**Viscous Fluid Flow**
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**Module - 07**
**Laminar Boundary Layers - I**
**Lecture - 03**

# Momentum Integral Equation for Flat Plate Boundary Layer

Hello, everyone. So, in last class, we learnt the Blasius solution for flow over flat plate using similarity transformation. Today, we will use another method to solve the flow over flat plate. This method is known as integral method or approximate method. Why it is approximate method because depending on the assumed velocity profile will get the final solution.

There are many situations where it is very difficult to obtain the exact solution; in that case we can use this integral method.

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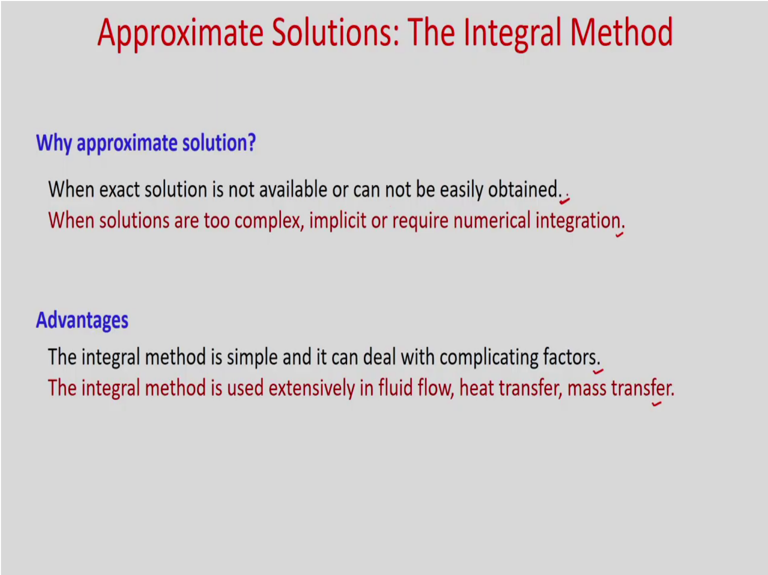


So, why do we need to use this approximate solution? When exact solution is not available or cannot be easily obtained; when solutions are too complex, implicit or require numerical integration. And, the advantages are the integral method is simple and it can deal with complicating factors. The integral method is used extensively in fluid flow heat transfer and mass transfer.

We will follow three procedures to use this integral method. First, we need to write down the governing equations, then you assume the velocity profile and express in terms of any unknown parameter; in this case, this unknown parameter is boundary layer thickness. And finally, you find this unknown parameter invoking the boundary conditions and you can write the velocity profile now using this known velocity boundary layer thickness.

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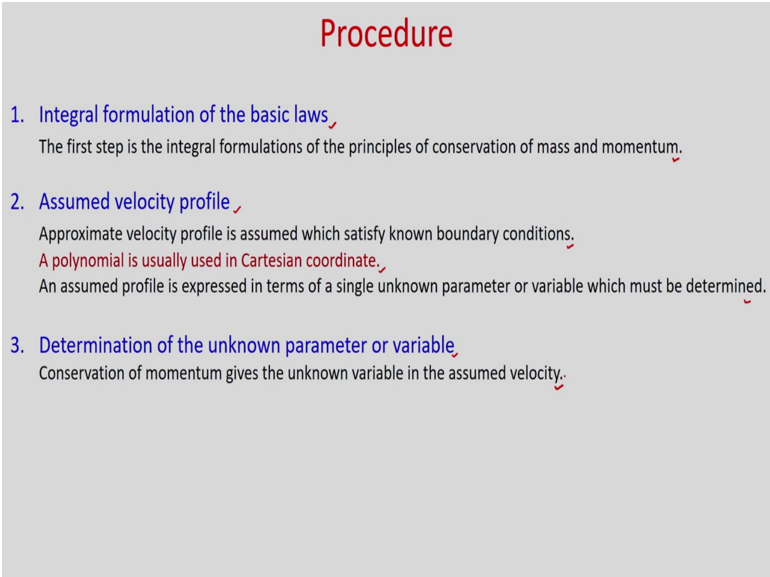


So, these are the three steps we will follow to use this integral method. The first step is the integral formulation of the principles of conservation of mass and momentum. Next step is to assume the velocity profile. Approximate velocity profile is assumed which satisfy non-boundary conditions. Generally, a polynomial is used in Cartesian coordinate and an assumed profile is expressed in terms of single unknown parameter or variable which must be determined.

Finally, we will determine the unknown parameter or variable. So, conservation of momentum gives the unknown variable in the assumed velocity.

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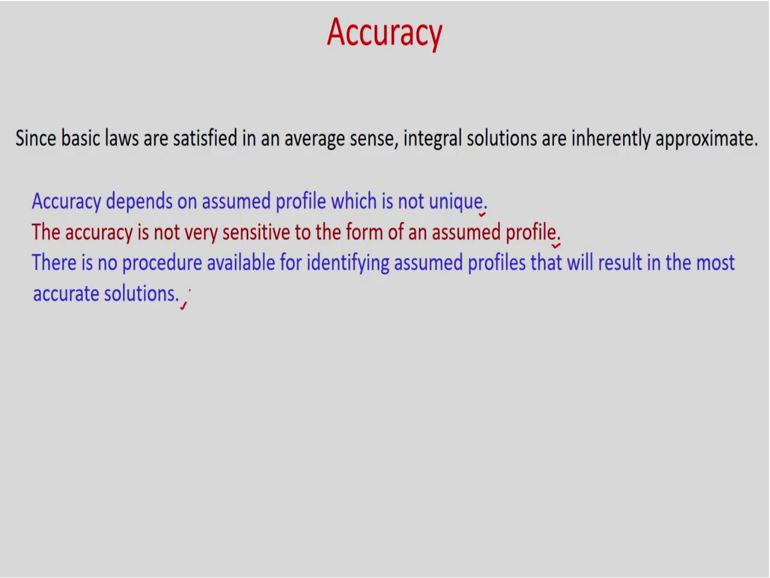


Since basic laws are satisfied in an average sense integral solutions are inherently approximate. Accuracy depends on assumed profile which is not unique. The accuracy is not very sensitive to the form of an assumed profile and there is no procedure available for identifying assumed profiles that will result in the most accurate solution.

