Viscous Fluid Flow

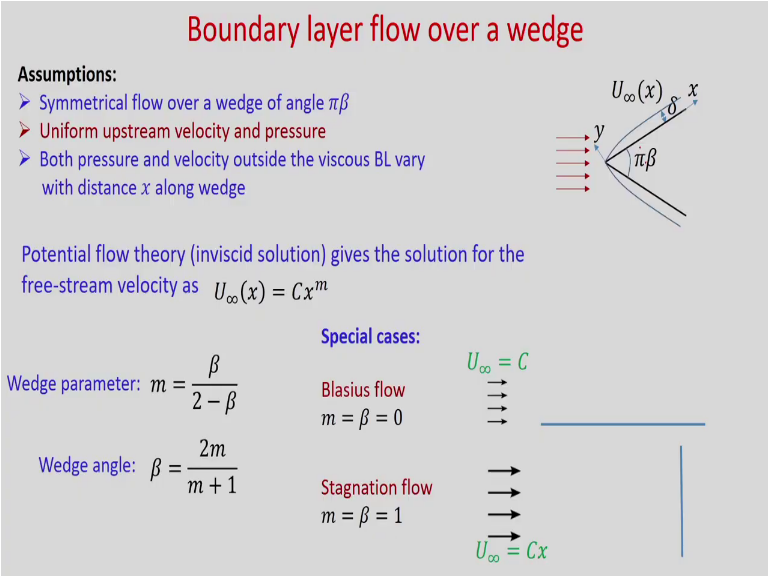
Prof. Amaresh Dalal
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

Module - 07
Laminar Boundary Layers - 1
Lecture - 04
Falkner-Skan equation: Boundary layer flow over a wedge

Hello everyone. So, we have already derived the Blasius equation, where we have considered flow over a flat plate and we know that the velocity outside the boundary layer is constant; that means free stream velocity is constant.

Today we will consider another situation, where this free stream velocity outside the boundary layer is not constant; it is function of axial direction. So, in today's class, we will consider Boundary layer flow over a wedge, where velocity U infinity is function of axial direction outside the boundary layer.

(Refer Slide Time: 01:22)



So, let us consider this wedge of angle pi beta, where you have velocity here uniform velocity. When it will pass over this wedge, so boundary layer will form from the leading edge of this wedge plate; x is the axial direction, y is the normal to this surface.

So, this boundary layer thickness delta is function of x. If you consider the velocity outside this boundary layer, so we will have U infinity which is no longer constant, it is function of the axial direction x. In this particular case you can see that, this free stream velocity U infinity will increase with axial direction x.

These are the assumptions we will consider for boundary layer flow over a wedge. We will consider symmetrical flow over wedge of angle pi beta, uniform upstream velocity and pressure, both pressure and velocity outside the viscous boundary layer vary with distance x along the wedge.

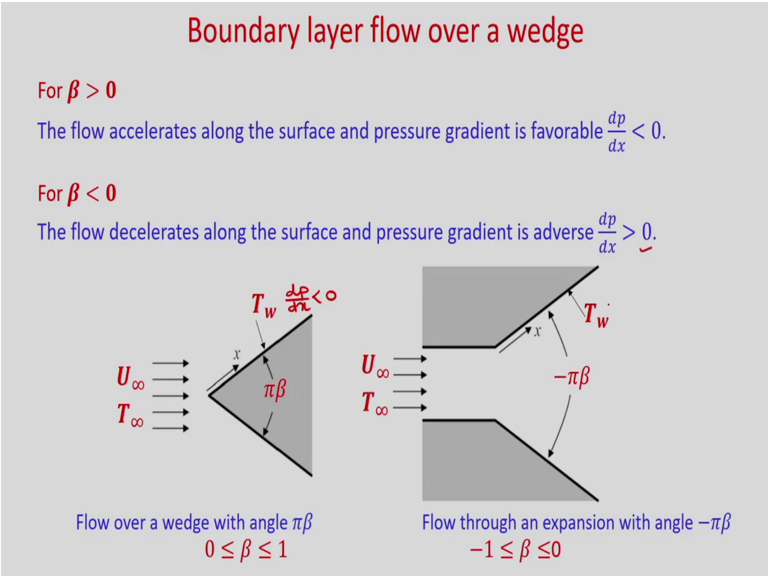
So, potential flow theory which is the inviscid solution gives the solution for the free stream velocity as U infinity is equal to Cx to the power m. So, from the fluid mechanics knowledge you know that from the potential flow theory that, this velocity U infinity will be some constant Cx to the power m, where m is known as wedge parameter, ok. So, the relation between wedge parameter and beta which is your wedge angle is given here.

So, m is equal to beta by 2 minus beta, and wedge angle you can express in terms of wedge parameter as 2 m by m plus 1. Now, let us consider two special cases; first case is that beta is equal to 0, ok. So, wedge angle is 0. So, if you put beta is equal to 0 here, so wedge parameter also will be 0. And in this particular case, obviously you can see that if beta becomes 0, it will be a flat plate. And what will be the free stream velocity in this case, as m is equal to 0, from this relation you will get U infinity as constant C.

