can write that 2 D approximately that you can see it will be rho minus infinity to plus infinity u U minus u d y and we know that u is equal to capital U minus velocity defect u 1 ok.

So, if you substitute it here, so we will get rho minus infinity to plus infinity u is U minus u 1 and this is u 1 d y. So, this you can see, you can write as rho integral minus infinity to plus infinity. So, it will be U u 1 minus u 1 square d y. So, obviously, u 1 is a very very small compared to the U. So, we can neglect this term. So, we can write 2 D is equal to rho U integral minus infinity to plus infinity u 1 d y. So, now, we will use the drag formula for the flat plate and we will find the constant A.

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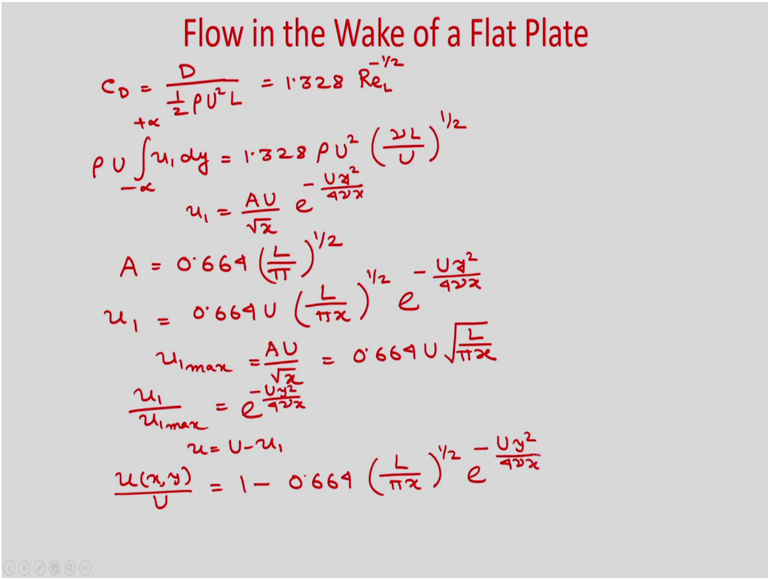

So, you can see that we have drag coefficient C D is equal to D divided by half rho U square into L per unit width. So, it will be 1.328 Reynolds number to the power minus half. So, this if you write, so Reynolds number based on L. So, we can write now rho U minus infinity to plus infinity u 1 d y should be 2 into D right.

So, 2 into D if you write, then you will get 1.328 rho U square and this will be nu L by U to the power half ok. So, now, if you substitute the velocity expression u 1 is equal to AU divided by root x e to the power minus U y square divided by 4 nu x. In this expression and if you integrate minus infinity to plus infinity, then you will get the constant A as 0.664 L by pi to the power half.

So, you can see, now you can write the velocity defect u 1 as 0.664 U L by pi x to the power half e to the power minus U y square divided by 4 nu x ok. So, here we can write that u 1 max is equal to AU by root x. So, this will be 0.664 U root L by pi x ok. So, we can write u 1 by u 1 max is equal to e to the power minus U y square divided by 4 nu x. Now, if we want to write the velocity u, then we can substitute this expression u is equal to U minus u 1.

So, u 1 is equal to U minus U, if you substitute it here, then we can write the velocity distribution u divided by capital U as 1 minus 0.664 L by pi x to the power half e to the power minus U y square by 4 nu x. So, from this expression, you can see obviously, velocity variation is happening exponentially ok. So, you can see that at y tends to infinity, obviously, the u will become capital U.

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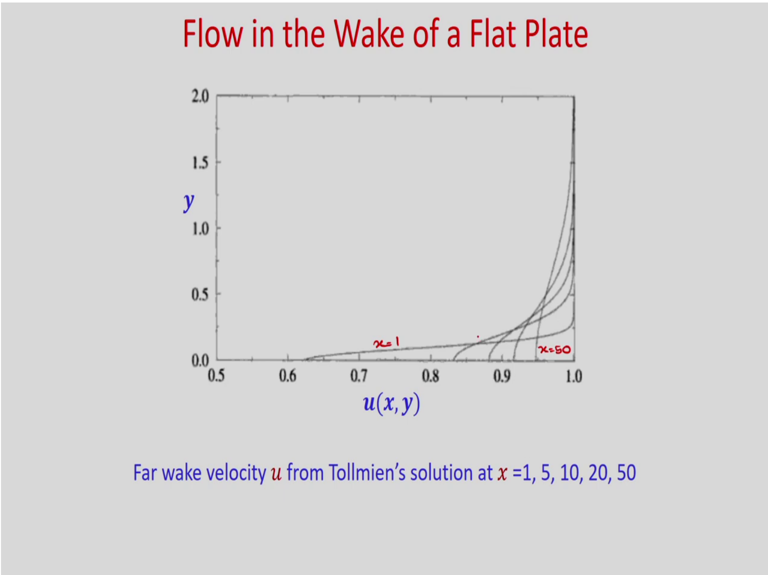