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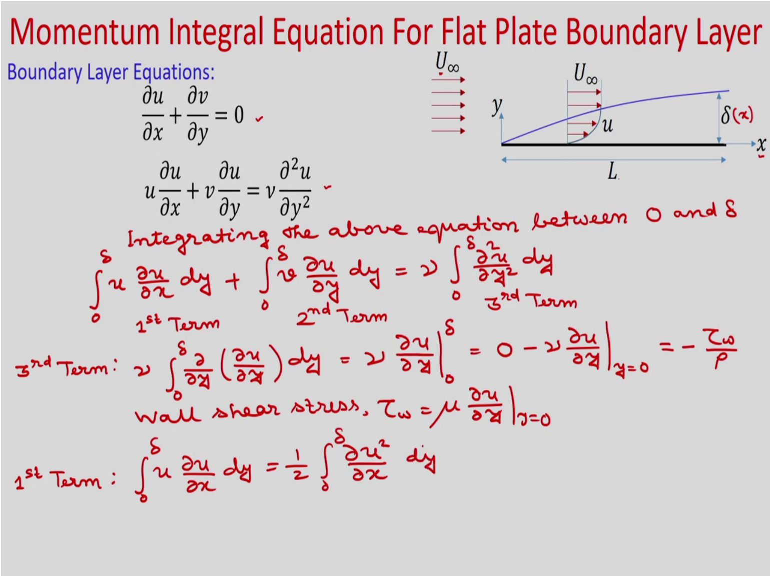


So, let us consider flow over a flat plate. So, this is the flat plate of length L, x is the axial direction, y is the normal direction and this is the velocity profile u inside the boundary layer and U infinity is the free stream velocity. So, we can write down these boundary layer equations this is the continuity equation and this is the momentum equation, where nu is the kinematic viscosity.

So, we will use this equation first and we will integrate it between 0 and delta. So, delta is the boundary layer thickness and you know that this is function of x. So, we will write the terms as integral 0 to delta u del u by del x dy plus integral 0 to delta v del u by del y dy is equal to nu.

Nu is constant; so, you can take it outside the integral. Integral 0 to delta del 2 u by del y square dy. So, this is the 1st term, this is the 2nd term in the left hand side and this is the 3rd term. So, now, we will evaluate this integral one by one. So, let us first consider the 3rd term. So, you can see the 3rd term is nu integral 0 to delta. So, this term we can write as del of del y del u by del y dy, ok. So, this now we can write as nu.

So, del u by del y and limits 0 to delta. So, now, you can see that at y is equal to delta; that means, at the edge of the boundary layer obviously, the velocity gradient is 0, right. So, we will get 0 at y is equal to delta and minus nu del u by del y at y is equal to 0, ok. If we define the wall shear stress tau w is equal to mu del u by del y at y is equal to 0, then you can see that this term del u by del y at y is equal to 0 you can write tau w by mu and here nu is mu by rho.

So, finally, this 3rd term we can write as minus tau w by rho ok. So, tau w is the wall shear stress. Now, let us consider the 1st term. We have integral 0 to delta u del u by del x dy. So, this we can write it as. So, this u will take inside this derivative ok. So, you can write half 0 to delta del u square by del x dy.

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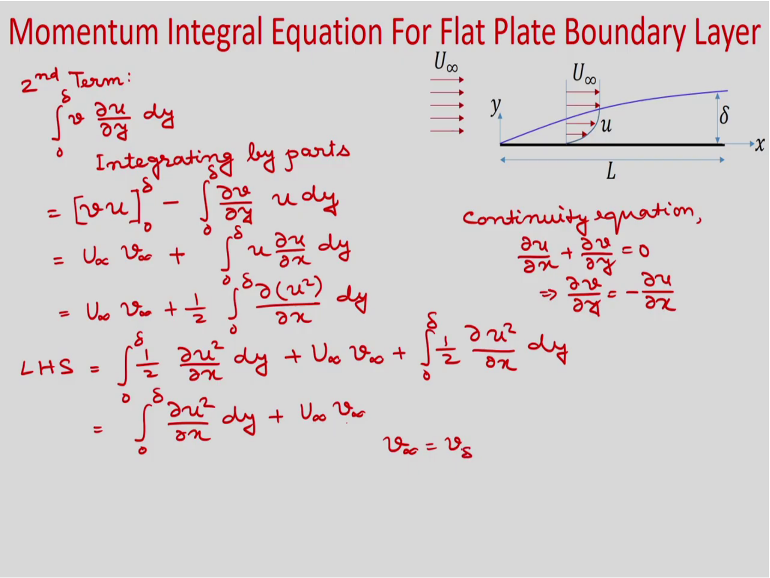


Now, let us consider the 2nd term. So, 2nd term, so, we have integral 0 to delta v del u by del y dy, ok. So, we can now use integration by parts. So, what we can write we can write v u and limits 0 to delta minus integral 0 to delta del v by del y u dy, ok. So, here you see vu limits 0 to delta. So, at y is equal to 0 obviously, velocities are 0. So, this will become 0 and at y is equal to delta. So, we need to write.