So, you can see for beta is equal to 0, it becomes Blasius flow, flow over a flat plate. Another case you consider where beta is equal to 1. So, if beta is equal to 1, here you can see that m will become 1; and if beta is equal to 1, so that means this will be pi angle, right. So, that means it will be a vertical flat plate. So, in this case if m is equal to 1, then velocity will become C x, so U infinity it will be C x.

So, if this flow situation arises, where flow is coming and hitting this vertical plate, this is known as stagnation flow. So, depending on the wedge angle, we will get different shape, whether it is positive or negative.

(Refer Slide Time: 05:02)



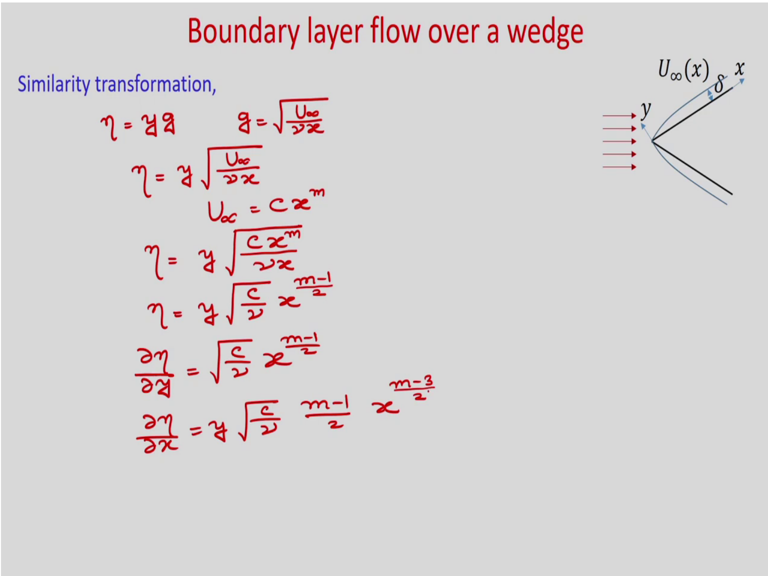
So, you can see here for beta greater than 0. So, if beta greater than 0; so this will be the situation, where situation is flow over a wedge with angle pi beta.

And beta varies between 0 and 1. So, in this particular case, you can see as flow takes over this wedge, then flow accelerates along the surface. Because if you see two different x location, then obviously you can see that due to the decrease in the flow area, flow will accelerate.

And pressure gradient is favorable; that means dp by dx will be less than 0 in this particular case. For beta less than 0 ok, so it will become flow through an expansion with angle minus pi beta, ok. As beta is less than 0 and beta varies between minus 1 and 0 and this will become flow through an expansion.

So, you can see here that, the flow decelerates along the surface and pressure gradient is adverse, so dp by dx will be greater than 0 in this situation. Now, we will use the similarity transformation and find the ordinary differential equation from this boundary layer equations for flow over a wedge. We will consider the same similarity parameter eta which we considered for flow over flat plate.

(Refer Slide Time: 06:41)

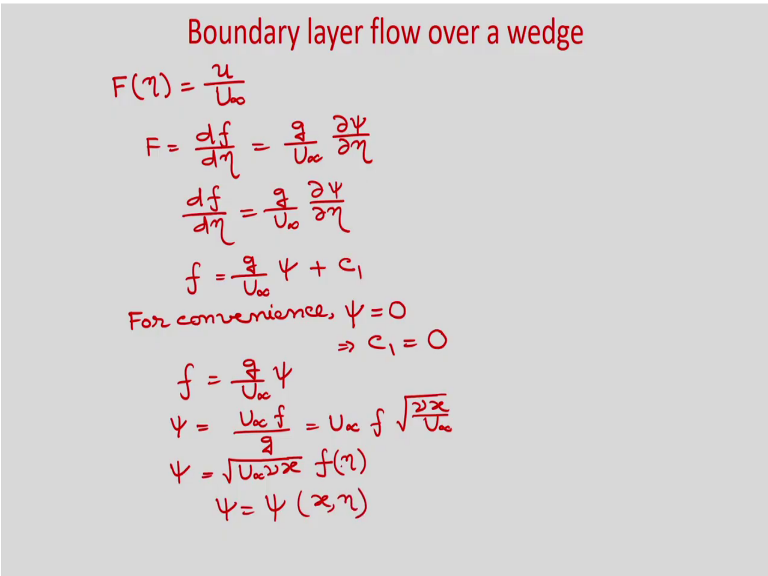


So, we will consider eta is equal to yg, where g is function of x and we have already derived that g is equal to root U infinity by nu x, ok. So, that means eta will be y root U infinity by nu x. In this case, U infinity is Cx to the power m, ok. So, if you put here, then you will get eta is equal to y root. So, U infinity we are putting Cx to the power m divided by nu x.

So, we will get eta is equal to y C by nu. So, nu is kinematic viscosity. So, we assume it constant, C is constant and we will get x to the power. So, m minus 1 divided by 2, because we have a root here, so m minus 1 divided by 2. So, now, we can calculate del eta by del y.

