Viscous Fluid Flow
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Module - 09
Laminar Free Shear Flows
Lecture - 01
Two-dimensional Laminar Jet

Hello everyone. So, till now we have considered wall bounded flows where we have considered boundary layer flow over a flat plate, then flow over a circular cylinder. But today, we will consider Free Shear Flows and also we will assume laminar and incompressible fluid flow.

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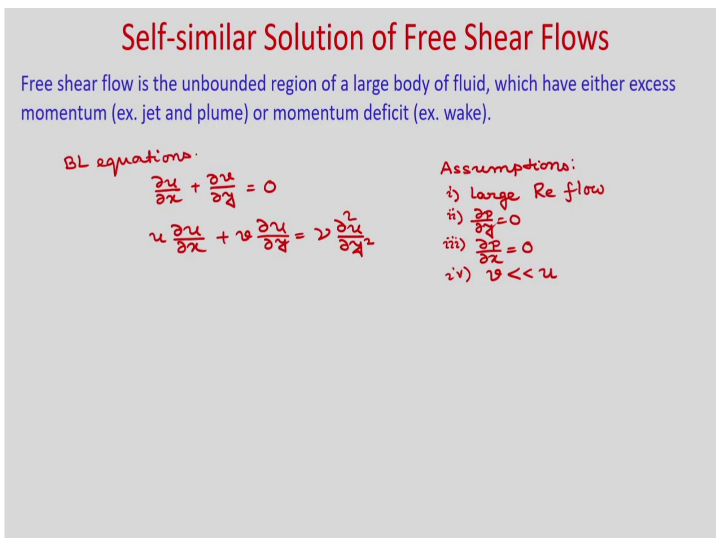

Free shear flow is the unbounded region of a large body of fluid which have either excess momentum examples are jet and plume or momentum deficit like wake. So, whatever governing equation we have derive for laminar boundary layer flows, if dominant velocity is in the x direction then those equations are valid for free shear flows. So, let us write down the governing equations.

So, we can have del u by del x plus del b by del y is equal to 0. So, this is the continuity equation and the boundary layer equation we can have as u del u by del x plus v del u by del y is equal to nu del 2 u by del y square. So; obviously, we are considering that dominant free shear velocity is u in the x direction. So, obviously, if you can see that these are the assumptions we have made.

So, one is large Reynolds number flows ok then boundary layer theory is valid and no separation then we are having del p by del y is equal to 0 and also it is open to the atmosphere and we can have the assumptions of del p by del x is equal to 0 and also we have v velocity is much much smaller than u so; obviously. So, these are the boundary layer equations.

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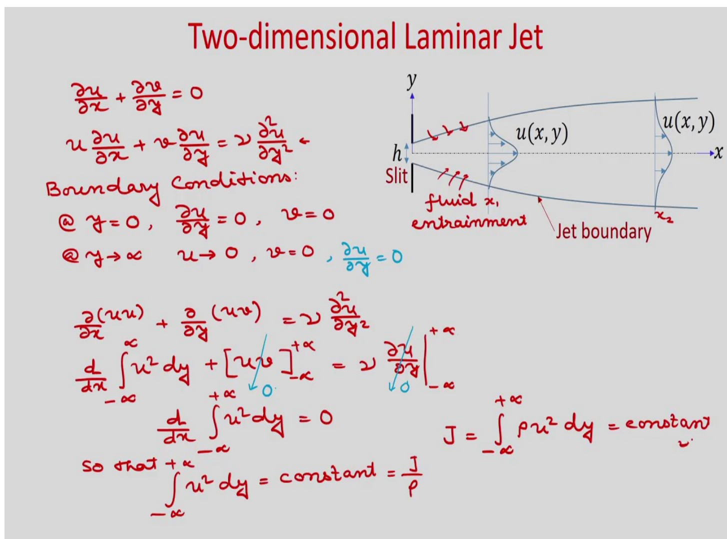

So, now we will consider two dimensional laminar jet and we will use similarity transformation technique to find the velocity distribution in the two dimensional laminar jet. So, consider this is the laminar jet and this is the small opening of height h and this is known as slit. When at high velocity this fluid will enter. So, there will be formation of jet and there will be center line.