So, at y is equal to delta you can see that we have free stream velocity U infinity, but v is unknown. So, you will write U infinity v infinity; v infinity is to be determined minus. So, you can see del v by del y. So, now, let us consider the continuity equation. So, what is your continuity equation? Del u by del x plus del v by del y is equal to 0. So, you can write del v by del y is equal minus del u by del x. So, this term del v by del y, now we will write minus del u by del x.

So, this minus and here 1 minus is there. So, we will write plus integral 0 to delta u del u by del x dy, ok. So, this we can write U infinity v infinity and this again we will write as half 0 to delta del. So, this u we are going to take inside this derivative. So, it will be u square del x dy.

So, now, let us simplify the left hand side. We have two terms, we have already written those two terms. So, left hand side we are going to get the 1st term we have written 0 to delta half del u square by del x dy and this is the term U infinity v infinity plus 0 to delta half del u square by del x dy. So, you can see this is half del u square by del x and this is also half del u square by del x.

So, these together will get integral 0 to delta del u square by del x d y plus U infinity v infinity. So, now, we need to find, what is v infinity; that means, v at y is equal to delta. So, for that, so, to evaluate this v at y is equal to delta again we will consider continuity equation.

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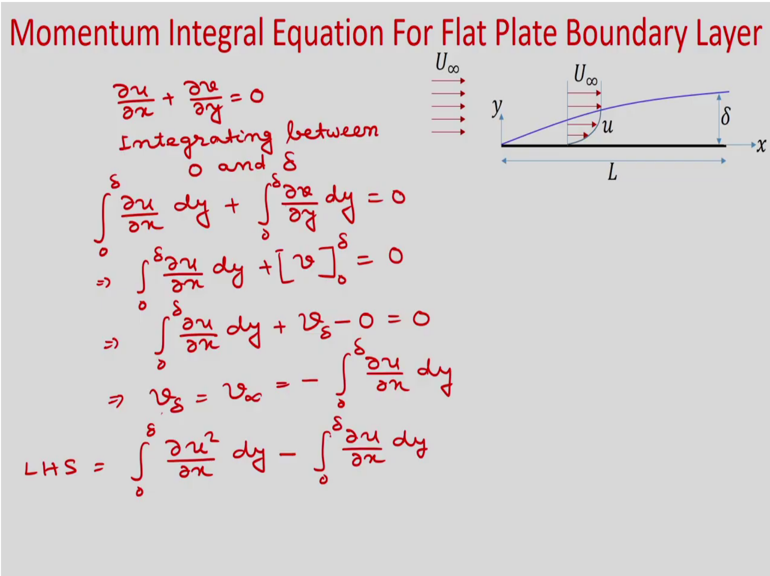


So, we have the continuity equation del u by del x plus del v by del y is equal to 0. So, we will integrate this equation between 0 and delta within the boundary layer. So, obviously, we can write 0 to delta del u by del x dy plus integral 0 to delta del v by del y dy is equal to 0, ok.

So, this we can write integral 0 to delta del u by del x dy and this you can see we can write v from 0 to delta is equal to 0. So, now, we can see we have 1st term as it is and this term you see at y is equal to 0 ok; that means, on the flat plate v is equal to 0, ok. So, we will get at y is equal to delta v delta; that means, minus. So, at y is equal to 0, this will be 0. So, is equal to 0.

So, v delta; that means, at y tends to infinity. So, v infinity we can write as minus integral 0 to delta del u by del x dy, ok. So, now, you can see that this v delta now we can substitute it here where this is your we have written v infinity which is your v delta.

So; that means, v infinity; that means, at v is at y is equal to delta. So, this now we can substitute it then we can write left hand side as integral 0 to delta, del u square by del x dy plus now we have v delta.

So, v delta is having minus. So, we will write minus integral 0 to delta del u by del x dy. So, now we can see that we have this derivative inside the integral. So, to evaluate this integral we will use Leibniz rule of integration.

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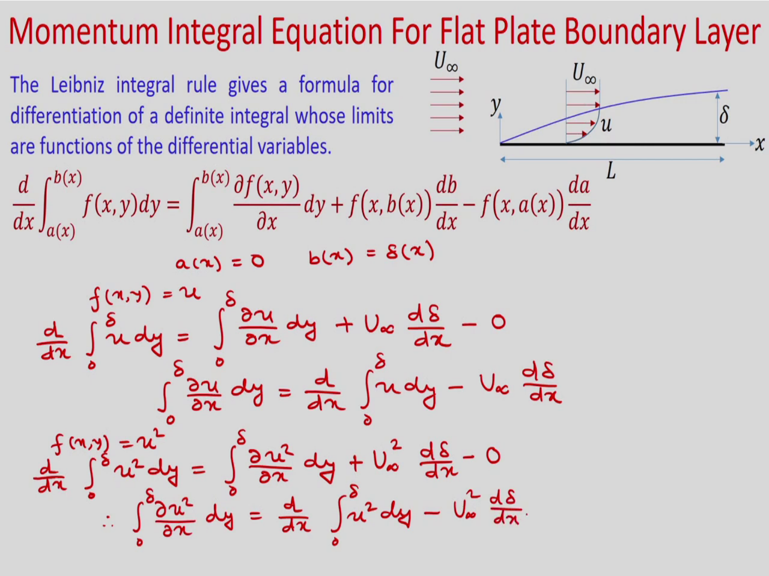

So, you can see the Leibniz integral rules. The Leibniz integral rule gives a formula for differentiation of a definite integral whose limits are functions of the differential variable. So, you can see d of dx integral a to b. So, where a, b are function of x. So, these are the limits, function of x and f x, y dy is equal to integral a x b x del f by del x dy plus f at the limits y is equal to b db by dx minus f x at y is equal to a da by dx.