So, it will be root C by nu x to the power m minus 1 by 2 and del eta by del x. So, from here you can see, it will be y root C by nu m minus 1 by 2 x to the power. So, m minus 1 by 2 minus 1, so it will become m minus 3 by 2. We also introduced another similarity variable F, which is function of eta as the ratio of u by U infinity.

(Refer Slide Time: 08:38)



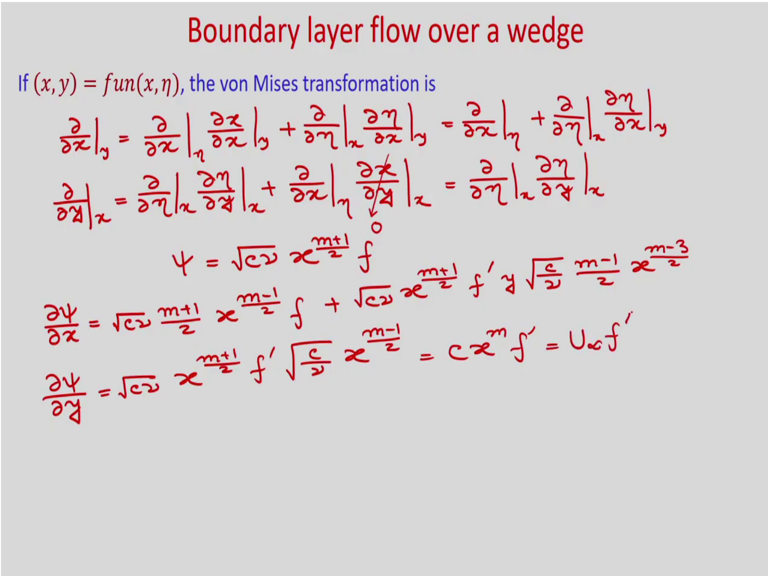
So, F is equal to u by U infinity. So, U infinity is the free stream velocity and we also showed earlier that, F is equal to df by d eta is equal to g by U infinity del psi by del eta, ok.

So, now you can see that, we can write as df by d eta is equal to g by U infinity del psi by del eta. So, now, if you integrate it, so integrate both side with respect to eta. So, what you will get? If you integrate it, you will get f is equal to g and U infinity is function of x. So, as we are integrating with respect to eta, so you can keep it outside. So, you will get psi plus some constant C 1.

So, we can assume for convenience that, the stream function on the surface of the wedge is 0, ok. So, in that case for convenience, let us consider that stream function value on the surface of the wedge as 0 ok, psi is equal to 0. So, obviously, f will be also 0. So, C 1 will be 0, ok. So, we can write f is equal to g by U infinity psi, psi equal to U infinity f by g, ok.

So, we know the value of g right, so U infinity f. So, we can write this as nu x by U infinity, ok. So, from here we can write stream function psi as root U infinity nu x f, ok. Now, f is function of eta and this is also function of x; so that means psi is function of x and eta, ok. So, we have expressed this psi as function of x and eta, because f is function of eta.

(Refer Slide Time: 11:20)



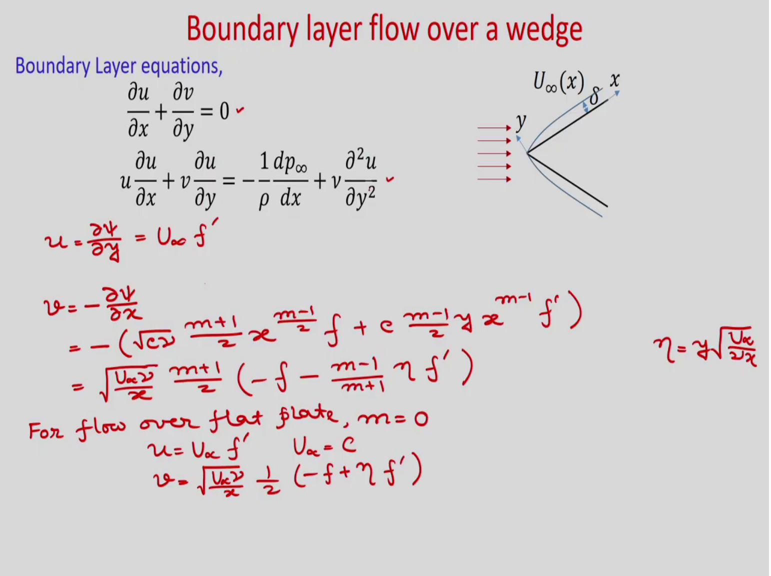
So, now we will use Von Mises transformation to find the derivative. So, if x, y is function of x eta; then the Von Mises transformation we can write as del of del x keeping y constant as del of del x keeping eta constant, then del x by del x keeping y constant plus del of del eta x del eta by del x y, ok. So, you can see this you can write this will be one, so it will be del of del x eta plus del of del eta x del eta by del x y.

Similarly, you can write del of del y keeping x constant as del of del eta x del eta by del y x plus del of del x eta and del x by del y x. So, here you can see that, this will become 0; because we are taking the derivative of x with respect to y, so this will become 0. So, this you can simplify and write as del of del eta x del eta by del y x, ok. And we have already written psi as root C nu x to the power m plus 1 by 2 f, where f is function of eta.