So, you can see that this is the jet boundary and there will be some fluid entrainment from here fluid entrainment from the jet boundary. So, physically jet spreads in outward direction with increasing x going to momentum diffusion due to viscosity and the velocity in the center line will decrease. So, you can see that we have considered two x location say let us say this is your location x 1 and location x 2. So, center line velocity will decrease as well as it will spread in outward direction.

So, if we neglect the fluid entrainment from the jet boundary then we can solve these governing equations easily. So, we have the governing equation continuity equation del u by del x plus del v by del y is equal to 0 and u del u by del x plus v del u by del y is equal to nu del 2 u by del y square. So, what are the boundary conditions for this case?

So, we are taking the axial direction is x along the center line and it is symmetric about the center line and this is the y direction. So, you can see that boundary conditions we have at y is equal to 0 we will have maximum velocity; that means, it is symmetric line. So, we can have del u by del y is equal to 0 as well as velocity v is 0 and as y tends to infinity; that means, far away you can see that velocity will be tending to 0.

So, u velocity will be tending to 0. This equation if you write in conservative form then we can write del of del x u u plus del of del y u v is equal to nu del 2 u by del y square. So, this is the conservative form of this momentum equation now if you integrate it from y is equal to minus infinity to y is equal to plus infinity at a particular x location then we can write as d of dx.

So, if you use Leibnitz theorem and you can convert it to minus infinity to plus infinity u square dy and this if you integrate then it will be just u v minus infinity to plus infinity and is equal to nu del u by del y minus infinity to plus infinity. So, you can see that at y tends to infinity v also is 0. So, you can see that as y plus minus infinity will have u v are 0.

So obviously, this term will become 0. So, this is 0 ok and you can see that at y tends to infinity obviously, we will have del u by del y also 0 because u is 0 right there will be no velocity gradient outside this z boundary. So, del u by del y also will be 0. So, this is also 0. So, from here you can see that we can write d of dx integral minus infinity to plus infinity u square dy is equal to 0.

So, if you integrate it. So, you can see that we can write so, that minus infinity to plus infinity u square dy is equal to constant ok constant and if you see the total flux of x momentum integrated on y at any location x, then we can get J is equal to integral minus infinity to plus infinity rho u square dy. So, you can see that for incompressible flow rho is constant.

So, you can take it outside. So, you can see that it will become J by rho; that means, J which is your momentum flux at any location x is constant. So, this is one important property ok. So, this is one important property where in the solution we will use that J which is your momentum flux in at any location x.

So, J, J is the momentum flux you can see that it is independent of x. So, it is constant at any station of x location. So, this is constant. So, we will use the similarity transformation technique to find the solution of this jet flow.

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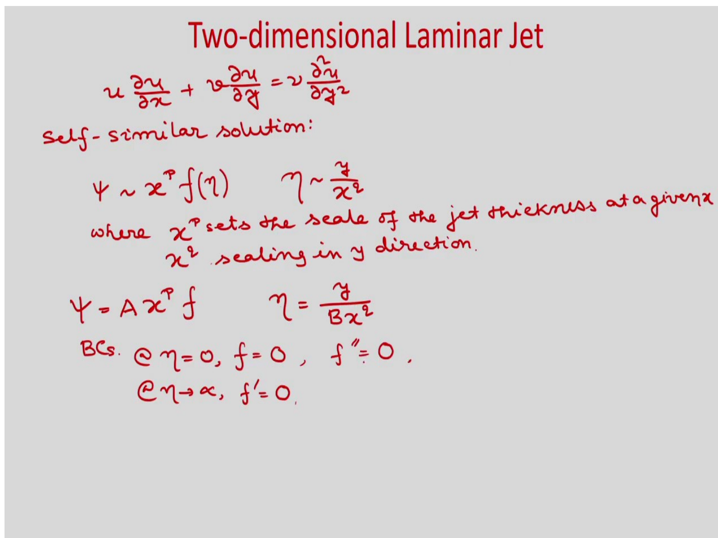

So, we will use the governing equation as u del u by del x plus v del u by del y is equal to nu del 2 u by del y square. So, we will use the self-similar solution. So, when can we use self-similar solution? So, if you see that any location x whatever velocity distribution you will get that those are varying only with a scale factor ok. So, if you use some suitable transformation variable then all the velocity profiles will fall in single curve.