Now, if you plot this velocity u with y at different x location, then you can see that far wake velocity u from Tollmien solution at different x location. So, this is u and this is y. So, you can see. So, this is for x equal to 1 position. So, how it is varying? Then, as you are going in the downstream; obviously, velocity defect is decreasing and you can see that this is at x is equal to 50 ok. So, from here, you can see that velocity defect is decreasing in the downstream direction and this is the velocity profile.

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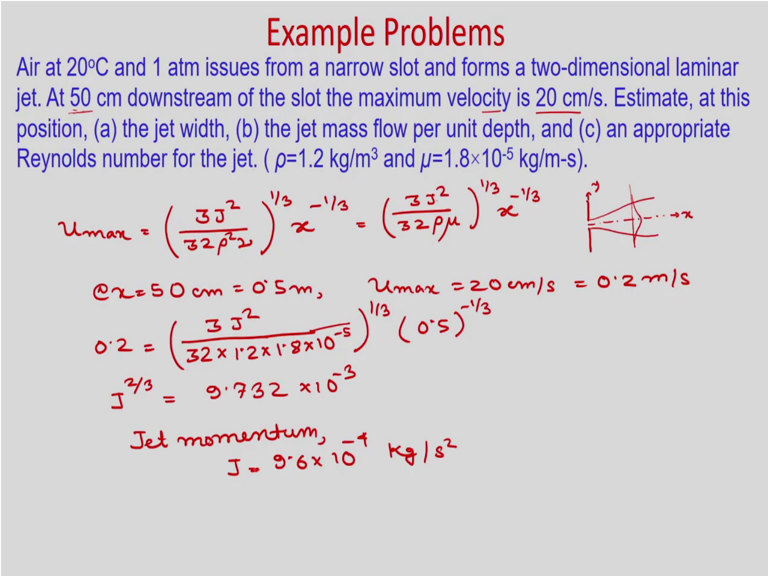

Now, let us solve two example problems based on these free shear flows. So, let us consider the first problem. Air at 20 degree centigrade and 1 atmosphere issues from a narrow slot and forms a two-dimensional laminar jet.

At 50 centimeter downstream of the slot the maximum velocity is 20 centimeter per second. Estimate at this position- a, the jet width; b, the jet mass flow per unit depth and c, an appropriate Reynolds number for the jet. The density and viscosity are given. So, you can see this problem of free shear jet.

So, we have already obtained the velocity distribution as well as the maximum velocity expression. So, you can see that this is the slot and from here, there will be jet coming and this is the central line and at any location, you can see this will be your velocity profile. So, this is x and in this direction, it is y. So, we know that the maximum velocity u max expression is 3 J square by 32 rho square nu to the power 1 by 3 x to the power minus 1 by 3.

So, this expression, we can write now as 3 J square divided by 32. So, here 1 nu mu by rho will get cancelled. So, you can write rho into mu to the power 1 by 3 x to the power minus 1 by 3. So, you can see that maximum velocity is given as 20 centimeter per second and the length is given as 50 centimeter ok. So, from here with the known parameters, we can calculate the jet momentum J ok.

So, you can see at x equal to 50 centimeter; that means, it is 0.5 meter; u max is 20 centimeter per second ok, so that means, 0.2 meter per second. So, if you substitute it here, we are going to get 0.2 is equal to 3 J square by 32 density is 1.2 into mu 1.8 into 10 to the power minus 5 to the power 1 by 3 and 0.5 to the power minus 1 by 3. So, from here, you find the value of J square ok.