So, now we can write using the Von Mises transformation del psi by del x, ok. So, you can see, now it will be root C nu m plus 1 by 2 x to the power m minus 1 by 2 f. So, del of del x we have written; now this one if you write, so plus root C nu x to the power m plus 1 by 2 f prime y C by nu m minus 1 by 2 x to the power m minus 3 by 2.

So, because we have written del eta by del x in the previous slide, so that we have written here. Similarly, del psi by del y you can write as this del of del eta x del eta del y. So, this if we write C nu x to the power m plus 1 by 2 f prime root C by nu x to the power m minus 1 by 2; so this you can write Cx to the power m f prime after simplification and Cx to the power m is nothing, but the free stream velocity U infinity f prime.

(Refer Slide Time: 14:45)



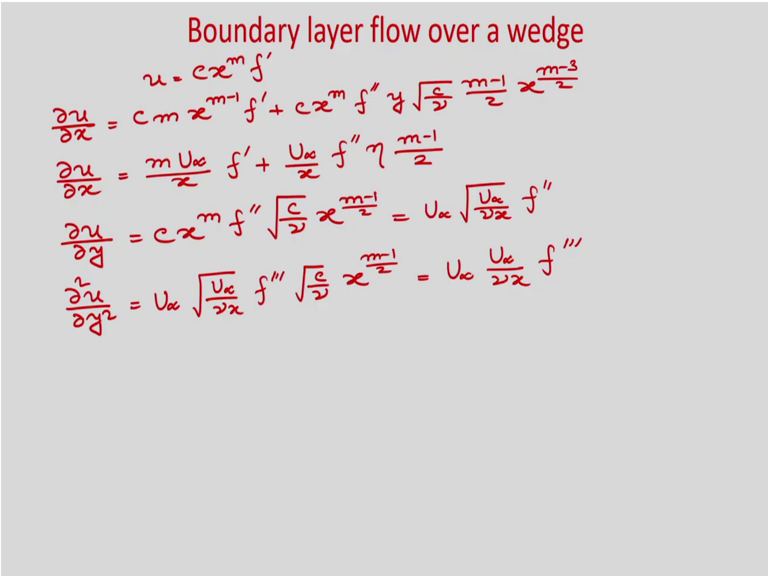
Now, let us consider the boundary layer equation with the pressure gradient term; because we are considering boundary layer flow over a wedge, so we should keep the pressure gradient term here. And let us write these in terms of the stream function. So, you can see this is the continuity equation and this is the boundary layer equation. So, we can write u is equal to del psi by del y, ok.

So, del psi by del y already we have written as U infinity f prime, ok. And similarly v we can write minus del psi by del x. So, we can write as minus root C nu, del psi by del x we have written in last slide m plus 1 by 2 x to the power m minus 1 by 2 f plus C m minus 1 by 2 y x to the power m minus 1 f prime. So, this we can write as root U infinity nu by x m plus 1 by 2 minus f minus m minus 1 by m plus 1.

And we can write in terms of eta, because eta we know that it is y root U infinity by nu x. So, here you will get eta f prime. So, if you consider the flow over flat plate; then obviously m will become 0. So, for flow over flat plate, let us see the expression of this u and v; m is equal to 0, ok.

So, you can see from here, anyway you will get u is equal to U infinity f prime, where U infinity will become C; because Cx to the power m, so m is 0. And v expression we will get as root U infinity nu by x, this will become half minus f; this will become 1, so plus eta it will become minus 1. So, this will become minus 1, then plus eta f prime. Now, let us calculate this gradient del u by del x, del u by del y and del 2 u by del y square.

(Refer Slide Time: 17:34)

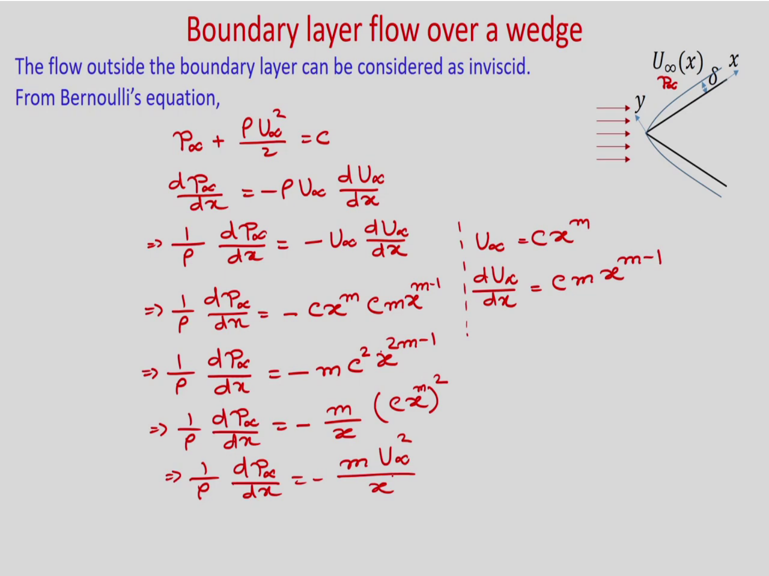


So, we know u is equal to Cx to the power m f prime. So, del u by del x we can write C m x to the power m minus 1 f prime plus Cx to the power m f double prime and del eta by del x. So, del eta by del x if you write, so it will be y root C by nu m minus 1 by 2 x m minus 3 by 2.