So, that is the self-similar solution. So, for this we can have the self-similar solution and we will use this similarity variable psi as varies as x to the power p f eta where eta is the similarity variable y by x to the power q and where x to the power p sets the scale of the jet thickness at a given x and x to the power q is the scaling in the y direction.

So, we can use that psi is equal to some constant A x to the power p f which is function of eta and eta is y by B into x to the power q. So, A B we will be choose in such a way that f and eta will be dimensionless. So, what are the boundary condition in terms of f? So, you can see that boundary conditions at eta is equal to 0 so; obviously, at eta is equal to 0 we can say that psi will be 0 and so, at y is equal to 0. So, one y is equal to 0.

So, we can have the symmetry line and psi will be constant. So, f will be 0 ok. So, for convenience we are taking this and f double prime will be also 0 ok at eta is equal to 0 at eta tends to infinity.

So, velocity will be 0. So, f prime will be 0. So, f double prime 0; that means, the velocity gradient del u by del y will be 0 because we are having maximum velocity at the center line. So, now we have defined the stream function psi and the similarity variable eta now let us use the momentum equation and right in terms of the stream function psi.

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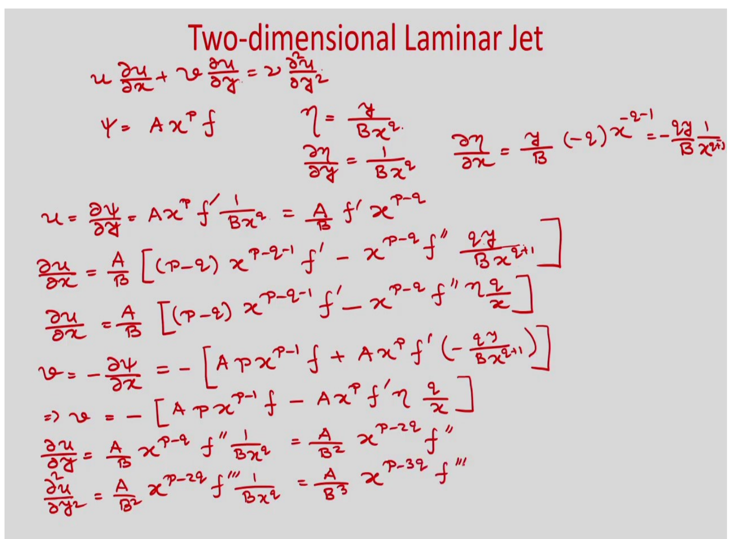

So, we have the momentum equation u del u by del x plus v del u by del y is equal to nu del 2 u by del y square and we have defined psi as A x to the power p f where f is function of eta and eta is y by B x to the power q. So, from here you can see you can write del eta by del y is equal to 1 by B x to the power q and del eta by del x we can write as y by B.

So, x to the power minus q. So, minus q and x to the power minus q minus 1. So, you can write it as minus q y by B 1 by x to the power q plus 1 ok. Now let us find the velocity u as del psi by del y. So, from here you can see that we can write A x to the power p del f by del eta. So, you can write f prime and del eta by del y. So, del eta by del y is 1 by B x q.

Then we can write del u by del x. So, you can see this is your u. So, you can write del u by del x. So, A by B you can take it outside the bracket. So, this we can write as A by B f prime x to the power p minus q. So, del u by del x if we write then it will be p minus q x to the power p minus q minus 1, f prime then we can have minus x to the power p minus q f double prime then del eta by del x.

So, del eta by del x is minus q y by B 1 by x to the power q plus 1. So, the that is why we have written minus sign here then we can write q y by B x to the power q plus 1. So, if you simplify it then we will get del u by del x as A by B p minus q x to the power p minus q minus 1 f prime minus. So, you can see here y by B x to the power q we can write as eta.