So, here for this particular problem we have a x equal to 0 because lower limit is equal to 0 and upper limit bx we have delta and delta is function of x. So, obviously, now if we choose the first function f x y is equal to u, then we can write from this Leibniz integral rule that d of dx integral 0 to delta u dy is equal to 0 to delta del u by del x dy.

Now, u at y is equal to delta; that means, at y is equal to delta we have U infinity. So, you can write plus U infinity v is delta. So, d delta by dx and obviously, U at y is equal to 0; that means, this is no slip condition. So, u is equal to 0. So, this will become 0 minus 0 ok. So, we can write integral 0 to delta del u by del x dy is equal to d of dx integral 0 to delta u dy minus U infinity d delta by dx.

So, next the function we will consider f x, y is equal to u square and we will use these Leibniz rule of integration. So, if f x, y is equal to u square, similarly we can write d of dx integral 0 to delta u square dy is equal to integral 0 to delta del u square by del x dy. Now, here you can see that this is u square, but at y is equal to delta. So, it is U infinity.

So, it will be plus U infinity square, d delta by dx minus; obviously, at y is equal to 0 velocity is 0. So, it will be minus 0. So, we can write now integral 0 to delta del u square by del x dy is equal to d of dx integral 0 to delta u square dy minus U infinity square d delta by dx.

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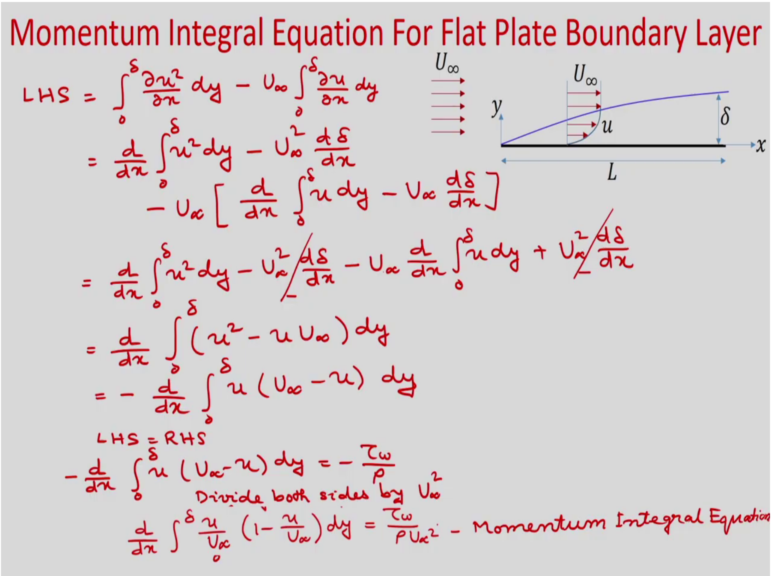


So, now let us simplify the left hand side and write the left hand side two terms as integral 0 to delta del u square by del x dy minus U infinity integral 0 to delta del u by del x dy. So, now, these we have evaluated using the Leibniz rule. So, now, let us put those values.

So, d by dx 0 to delta u square dy minus U infinity square d delta by dx and we have for this term minus U infinity d of dx integral 0 to delta u dy minus U infinity d delta by dx. So, now, we can simplify it. You can see we can write d of dx integral 0 to delta u square dy minus U infinity square d delta by dx.

Now, let us multiply this U infinity with these two terms. So, we will get minus U infinity d of dx integral 0 to delta u dy and this will be plus U infinity square d delta by dx ok. So, you can see these term and this term will get cancelled. So, finally, we will get d of dx integral 0 to delta u square and here U infinity is constant, right.

So, you can take inside the integral. So, you can write minus u U infinity dy ok. So, you can write minus d of dx. So, we are we have to write this term first. So, integral 0 to delta we are taking u outside. So, it will be U infinity minus u dy. We have already evaluated the right hand side term which is the 3rd term in terms of the wall shear stress.

So, finally, these integrated this momentum equation we can write as. So, if we equate left hand side is equal to right and side; that means, right hand side we have this 3rd term. So, we can write as minus d of dx integral 0 to delta u U infinity minus u dy is equal to minus tau w by rho ok.

So, we can write this equation and divide both side by U infinity square and minus minus will get cancel. So, you can write d of dx integral 0 to delta u by U infinity 1 minus u by U infinity dy is equal to tau w by rho infinity square. So, this equation is known as momentum integral equation. So, now, we have written the integral form of the governing equation and we have derived the momentum integral equation, next step is to assume the velocity profile.

