Viscous Fluid Flow
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Module - 07
Laminar Boundary Layers - I
Lecture - 01
Derivation of Boundary Layer Equations

Hello everyone. So, in today's class, we will consider the boundary layer flows; first let us know, what is boundary layer. The boundary layer of a flowing fluid is a thin layer near to a solid surface and the flow near to the solid surface is known as boundary layer flows.

In 1904, German scientist Ludwig Prandtl first introduced the concept of boundary layer and derived the equations for boundary layer flows dropping some terms from the Navier Stokes equations. So, today we will start with the Navier Stoke equations and we will see when we can drop some terms from the Navier Stoke equations and which are the terms we can drop from the Navier Stoke equations. Then we will derive the boundary layer equations.

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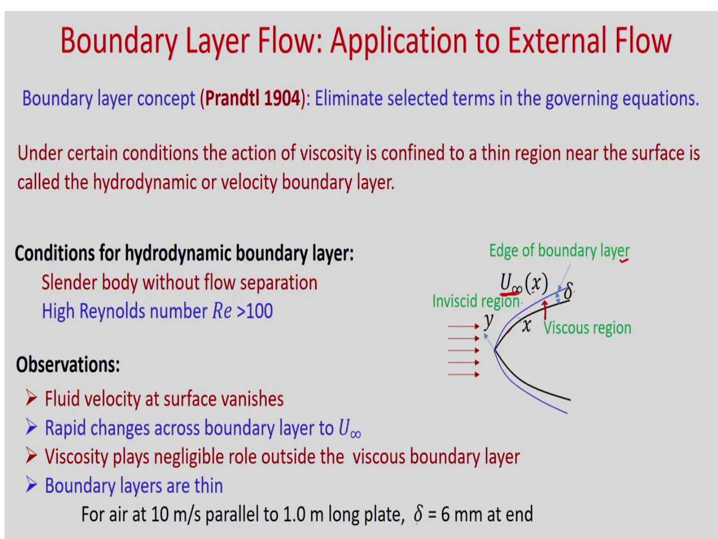

So, let us consider flow over solid surface. So, you can see, this is one surface, this is known as cylinder body and flow is taking place. Then near to the surface one thin layer will be formed, where inside of this thin layer, viscosity effect will be there, which is known as viscous region. And outside this thin layer, the viscous effect can be neglected and that region is known as inviscid region.

So, you can see that under certain conditions, the action of viscosity is confined to a thin region near the surface is called the hydrodynamic or velocity boundary layer. So, what are the conditions for hydrodynamic boundary layer? First thing is that, cylinder body without flow separation and it should be high Reynolds number flow; that means Reynolds number should be greater than 100 or so.

And the thin layer where the distance from the wall, this is known as boundary layer thickness and it is denoted by delta. And this line is a fictitious line, where inside we have a viscous region and outside we have inviscid region, and this edge is known as edge of boundary layer.

So, what are the observation we have for flow over a surface? So, you can see that, fluid velocity at surface obviously it will be 0, so it vanishes. And rapid changes across boundary layer to U infinity; because you can see that at the outside of this thin layer, obviously the this is inviscid region and the velocity will be the free stream velocity U infinity.

After that if you go normal to this direction, there will be no change in this velocity. However, in the direction of this surface if it is x along the surface and y is normal to the surface; then along the surface depending on the configuration of the surface, U infinity may change with x. However, U infinity is not function of y; because it is inviscid region, so there will be no change of velocity U infinity normal to this surface outside this boundary layer.

Viscosity plays negligible role outside the viscous boundary layer and the boundary layers are very thin, ok. So, if you consider the characteristic length as the length of this surface; then you can see for air at 10 meter per second parallel to 1 meter long plate, ok. So, if you consider a flat plate and we have flow up air at a speed of 10 meter per second; then at the end of this flat plate, you will get the boundary layer thickness as 6 millimeter.

So, you can see that 1 meter long plate at the end of the plate, you will get the boundary layer thickness as 6 millimeter. So, obviously you can see that 6 millimeter is very very small compared to the length of the plate, which is 1 meter. So, we can assume that boundary layer thickness is very very small compared to the its length.