So, we can write minus x to the power p minus q f double prime eta q by x. Now we can write v, v as minus del psi by del x ok. So, v is equal to minus del psi by del x. So, from here you can see that we can write minus. So, del psi by del x. So, it will be A p x to the power p minus 1 f then plus A x to the power p f prime then del eta by del x.

So, del eta by del x is this one minus q y by B x to the power q plus 1. So, if you simplify it you will get v is equal to minus A p x to the power p minus 1 f minus A x to the power p f prime eta q by x then let us find del u by del y. So, del u by del y ok. So, this is your u. So, from here del u by del y if you find.

So, it will be A by B x to the power p minus q f double prime and del eta by del y del eta by del y is 1 by B x to the power q. So, if you simplify it you will get A by B square x to the power p minus twice q f double prime and now let us find this one. So, del 2 u by del y square is equal to. So, you take derivative with respect to y. So, it will be A by B square x to the power p minus twice q f triple prime.

Then del eta by del y; that means, 1 by B x to the power q. So, you will get A by B cube x to the power p minus 3 q f triple prime. Now we have evaluated the velocities u v and the other gradient. So, now, let us put it in the momentum equation.

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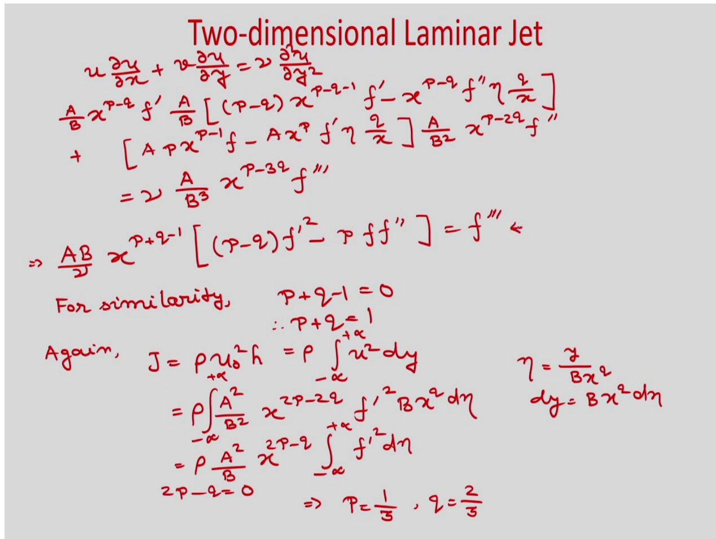

So, we have equation u del u by del x plus v del u by del y is equal to nu del 2 u by del y square. So, now, let us put all the values. So, if you put it we have written u as A by B x to the power p minus q f prime del u by del x A by B p minus q x to the power p minus q minus 1 f prime minus x to the power p minus q f double prime eta q by x.

Then we have v. So, v as A p x to the power p minus 1 f minus A x to the power p f prime eta q by x then we have del u by del y. So, it will be A by B square x to the power p minus twice q f double prime and right hand side we have del 2 u by del y square. So, it will be A by B cube x to the power p minus 3 q f triple prime. If you simplify it you will get AB by nu x to the power p plus q minus 1 p minus q f prime square minus p f f double prime is equal to f triple prime.

So, if you see this equation. So, we use the self-similar solution and we assume that self-similar solution exist for this flow then; obviously, this equation should be independent of x and to make it independent of x you can see the this power p plus q minus 1 must be 0 ok. So, for similarity p plus q minus 1 should be 0 that means, p plus q is equal to 1 and also we know J is equal to rho u naught square h where u naught is the inlet velocity is equal to rho integral minus infinity to plus infinity u square dy.

So, J is equal to we can write u square u we know this is minus infinity to plus infinity A square by B square x to the power twice p minus twice q f prime square dy. So, dy. So, we have eta is equal to y by B x to the power q. So, dy you can write B x to the power q d eta. So, it will be B x to the power q d eta.