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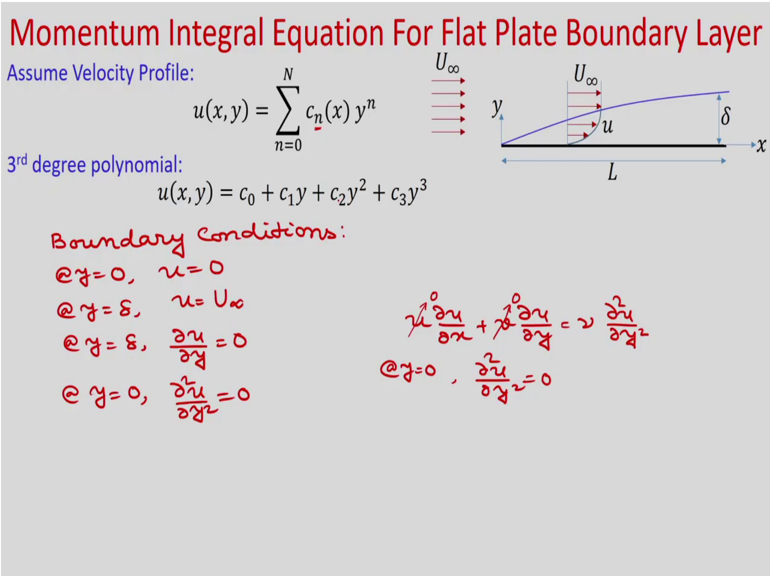


So, for laminar flow generally polynomial is used and if we write the polynomial of nth order, then you can see assume. So, you can assume the velocity profile as u x, y as summation of n is equal to 0 to n c n y n, where c n is the coefficient and it is function of x. So, if we use 3rd degree polynomial so, for the velocity profile then u x, y you can write as c naught plus c 1 y plus c 2 y square plus c 3 y cube.

Here we have 4 coefficients which are function of x and those are unknown. So, we need to find these unknowns and we need 4 boundary conditions because we have 4 unknowns. So, we need 4 boundary conditions, 3 boundary conditions are easy to write because you know that at y is equal to 0 we have no slip boundary condition. So, u is equal to 0.

At y is equal to delta; that means, at the edge of the boundary layer we have the velocity profile u is equal to infinity and also the velocity gradient is 0; that means, del u by del y at y is equal to delta is equal to 0, but another boundary condition we need. So, that will derive the boundary condition from the governing equation invoking that at y is equal to 0, velocity is 0.

So, let us write first the boundary conditions. So, at y is equal to 0 u is equal to 0; at y is equal to delta u is equal to U infinity; at y is equal to delta del u by del y is equal to 0, velocity gradient is 0 ok. Now, we need to write another boundary condition. So, for that we will use the governing equation u del u by del x plus v del u by del y is equal to nu del 2 u by del y square.

So, this is derived boundary condition and we will put at y is equal to 0; obviously, you can see that u is 0, at y is equal to 0 v is 0. So, from here you can see the boundary condition will become del 2 u by del y square is equal to 0. So, we have another boundary condition at y is equal to 0 del 2 u by del y square is equal to 0 and it is known as derived boundary condition. So, we have this polynomial now you invoke these boundary conditions.

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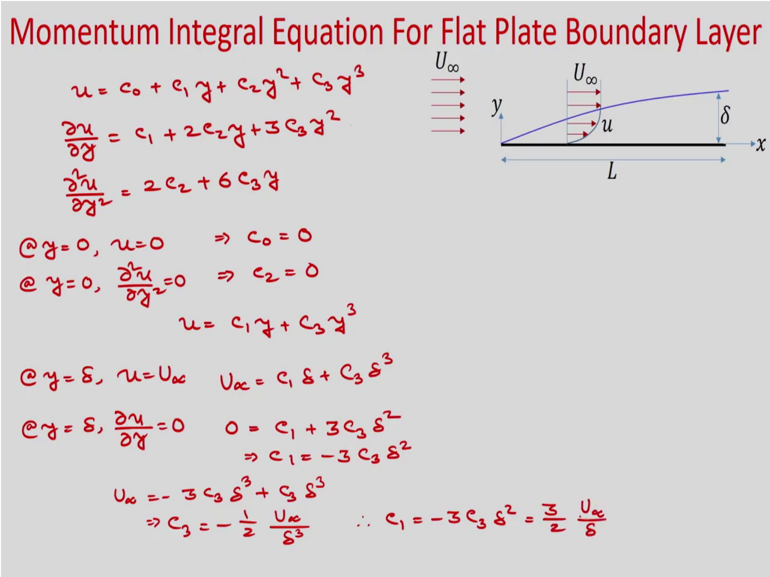


So, we have u is equal to c naught plus c 1 y plus c 2 y square plus c 3 y cube, ok. So, you can write del u by del y is equal to c 1 pus twice c 2 y plus 3 c 3 y square and del 2 u by del y square we can write as twice c 2 plus 6 c 3 y. So, boundary conditions at y is equal to 0 u is equal to 0 ok.

So, if you put at y is equal to 0 u is equal to 0, that will give c 0 as 0; at y is equal to 0 we have del 2 u by del y square is equal to 0 ok. So, if you put here you will get c 2 is equal 0. So, these two coefficients are 0. So, we can write now u is equal to c 1 y plus c 3 y cube.

So, now let us apply the next boundary condition at y is equal to delta u is equal to U infinity. So, from here you can see that it will be U infinity is equal to c 1 delta plus c 3 delta cube and we have at y is equal to delta del u by del y is equal to 0. So, from here you can see that c 2 is

0. So, you will get left hand side del u by del y is equal to 0; in right hand side c 1, c 2 is 0. So, 3 c 3 delta square.

So, now, you solve these two equations and find c 1 and c 3. So, from here you can see that c 1 we can write as minus 3 c 3 delta square and from here you can see we have U infinity c 1 is minus 3 c 3 delta square and delta. So, it will be delta cube plus c 3 delta cube. So, from here you can see c 3 will be minus half U infinity by delta cube.