So, you can see you can write J to the power 2 by 3 is equal to 9.732 into 10 to the power minus 3 and from here, jet momentum you will get J as 9.6 into 10 to the power minus 4 kg per second square ok. Now, we will calculate the jet width because jet width, we can express in terms of jet momentum and this jet momentum value, we will use.

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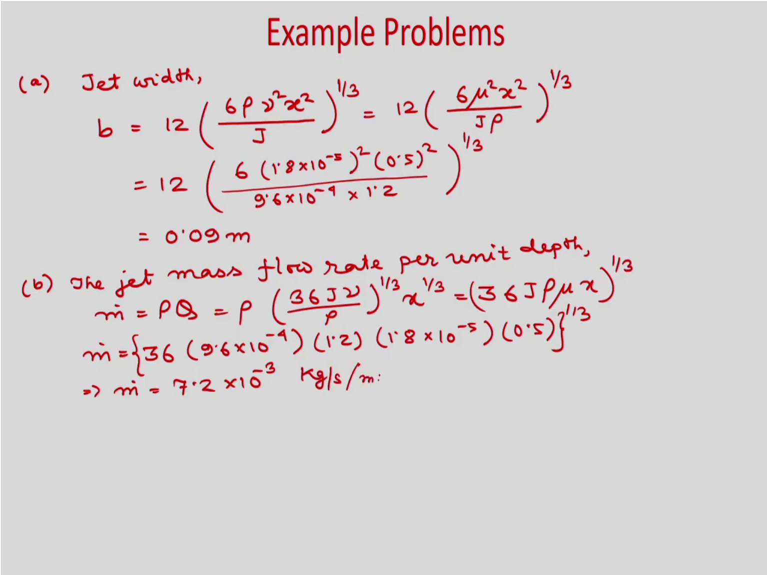

So, jet width b is equal to 12 into 6 rho nu square x square divided by J to the power 1 by 3 ok. So, this we can write as with known parameters 6 mu square x square divided by J rho to the power 1 by 3. So, now, you can see b will be 12 6 into mu square. So, 1.8 into 10 to the power minus 5 square; x square, so it is 0.5 square divided by J, 9.6 into 10 to the power minus 4 and rho, it is 1.2 ok to the power 1 by 3. So, from here, you will get it is as 0.09 meter.

So, next we need to calculate the jet mass flow per unit width. So, we will use the formula for jet mass flow rate per unit depth m dot as rho Q and Q we have already calculated as 36 J nu by rho to the power 1 by 3 x to the power 1 by 3. So, this we can express as 36 J rho mu x to the power 1 by 3. So, if you put all the values, then you will get 36 into J, 9.6 into 10 to the power minus 4, rho 1.2. Then, u; sorry this is mu, 1.8 into 10 to the power minus 5 into 0.5 to the power 1 by 3.

So, we will get m dot as 7.2 into 10 to the power minus 3 kg per second per meter ok; per unit depth we have written, so per meter we will not write here. Now, we need to calculate the suitable Reynolds number. So, Reynolds number we can define based on maximum velocity or based on the mass flow rate. So, let us see that Reynolds number, we can write based on different parameters as.

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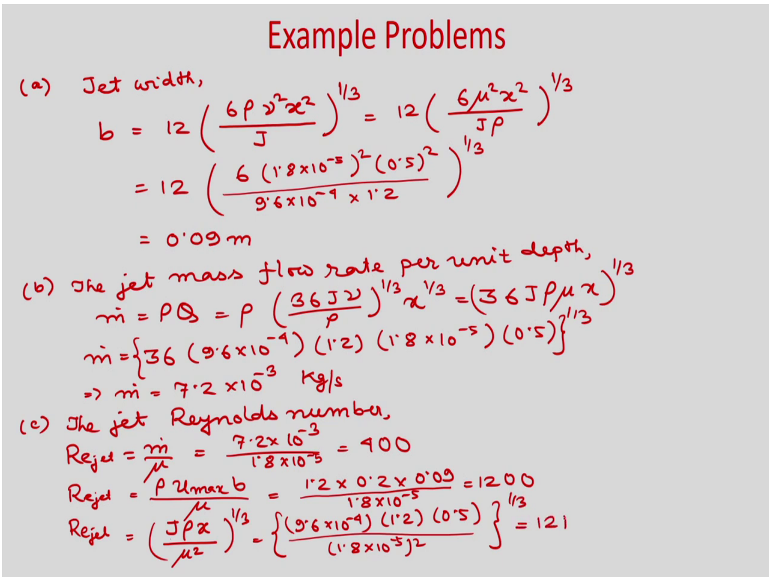