So, if you see this we can write it as rho A square by B x to the power twice p minus q this we can take it outside because we are integrating with respect to eta it will be minus infinity to plus infinity f prime square d eta. So, this J as we have shown that it will be constant right. So, to make it constant this power of x must be 0.

So, you can see that it will be constant for all x if twice p minus q is 0. So; that means, twice p minus q is equal to 0 ok. So, we have p plus q is equal to 1 and twice p minus q is equal to 0 that will give you p is equal to 1 by3 and q is equal to 2 by 3.

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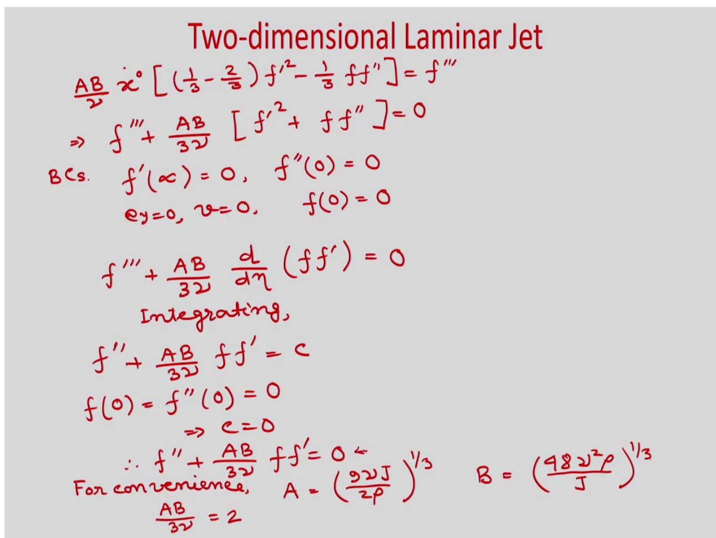

So, now, if you put p and q value in this equation then we will get AB by nu. So, x to the power 0 1 by 3 minus 2 by 3 f prime square minus 1 by 3 f f double prime is equal to f triple prime. So, further if you simplify we will get f triple prime plus AB by 3 nu f prime square plus f f double prime is equal to 0.

So, with the boundary conditions f prime at infinity ok that is the velocity is 0 and f double prime; that means, the velocity gradient at center line is 0 and v is also 0 right at y is equal to 0 v is equal to 0. So, from here you will get that the stream function will be constant at the centre line. So, and that we are making as 0.

So, f 0 is 0. So, now, you can see that this equation we can rewrite as f triple prime plus AB by 3 nu d of d eta f f prime is equal to 0 ok. If we integrate it we can write after integrating f double prime plus AB by 3 nu f f prime is equal to constant ok. So, we have the boundary conditions that f at eta is equal to 0 and f double prime at eta is equal to 0 is 0. So, that will give c is equal to 0.

So, we will get the equation finally, f double prime plus AB by 3 nu f f prime is equal to 0. So, now, you can see that in this equation these AB are present and to have this f and eta as dimensionless we have to choose the value of AB such way that f and eta will become dimensionless.

So, for convenience let us choose the value of A as 9 nu J divided by 2 rho to the power 1 by 3 and B as 48 nu square rho divided by J to the power 1 by 3 ok. So, if you choose that then the solution will also become easy. So, from here you can see AB by 3 nu if you calculate it will become 2 ok.

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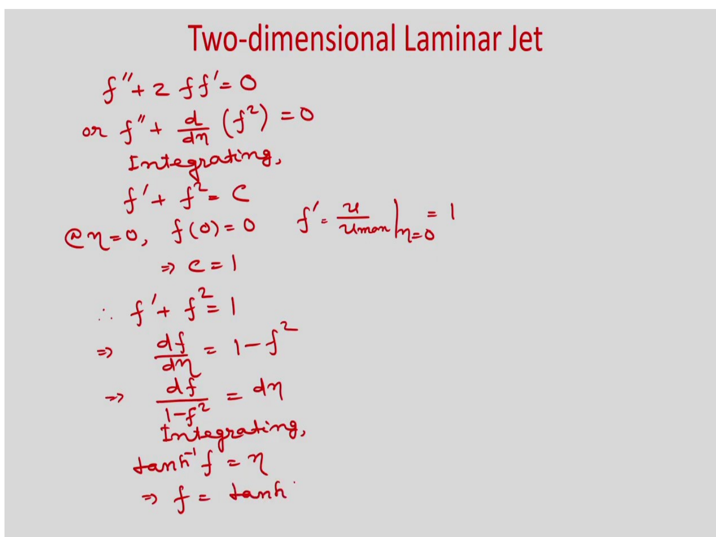