So, we will get c 1 as minus 3 c 3 delta square and this is minus c 3. So, it will be 3 by 2 you infinity by delta. So, we have found these 4 coefficients c naught and c 2 are 0 and we have nonzero values for c 1 and c 3. So, if you put these coefficients in terms of delta then we will get the velocity profile in terms of one unknown variable that is delta boundary layer thickness.

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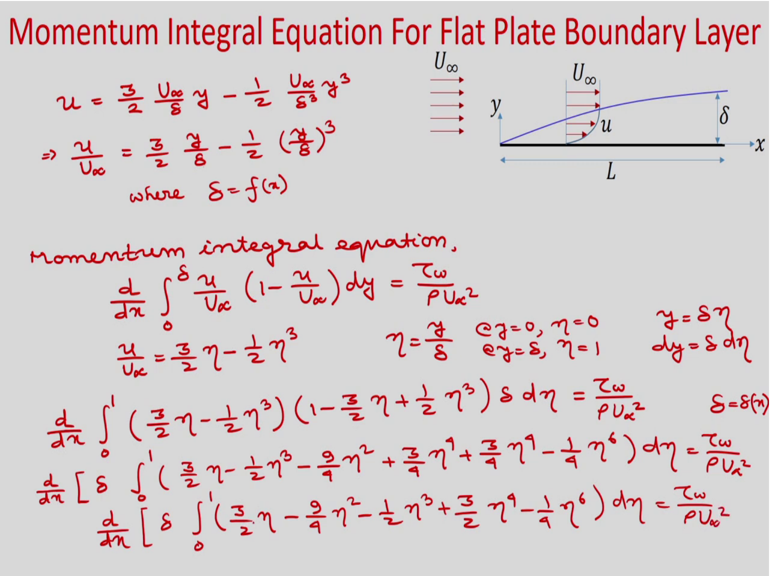

So, you can see we have u is equal to 3 by 2 U infinity by delta y minus half U infinity by delta cube y cube. So, you can write u by U infinity as 3 by 2 y by delta minus half y by delta cube where delta is function of x ok. So, you can see that the velocity profile now we have expressed in terms of one unknown parameter that is delta. Now, we need to find the value of delta.

So, next we will put this assumed velocity profile which we have expressed in terms of one unknown parameters in the momentum integral equation and then we will find this unknown parameter delta. So, we have the momentum integral equation we have already derived d of dx integral 0 to delta u by U infinity 1 minus u by U infinity dy is equal to tau w by rho infinity square.

And, we have this velocity profile and we can write this velocity profile u by U infinity in terms of eta where we will define eta as y by delta ok. So, if you put eta is equal to y by delta. So, you can write 3 by 2 eta minus half eta cube ok.

So, eta is equal to y by delta; that means, the limits now in terms of eta you can see that at y is equal to 0; obviously, eta is equal to 0 and at y is equal to delta eta is equal to 1, and we have written y is equal to delta eta. So, you can write dy is equal to delta d eta ok. So, now, all these you put in this expression.

So, we will get this momentum integral equation you can see d of dx integral 0 to delta. So, this term will be there 3 by 2 eta minus half eta cube and this will be 1 minus u by infinity. So, it will be 1 minus 3 by 2 eta plus half eta cube and dy is equal to delta d eta. So, it will be delta d eta and this limit will be eta is equal to 0 to 1, and right hand side we have tau w by row U infinity square.

Now, let us find this integral and find the value of delta. So, you can see that if you multiply this then you will get d of dx delta you can see that you can take it outside right outside this integral because delta is function of x right delta is function of x. So, obviously, as you are integrating over this d eta so, you can take outside this integral this delta.

So, you can write delta integral 0 to 1, now you multiply this ok. So, if you multiply this you will get 3 by 2 eta minus half eta cube this if you use. So, it will be minus 9 by 4 eta square and this will be plus 3 by 4 eta to the power 4 and with these if you multiply it will be plus 3 by 4 eta to the power 4 and this will be minus 1 by 4 eta to the power 6 ok.

So, it will be d eta is equal to tau w by row U infinity square. So, you can write d of dx delta integral 0 to 1. So, this term is 3 by 2 eta we have eta square minus 9 by 4 eta square, then eta cube only we have minus half eta cube eta to the power 4. So, these two terms together we can write 3 by 2 eta to the power 4 and we have minus 1 by 4 eta to the power 6 d eta is equal to tau w by row U infinity square. So, now, you can integrate this ok.

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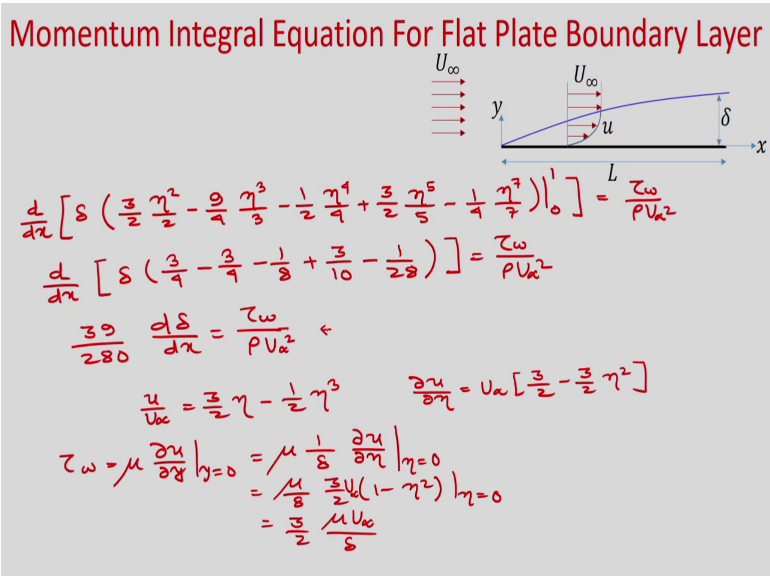


So, after integrating we can write d of dx delta. So, after integration 1st term will be 3 by 2 eta square by 2 minus 9 by 4 eta cube by 3 minus half eta to the power 4 by 4 plus 3 by 2 eta to the power 5 by 5 minus 1 by 4 eta to the power 7 by 7 and limits 0 to 1 is equal to tau w by row U infinity square.