So, the jet Reynolds number Re jet ok, based on mass flow rate we can write m dot by mu. So, it will be 7.2 into 10 to the power minus 3 divided by 1.8 into 10 to the power minus 5. So, it will be 400 ok. Re jet, if you express in terms of the maximum velocity and the depth, then you can write as 1.2 into 0.2 into 0.09 divided by 1.8 into 10 to the power minus 5. So, it will be 1200.

And Reynolds number based on this jet momentum, so it can be expressed as J rho x divided by mu square to the power 1 by 3. So, we can write as 9.6 into 10 to the power minus 4, 1.2, x is 0.5 divided by 1.8 into 10 to the power minus 5 whole square to the power 1 by 3. So, if you calculate, you will get 121.

So, obviously, from this expression you can see that Reynolds number we can define based on sub parameter and obviously, you will get different numbers ok. But these 3 Reynolds numbers are not same, as these are not defined based on the same parameter ok. So, next let us consider another problem for the flow near the wake of a flat plate ok.

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Example Problems

Air at 20°C and 1 atm flows at 1 m/s past a slender 2D body of length L=30 cm, whose drag coefficient is 0.05 based on 'plan' area boundary layer. Assuming laminar wake flow at a point 3.5 m downstream of the trailing edge, estimate (i) maximum wake velocity defect and (ii) the 1% wake thickness. (ρ=1.2 kg/m^3 and μ=1.8×10^{-5} kg/m-s)

$$\frac{u_1}{U_0} = C_D \left(\frac{Re_L}{16\pi}\right)^{1/2} \left(\frac{L}{x}\right)^{1/2} \exp\left(-\frac{U_0 y^2}{4x\nu}\right)$$

where U_0 = free-stream velocity, u_1 = velocity defect, y = distance from the axis of symmetry and $Re_L = \frac{U_0 L}{\nu}$ is the body Reynolds number.

So, let us read out the problem. Air at 20 degree centigrade and 1 atmosphere flows at 1 meter per second past a slender 2-dimensional body of length L is called to 30 centimeter, whose drag coefficient is 0.05 based on 'plan' area boundary layer. Assuming laminar wake flow at a point 3.5 meter downstream of the trailing edge estimate maximum wake velocity defect and the 1 percent wake thickness, density and viscosity are given.

And the expression of this velocity defect is given, where U naught is the free-stream velocity, u 1 is the velocity defect, y is the distance from the axis of the symmetry and Reynolds number based on U naught L by nu is the body Reynolds number.

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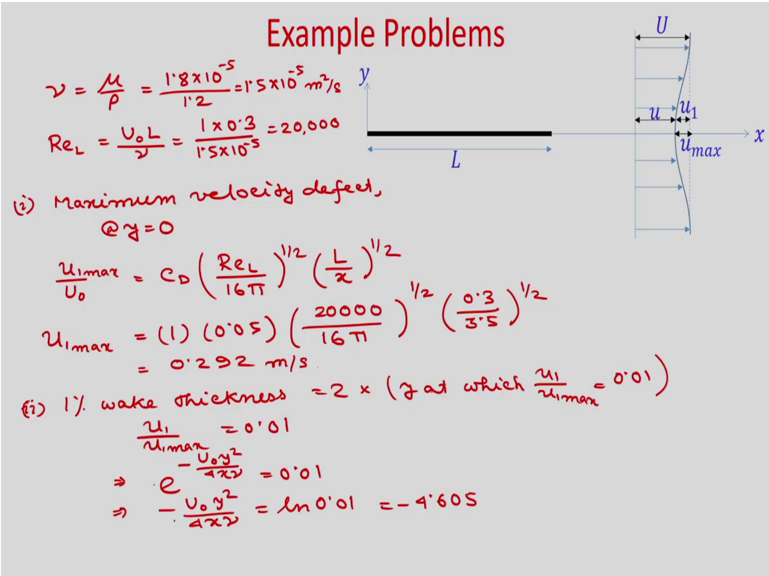

So, you can see this is the problem, where length of the plate is given and velocity defect u 1 expression is given. So, we can see that mu is given. So, nu we can calculate as mu by rho is equal to 1.8 into 10 to the power minus 5 divided by 1.2. So, it will be 1.5 into 10 to the power minus 5 meter square per second ok.