And if you simplify, you will get del u by del x as m u infinity divided by x f prime plus U infinity by x f double prime; here you can write eta m minus 1 by 2. Now, you can write del u by del y. So, it will be Cx to the power m f double prime and then del f by del y. So, del f by del y you can write root C by nu x to the power m minus 1 by 2. So, if you simplify it, you will get U infinity root U infinity by nu x f double prime.

And del 2 u by del y square, from here again you can write U infinity root U infinity by nu x f triple prime ok and again eta by del y. So, it will be root C by nu x to the power m minus 1 by 2. So, this you can simplify and write U infinity; U infinity by nu x f triple prime. Now, we have evaluated the velocity and velocity gradient. Now, let us calculate the pressure gradient, right.

(Refer Slide Time: 19:35)



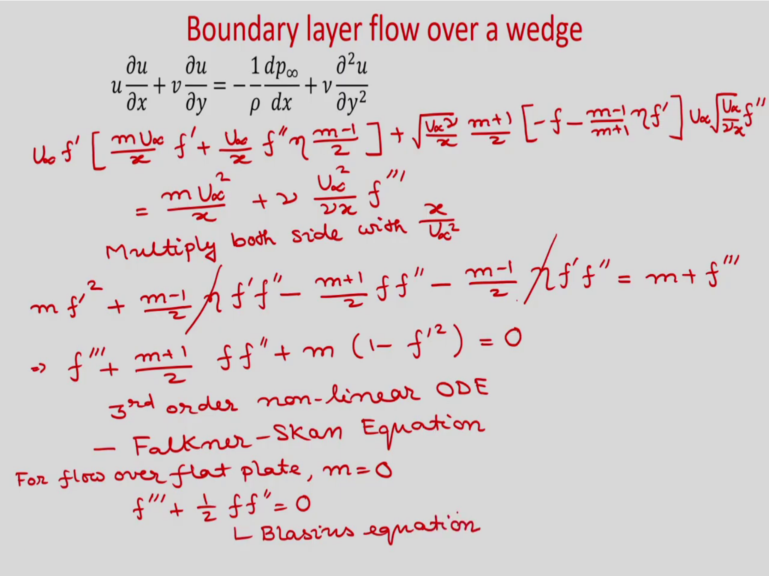
So, you know the flow outside the boundary layer can be considered as inviscid. So, from Bernoulli's equation, we can write p infinity plus rho U infinity square by 2 is equal to c. So, outside we have pressure p infinity, right. And this we have already derived in this module. So, this is the Bernoulli equation.

Now, we can write dp infinity by dx is equal to minus rho U infinity d U infinity by d x; because u infinity is function of x. So we can write 1 by rho dp infinity by dx is equal to minus U infinity d infinity by dx. And we have U infinity as Cx to the power m, so d U infinity by dx will be C m x to the power m minus 1.

So, if you put it here, then we can write 1 by rho dp infinity by dx is equal to minus. So, U infinity is Cx to the power m and d U infinity by dx is C m x to the power m minus 1. So, we can write 1 by rho dp infinity by dx is equal to minus m C square, then we have x 2 m minus 1 or you can write 1 by rho dp infinity by dx is equal to minus. So, we will just divide by x.

So, will multiply with x here. So, it will become x to the power 2 m, that means we will get Cx to the power m square; that means 1 by rho dp infinity by dx we will get as minus m U infinity square by x, ok.

(Refer Slide Time: 21:46)



So, now let us consider the boundary layer equation and put all the values of velocity, velocity gradient and the pressure gradient. So, if you put it here. So, u we have written as U infinity f prime del u by del x m U infinity by x f prime plus U infinity by x f double prime eta m minus 1 by 2 plus. So, v v is root U infinity nu by x m plus 1 by 2 minus f minus m minus 1 divided by m plus 1 eta f prime.

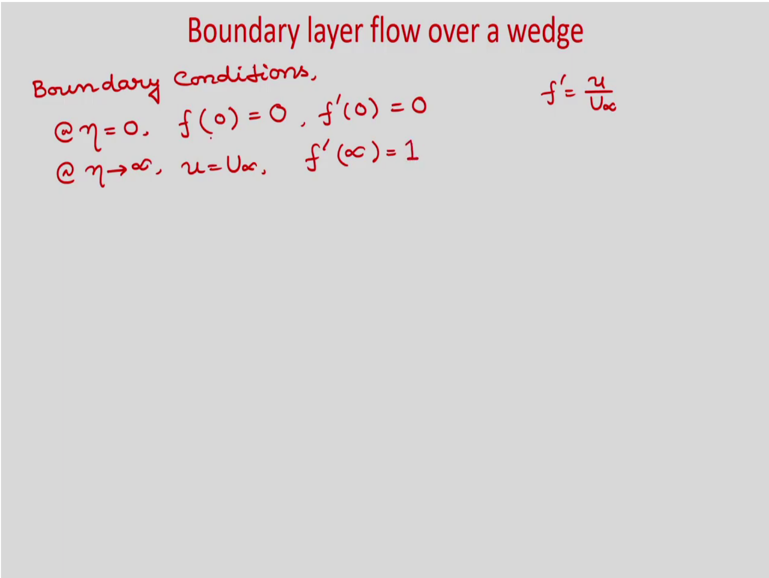
And del u by del y, it is U infinity root U infinity by nu x f double prime, ok. And in right hand side minus 1 by rho dp infinity by dx we have written as m U infinity square by x plus nu del 2 u by del y square, it is U infinity square by nu x f triple prime, ok. So, multiply both side with x by U infinity square, ok. So, what you will get?