So, obviously, at eta is equal to 0 if you put it will become 0. So, eta is equal to 1, if you put then you will get d of dx delta. So, it will be 3 by 4. So, it will be minus 9 by 4 and divided by 3. So, it will be minus 3 by 4, it will be minus 1 by 8, this will be plus 3 by 10 and this will be minus 1 by 28 is equal to tau w by rho U infinity square.

So, now if you see it you will get this will be 39 by 280. So, 39 by 280 and it will be d delta by dx is equal to tau w by rho U infinity square. So, we have already expressed the velocity profile in terms of the boundary layer thickness delta. So, from there we can find the tau w in terms of delta.

So, tau w is the wall shear stress. So, we have the velocity profile right as u by U infinity is equal to 3 by 2 eta minus half eta cube, right. So, wall shear stress tau w now we can write mu del u by del y at y is equal to 0. So, in terms of eta if you write; so, mu 1 by delta del u by del eta at eta is equal to 0.

So, now you can see if you put it del u by del eta, so, what you will get? Del u by del eta from here you can see it will be U infinity 3 by 2 minus 3 by 2 eta square ok. So, this we can write now mu by delta. So, it will become at eta is equal to 0, right. So, 3 by 2 1 minus eta square at eta is equal to 0.

So, and 1 U infinity will be there. So, from here you can see it will be 3 by 2 mu U infinity by delta. So, this you put it in this expression ok. So, we will get 39 by 280 d delta by dx is equal to tau w by row infinity square. So, tau w is 3 by 2 mu infinity by delta, ok.

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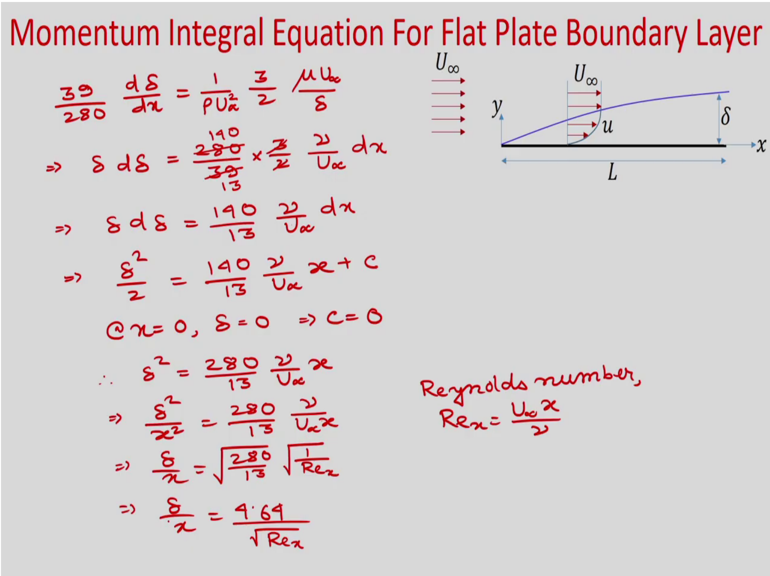


So, now if you rearrange it, so, you will get delta you take in the left hand side. So, delta d delta and in the right hand side you take all other terms. So, it will be 280 by 39 into 3 by 2 and we have. So, mu by row that we can write as nu and this U infinity, this U infinity square.

So, you will get nu by U infinity dx ok. So, 3 13 to 140. So, we can write as delta d delta is equal to 140 by 13 nu by U infinity dx. Now, if you integrate it we can write delta square by 2 is equal to 140 by 13 and nu and U infinity are constant, right. So, nu by infinity x plus constant c integration constant.

Now, we can find this integration constant by invoking that at the wall; that means, y is equal to 0. This integration constant c now we can find from at the leading edge at x equal to 0, what is the boundary layer thickness? So, at x equal to 0 boundary layer thickness is 0, right. So, from there we can find the value of integration constant.

So, at x equal to 0 or x tends to 0, we have delta tends to 0 right. So, that means, if you put it here. So, that will give c is equal to 0. So, from here you can see you can write delta square is equal to 280 by 13 nu by U infinity x.

So, from here you can see you can write as delta square by x square is equal to 280 by 13 nu. So, x square we have divided. So, it will be U infinity x. Now, you define the Reynolds number based on x ok. So, Re x as U infinity x by nu then we can write delta by x as root 280 by 13 root.

So, U infinity x by nu; that means, 1 by Re x ok. So, if you evaluate it you will get around delta by x is equal to 4.64 root Re x. So, you can see that we have already know the delta by x from the exact solution in last class we have derived. So, delta by x we have written is equal to 5 by root Re x, but when we assumed 3rd degree polynomial of this velocity profile, then this is one approximate solution because we assume the velocity profile and we are getting close to this 5 right 4.64 and we have 5 in the numerator when we write delta by x.