So, the equation this equation now, will become f double prime plus twice f f prime is equal to 0. Again we can write f double prime plus d of d eta f square is equal to 0 this we can rewrite like this. So, now integrating we will get f prime plus f square is equal to constant.

So, you can see at eta is equal to 0 we have f 0 is 0 and f prime we are defining as u by u max at eta is equal to 0 ok. So, f prime at eta is equal to 0 ok because maximum velocity will occur at center line. So, this will be 1 so that means, c will become 1. So, we can write f prime plus f square is equal to 1 we can write then this is df by d eta. So, d f by d eta is equal to 1 minus f square.

So, you can write d f divided by 1 minus f square is equal to d eta. So, integrating we will get tan hyperbolic inverse f is equal to eta ok so that means, f will become tan hyperbolic eta. So, what is the physical significance of f? So, f is having the physical significance of stream function and f prime is the velocity.

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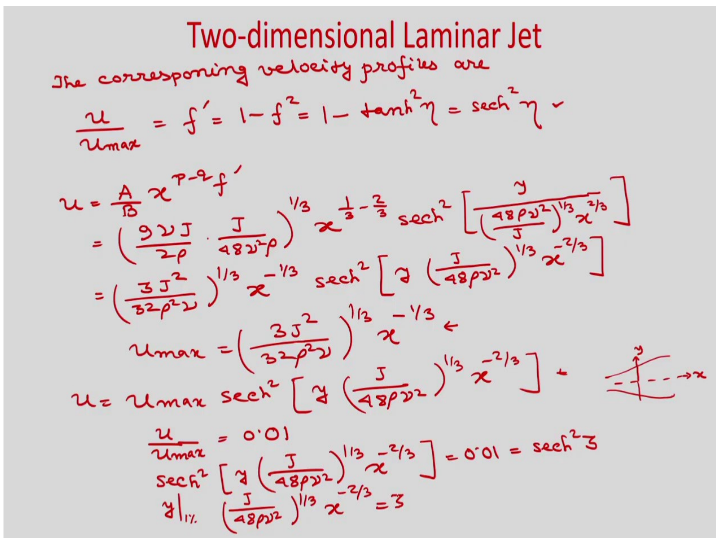

So, now we have the solution f prime is equal to tan hyperbolic eta now, we can write the corresponding velocity profiles corresponding velocity profiles are u ok by u max ok u by u max is f prime and f prime you have seen that it is 1 minus f square and 1 minus f square. So, f is already we have written in terms of eta. So, 1 minus tan hyperbolic square eta; that means, sec hyperbolic square eta.

So, you can see that this will give the this velocity u which is function of x and y. Now velocity u already we have written as A by B x to the power p minus q f prime and p q A B value already we have assumed. So, you can write 9 nu J divided by twice rho and v as 48 nu square rho by J. So, you can write this and to the power 1 by 3 x to the power 1 by

3 minus 2 by 3 and we have sec hyperbolic square y by 48 rho nu square divided by J to the power 1 by 3 x to the power 2 by 3.

So, this is the eta, eta we have written as y by B x to the power q ok. So, if you simplify it. So, you will get 3 j square divided by 32 rho square nu to the power 1 by 3 x to the power minus 1 by 3 sec hyperbolic square y J divided by 48 rho nu square to the power 1 by 3 x to the power minus 2 by 3.