And Reynolds number based on U naught L divided by nu, we can write as 1 into 0.3 divided by 1.5 into 10 to the power minus 5 as 20000 ok. So, now we can calculate the maximum velocity defect. So, where we will get this maximum velocity defect? At y is equal to 0.

So, we have the expression of u 1 by U naught, if you put, so you can see this is the expression. The Reynolds number we have calculated and if you put at y is equal to 0, then this term will become 1. So, you can write u 1 by U naught; that means, u 1 max as C D Reynolds number divided by 16 pi to the power half L by x to the power half.

So, from this expression, u 1 max we can write as; so, U naught is 1, 1, then C D is 0.05, Reynolds number 20000 divided by 16 pi to the power half, then L by x, it will be 0.3 and at length 3.5, we need to calculate. So, 0.3 by 3.5 to the power half. So, if you calculate it, you will get u 1 max as 0.292 meter per second.

So, now we need to calculate the wake thickness. So, we can calculate as 1 percent wake thickness is 2 into y at which u 1 by u 1 max is 0.01 ok. So, you can see that u 1 by u 1 max is 0.01. So, u 1 by 1 max, we know that it will be exponential minus U naught y square by 4 x nu will be 0.01. So, it will be minus U naught y square by 4 x nu is equal to ln 0.01 and it will get you will get minus 4.605 ok.

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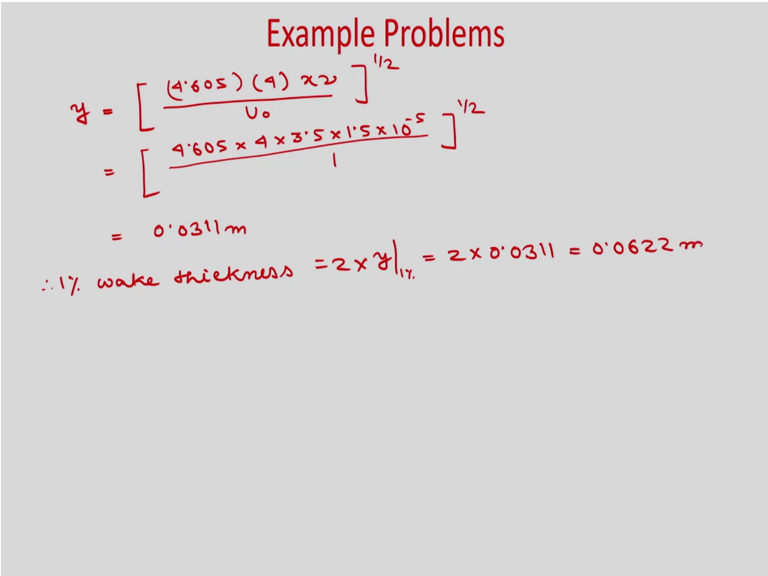

So, from here, if you express y, then we will get as 4.605 4 x nu divided by U naught to the power half. So, now, you put the values. So, you will get 4.605 into 4 into x as 3.5 nu 1.5 into 10 to the power minus 5 and U naught is 1 to the power half. So, this you will get as 0.0311 meter. So, 1 percent wake thickness will be. So, both side you have to consider. So, you will get 2 into y, where u 1 by u 1 max become 1 percent. So, 2 into 0.0311 and that will be 0.0622 meter.

So, in today's class, we considered the free shear layers for the flow behind a wake of the flat plate of finite length.

Here, we define the velocity defect as the difference of free stream velocity and the velocity u in the wake, then we considered the boundary layer equation and this boundary layer equation with the simplification, we have written in terms of the velocity defect. So, with the proper bounding condition, it can be solved and we wrote the velocity defect expression, where one unknown parameter was there, that is constant A.

Then, we equated the drag of the wake with the momentum deficit at any location x and from there, we found the value of this constant A and from there, we expressed the velocity distribution u in terms of x and y. Then, we have shown the velocity profile at different x location and obviously, you can see that this velocity varies exponentially in the y direction. Later, we solved two example problems from this free shear jet as well the wake behind the flat plate.

Thank you.