So, you will get m f prime square. So, if you multiply this with these terms, plus m minus 1 by 2 eta f prime f double prime minus. So, here you can write m plus 1 by 2 f f double prime; then we will get minus m minus 1 by 2 eta f prime f double prime and equal to here you will get m plus f triple prime. So, you can see that this term and this term is same with a minus sign, so we can cancel it.

So, we will get f triple prime plus m plus 1 by 2 f; f double prime plus m 1 minus f prime square is equal to 0. So, you can see that, we started with the partial differential equation and using similarity transformation. We transform the PDE to ODE. And you can see that, this is third order non-linear ordinary differential equation, this equation is known as Falkner Skan equation. So, you can see that this is 3rd order non-linear ordinary differential equation and this equation is known as Falkner Skan equation.

So, you can see that now for flow over flat plate m is equal to 0, right. So, special case for flow over flat plate, horizontal flat plate; so in that case m is equal to 0. So, this equation you can write as f triple prime. So, m is equal to 0, so it will be half f f double prime and this will become 0, so equal to 0. So, this we have already derived which is known as Blasius equation.

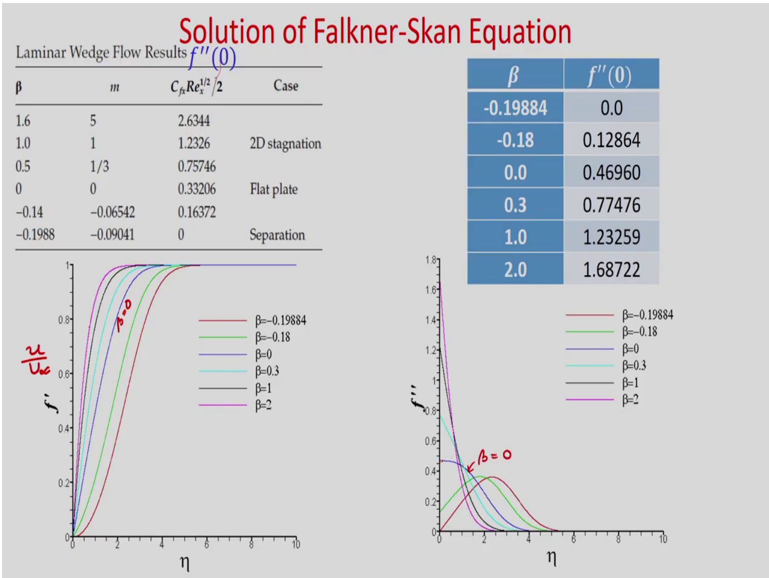
(Refer Slide Time: 26:03)



Now, what about the boundary conditions ok? So, obviously, at y is equal to 0; that means on the wedge surface, the stream function is 0 and velocities are 0. So, we can write boundary conditions. So, y is equal to 0; that means eta will become 0, ok. So, we can see that the stream function will be 0; that means f at eta is equal to 0 will be 0. And the velocity u is equal to 0; that means we know that, f prime is u by U infinity.

So, f prime 0 will be 0, this is eta is equal to 0. Similarly, at eta tends to infinity; that means at the edge of the boundary layer, we have u as U infinity, right. So, you will get f prime from here infinity is equal to 1. So, these are the boundary conditions. So, this Falkner Skan equation if you numerically solve, then you will get the velocity distribution f prime as well as the shear stress distribution f double prime.

(Refer Slide Time: 27:40)



So, you can see that for the case of laminar wedge flow for different values of m; shear stress in terms of this drag coefficient and Reynolds number, it is tabulated here. So, this is the representation of f double prime 0; that means at eta is equal to 0, the del 2 u by del y square, ok.

So, that is the value f double prime. So, you can see that from the numerical solution, you have to guess this value and you have to see that at eta tends to infinity, f prime should become 1, right.

So, you can see that for different values of m, here you can see that f double prime 0 value, ok. So, for a special case you know that m is equal to 0; that means it is flow over flat plate and we have already shown that f double prime is 0 is 0.33206. And if it is a m is equal to 1, then it is 2 D stagnation flow, so f double prime 0 you will get 1.2326.

And you can see f double prime 0 will become 0 for m is equal to minus 0.09041, where beta is equal to minus 0.1988. So, in this case this is the point where flow will separate; that means the del 2 u by del y square at the surface is 0. So, flow will just separate at this point.