So, if we represent this as u max. So, we can write u max is equal to 3 J square by 32 rho square nu to the power 1 by 3 x to the power minus 1 by 3. So, you can see that u max if you represent like this. So, you can write u is equal to u max sec hyperbolic square y J by 48 rho nu square to the power 1 by 3 x to the power minus 2 by 3 ok.

So, you can see this is nothing but sec hyperbolic eta square and this from this equation you can see that u is equal to u max sec hyperbolic square eta and that we have written and u max expression is this. So, now, let us find the width of the jet. So, we will represent the thickness of this width as the velocity near to the jet boundary it becomes one percent of the center line velocity u max.

So, if you define in that way, then you can see from here your u max ok u max you can see it will decay right with x to the power minus 1 by 3 right. So, the maximum velocity decays like x to the power minus 1 by 3. So, from here you can see that u by u max will become 1 percent right. So, it will be 0.01 ok. So, from here we can write sec hyperbolic square y J by 48 rho nu square to the power 1 by 3 x to the power minus 2 by 3.

So, this will become 0.01 and this is sec hyperbolic square 3. So, from here you can see that y location. So, we are considering this is the center line. So, this is the jet boundary. So, in one way we are finding this y.

So, both way it will be just multiplied by 2 ok. So, when we are finding the thickness of the jet. So, you can see from here you can write y where u is becoming 1 percent of the u max, J divided by 48 rho nu square to the power 1 by 3 x to the power minus 2 by 3 is equal to 3 ok.

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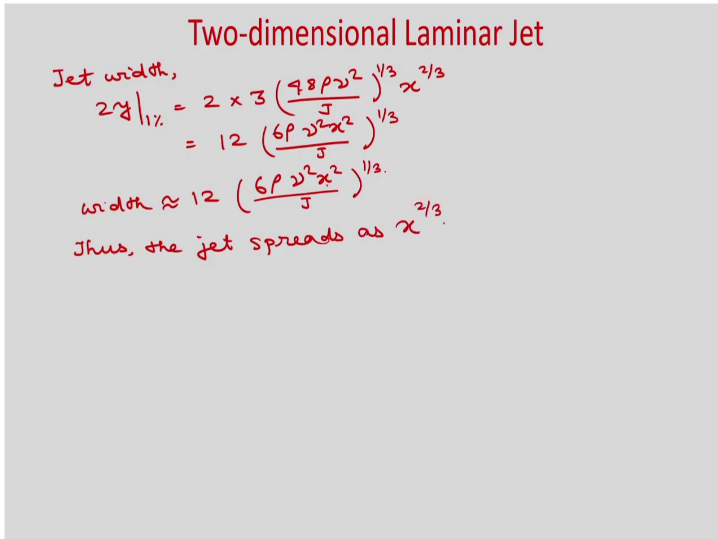

So, from here you can find the jet width as 2y where u is becoming 1 percent of the u max as 2 into 3 48 rho nu square divided by j to the power 1 by 3 x to the power 2 by 3 ok. So, this will become 12 rho nu square x square by J to the power 1 by 3 ok. So, we can see the width will we be 12 into 6 rho nu square x square by J to the power 1 by 3.

So, you can see that this width with x it will spread as x to the power 2 by 3. So, thus the jet spreads as x to the power 2 by 3. So, now, we are interested to find the volume flow rate at any location x ok. So, that just we need to integrate this velocity from y minus infinity to infinity.

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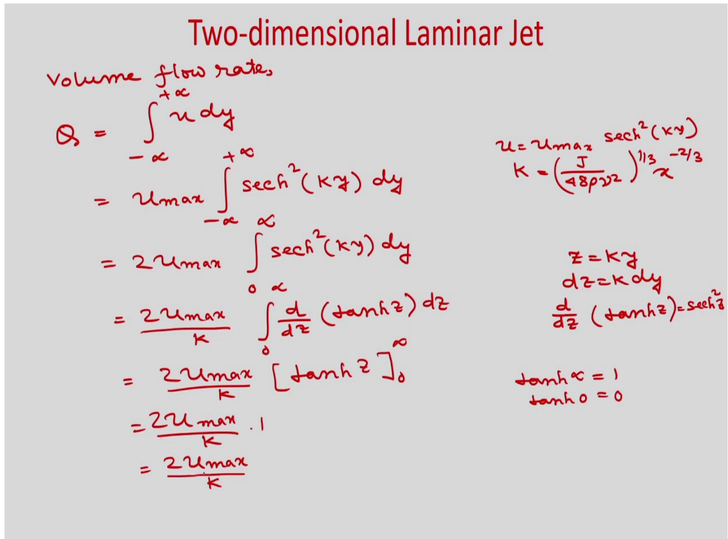