So, it is very close. Now, let us find the other parameters like shear stress and skin friction coefficient.

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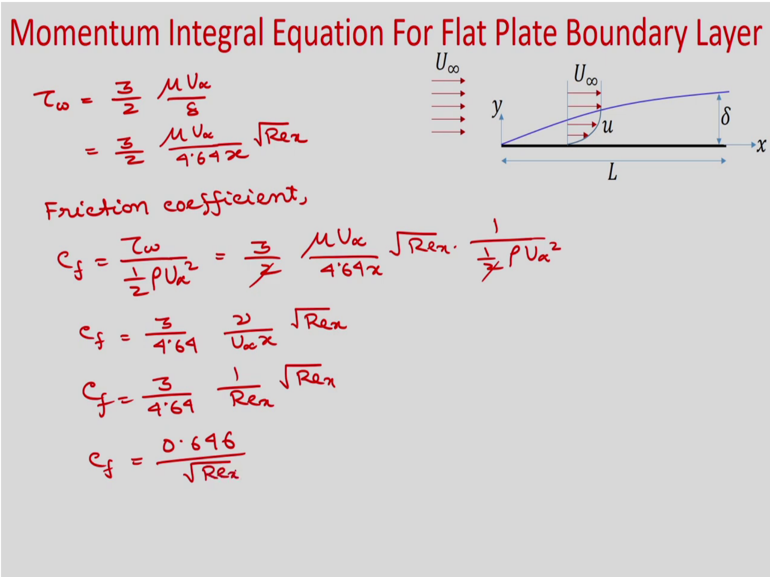


So, if we know tau w already we have written 3 by 2 mu U infinity by delta ok. So, if you put the delta value here. So, you will get 3 by 2 mu U infinity and delta is 4.64 by root Re x right into x. So, now, let us write the skin friction coefficient which is non-dimensional representation of the shear stress. So, you can write friction coefficient C f is equal to tau w by half rho U infinity square. So, this if we put it here. So, we will get 3 by 2 mu U infinity by 4.64 x root Re x and we have 1 by half row U infinity square ok.

So, these 2, 2 will get cancel. So, we will get C f is equal to 3 by 4.64 and this rho and mu it will be nu and U infinity square 1 nu infinity in the numerator. So, you will get divided by U infinity x root Re x and nu by infinity x is equal to Re x right. So, it will be 1 by Re x and this is root Re x.

So, now you can write C f is equal to 0.646 divided by root Re x ok. So, if you remember from the exact solution we found this skin friction coefficient as 0.664 right divided by root Re x. So, you can see that from integral solution delta by x we found 4.64 by root Re x whereas, from the exact solution Blasius solution we got delta by x as 5 by root Re x.

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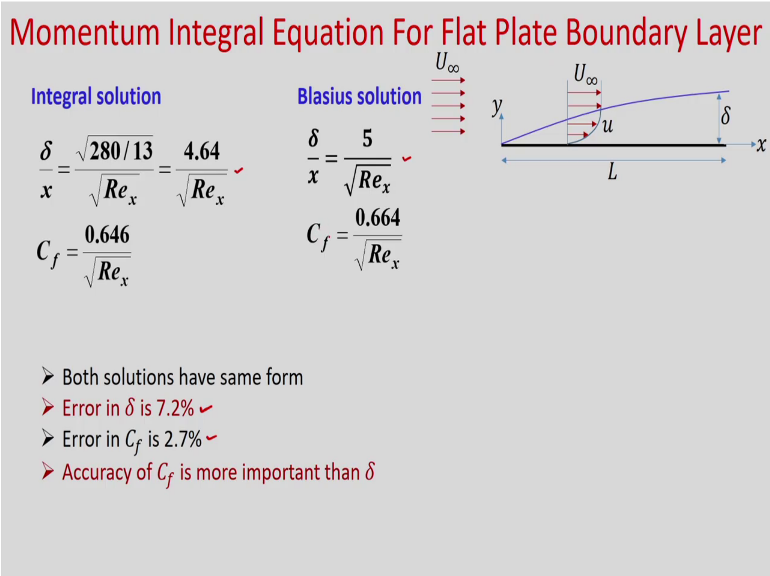


And, C f we found today from the integral solution as 0.646 divided by root Re x and in last class we found from the Blasius solution C f as 0.664 by root Re x. So, you can see that both solutions are having the same form, right. It varies as 1 by root Re x and error in delta if you see that here you are getting 5 and here you are getting 4.64.

So, error in delta if you find then it will be 7.2 percent only and error in C f you can see that it is very less compared to the boundary layer thickness. So, it is 2.7 percent. So, obviously, in design point of view C f is more important than delta and from here; obviously, you can also calculate the average friction coefficient C L.

So, in today's class we used integral method to solve the flow over flat plate. So, there are three steps first step is to write the integral form of the governing equation, next step is to assume the velocity profile and express in terms of one unknown parameter, boundary layer thickness delta and then find the delta and from delta you can get the velocity profile. And, hence you can find the boundary layer thickness and the skin friction coefficient.

So, first after integrating the governing equation we wrote the momentum integral equation and we use 3rd degree polynomial of the velocity profile and from there we found the four coefficients and then we expressed this velocity profile in terms of delta. And, invoking these velocity profile in the momentum integral equation we have found the unknown parameter delta.

And, from there we have found the skin friction coefficient and later we have found that based on what polynomial you are using you will get different solution in the boundary layer thickness delta by x and hence you will get different value of skin friction coefficient both average and local.

Thank you.