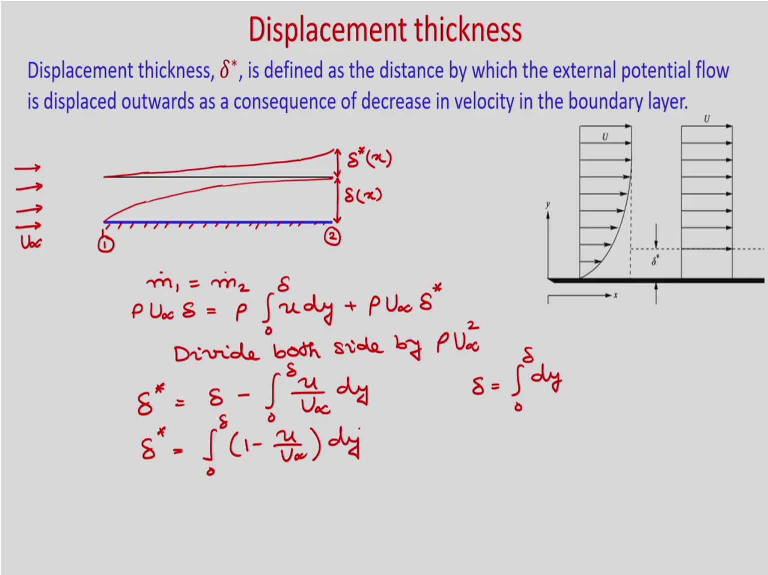
So, after that the boundary layer theory will not be valid, if flow separates. So, here we have plotted the velocity distribution f prime; that means u by U infinity with eta. So, you can see that for different values of beta. So, this is beta 2; then this is the beta 1, which is the case of 2 D stagnation flow and this blue color, it is for flow over flat plate, beta is equal to 0, ok.

And you know that at eta tends to eta is equal to 5, it becomes almost 99 percent of the free stream velocity. Here we have plotted the f double prime; that means del 2 u by del y square ok versus eta. So, obviously outside the boundary layer, the shear stress will become 0; because it will have the same free stream velocity U infinity. So, you can see that as eta tends to infinity, this f double prime becomes 0.

However, at the surface we will get the maximum value, because maximum shear stress will occur at the surface. So, these are the values for different values of beta f double prime at eta is equal to 0. And this is the case, you can see this blue color beta is equal to 0; that means flow over flat plate. So, here it is f double prime 0 is 0.33206. Now, we will discuss about two different thickness, that is displacement thickness and momentum thickness and then we will discuss about the shape factor

So, first let us discuss about the displacement thickness. So, what is displacement thickness? So, if you consider just parallel flow without any presence of wall, then streamline will be parallel.

(Refer Slide Time: 31:42)



So, let us say that we have uniform velocity and we do not have any plate. So, you do not consider this blue color plate, let us say that these are the; this is the streamline and this is this will be horizontal, ok. Now, if you bring this flat plate here; so obviously there will be formation of boundary layer.

So, there will be formation of boundary layer like this, ok. So, due to formation of this boundary layer you know that, this streamline will be no longer parallel to this plate or it will not be horizontal. If you consider two different x location to keep the mass flow rate same, it has to be deflected outward direction

So, let us say that due to this presence of this wall, this horizontal stream line is deflected like this. So, at any location x, you can see there will be deflection of this stream line and this is the streamline in the presence of this flat plate. So, obviously you can see this is the boundary layer thickness at location x ok and it is function of x. And due to the presence of this flat plate, the deflection of the stream line is known as displacement thickness.

So, this is known as delta star, which is also function of x. So, you can see that displacement thickness delta star is defined as the distance by which the external potential flow is displaced outward as a consequence of decrease in velocity in the boundary layer, ok.

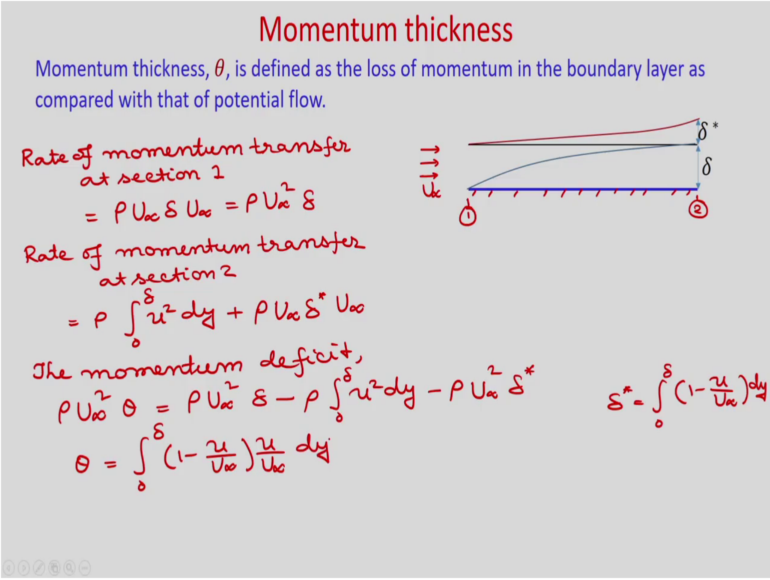
So, in this figure if you consider that, if we have this boundary layer flow; if you represent or if you shift this flat plate at a distance delta star. So that the velocity will become uniform, that means the mass flow rate will remain same.