So, we can find the volume flow rate ok per unit width we are considering, because we are considering two dimensional case, so, per unit width -if you consider them Q will become integral minus infinity to plus infinity u dy and u we have written as u max integral minus infinity to plus infinity sec hyperbolic square let us write ky dy where we know that u is equal to u max sec hyperbolic square ky where k is equal to j by 48 rho nu square to the power 1 by 3 x to the power minus 2 by 3 ok.

So, this we can write as 2 u max integral 0 to infinity ok sec hyperbolic square ky dy. Now we can write z is equal to ky. So, dz will become k dy. So, and we can write d of dz of tan hyperbolic z ok. So, this is nothing but sec hyperbolic square z. So, this now we can write as twice u max divided by k integral 0 to infinity d of dz tan hyperbolic z and dz.

So, this dz is equal to k dy. So, we have divided k here and we can write twice u max by k tan hyperbolic z from 0 to infinity. So, you can see twice u max divided by k. So, tan hyperbolic infinity ok tan hyperbolic infinity is 1 and tan hyperbolic 0 is 0. So, from here you can see that it will become 1. So, we can write as twice u max by k.

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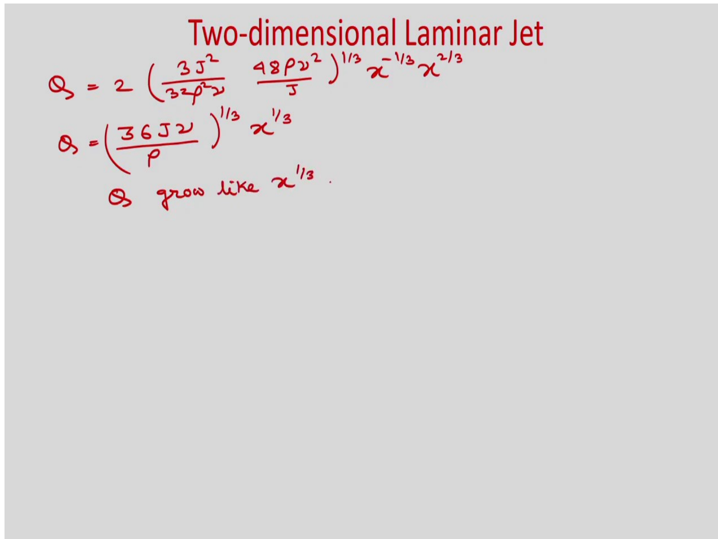

So, now if you put the value of k and u max we can write the volume flow rate as 2 3J square divided by 32 rho square nu 48 rho nu square by J to the power 1 by 3 x to the power minus 1 by 3 and x to the power 2 by 3. So, you can write Q as 36 J nu divided by rho to the power 1 by 3 x to the power 1 by 3.

So, from this expression you can see that volume flow rate increases in the axial direction x with x to the power 1 by 3 so; that means, Q grow like x to the power 1 by 3. So, in today's lecture we considered laminar two dimensional jet where in z direction there is no variation of velocity and any gradient of any property is 0. So, x is the axial direction and at x equal to 0 we have a high velocity from the slit of height h.

So, this uniform velocity is considered as u naught and when it will pass through this jet obviously, due to momentum diffusion this jet width will increase in the axial direction, but we have shown one important property here that the momentum flux ok becomes constant along x ok.

So, from using that property we have found the velocity distribution of the jet where u becomes almost 1 percent of the center line velocity u max and then finally, we have calculated the volume product per unit width.

Thank you.