Here also you can see as it is a streamline deflected, so if this is the location 1; whatever mass flow rate will be there at location 2, mass flow rate will remain same, because there will be no flow across the stream line. So, let us consider the mass flow rate at section 1. So, this will be equal to mass flow rate at section 2.

So; obviously, you can see this is the distance delta, so as per this figure it will be rho U infinity delta is equal to. So, there will be velocity distribution. So, we need to integrate 0 to delta u d y and this is the deflection and it is outside the boundary layer. So we have the free stream velocity U infinity, so it will be rho U infinity delta star, ok. So, now, divide both side by rho U infinity square. So, and rearrange delta star, you will get as delta minus integral 0 to delta u by U infinity d y, ok.

And delta also you can represent as integral 0 to delta d y. So, if you represent here, then we can write this displacement thickness delta star as integral 0 to delta 1 minus u by U infinity d y. So, now, you can see that due to the presence of this flat plate; there will be deflection in the streamline and there will be deficit in the momentum in the downward direction.

(Refer Slide Time: 36:00)



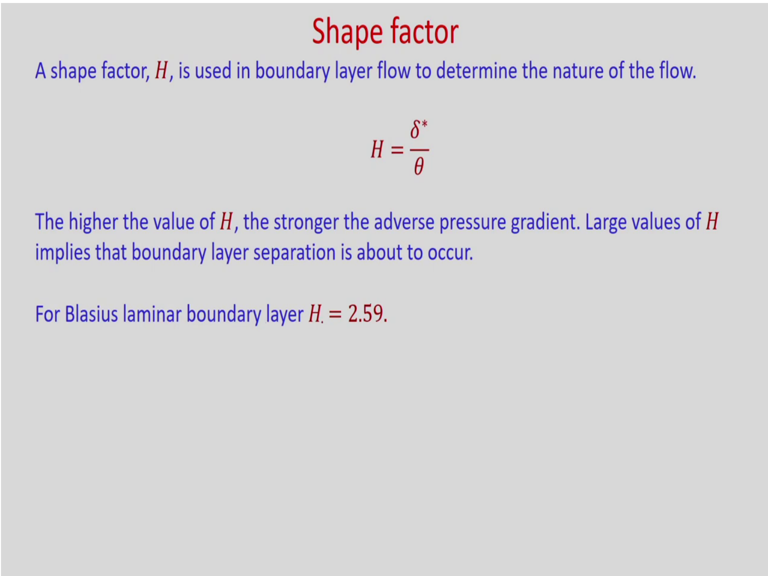
So, if we consider this as flat plate and we have free stream velocity U infinity. So, this is the stream line before the presence of this flat plate; but due to presence of flat plate, this is deflected as this. So, this is the delta star and this is the boundary layer thickness delta; so at location 1 whatever momentum you have, at location 2 there will be some deficit.

So, now, this deficit in momentum is represented as momentum thickness delta. So, you can see that rate of momentum transfer at section 1 is equal to. So, rho U infinity delta will be the mass flow rate into U infinity, so this is the momentum. So, it will be rho U infinity square delta.

And rate of momentum transfer at section 2. So, you can see. So, there will be velocity distribution u; so it will be rho integral 0 to delta u d y and u is the velocity, so it will be u square d y. So, that is the rate of momentum transfer up to this and from here you can see there is a deflection delta star. So, you will get rho, so U infinity. So, this is the mass flow rate into the velocity U infinity, ok.

So, now the momentum deficit, we can represent as rho U infinity square theta is equal to rho U infinity square delta minus rho integral 0 to delta u square d y minus rho U infinity square delta star, ok. And delta star we have already represented as integral 0 to delta 1 minus u by U infinity d y, ok. So, if you simplify it and divide by rho infinity square, then we can write theta as integral 0 to delta 1 minus u by U infinity u by U infinity d y.

(Refer Slide Time: 39:03)



So, now we will define shape factor. A shape factor H is used in the boundary layer flow to determine the nature of the flow. So, it is represented as H is called to delta star; delta star is the displacement thickness divided by theta, theta is the momentum thickness. So, this is called shape factor; because this H depends on solely on the velocity profile, ok.

So, the higher the value of H, the stronger the adverse pressure gradient and larger values of H implies that boundary layer separation is about to occur. And for Blasius laminar boundary layer H will be 2.59. So, in the next module, we will discuss more about the shape factor. In today's class, we considered boundary layer flow over a wedge and we have used similarity transformation to derive the Falkner Skan equation.

Then we discuss about the boundary conditions and if you solve numerically this Falkner Skan equation; then you can see that for different values of m, the value of the shear stress at the wall and after a certain negative value of m, the f double prime 0 becomes 0.

So, obviously at that point, the separation will occur; so boundary layer theory will not be valid once the flow separates. We have also shown the velocity distribution over this wedge for different values of beta. Next we discussed about the displacement thickness, momentum thickness and the shape factor.

Thank you.