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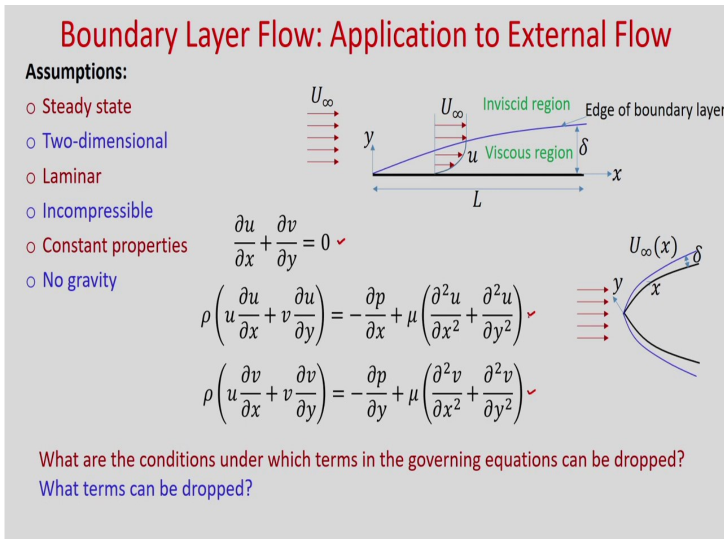

So, first let us write down the governing equations of fluid flow. So, we will assume steady state, laminar, incompressible fluid flow with constant properties; we will neglect the gravity term. So, this is the continuity equation for two dimensional case and this is the x momentum equation and this is the y momentum equation.

So, we are writing in general this equation in two dimensional case. So, now, the questions are what are the conditions under which terms in the governing equation can be dropped? And the second question is, what terms can be drop from these equations?

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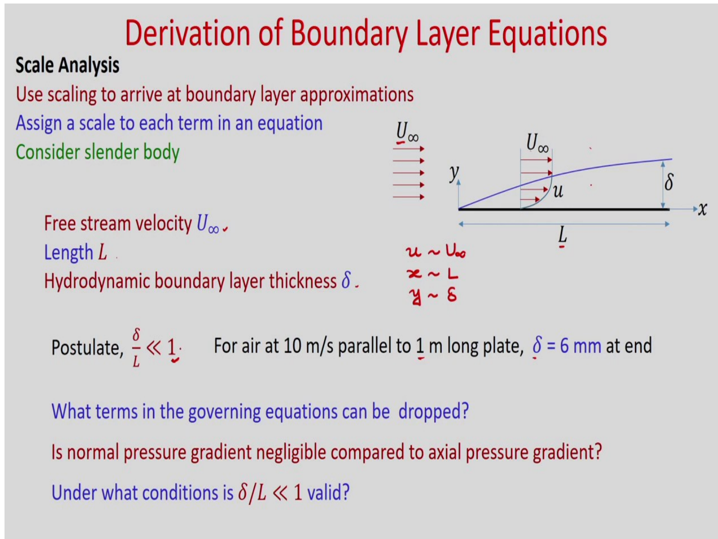

So, to answer the first question we will use scale analysis, which is known as order of magnitude analysis. So, what we do in this analysis? We assign some scale to each parameters and we will see the order of magnitude of each term in the governing equations. So, scaling is used to estimate the order of magnitude of each term in the Navier Stoke equation and we drop the terms of a higher order. And in this procedure, a scale is assigned to each variable in an equation.

So, you can see that if we consider this flow over a surface, where free stream velocity is U infinity; x is the in the direction of the surface and y is normal to the surface and characteristic length is L. And characteristic velocity we will consider as free stream velocity; here obviously you can see outside this boundary layer, we will have the free stream velocity U infinity where inside we have viscous effect, but outside we have inviscid region.

And this is the edge of the boundary layer and delta is the boundary layer thickness. So, let us assign the scale of each term. So, let us say that we will consider free stream velocity U infinity and length L and hydrodynamic boundary layer thickness is delta. So, obviously you can see that, we will assign the scale of velocity u as order of U infinity, ok.

And for the x, we will assign the scale of L and for y will assign the scale of delta, which is your boundary layer thickness. So, now, we have already shown that for air at 10 meter per second parallel to 1 meter long plate, delta is 6 millimeter at the end; so obviously this boundary layer thickness is very very small compared to the length of the plate. So, we can postulate that delta by L is much much less than 1.

So, now we will answer of the following questions, what terms in the governing equations can be dropped? And is normal pressure gradient negligible compared to axial pressure gradient? And under what conditions is delta by L much much less than 1 valid? So, obviously we can see that, when we have delta by L much much less than 1; we can drop some terms from the Navier Stoke equations. So, first let us consider the continuity equation and find the scale for the velocity v.

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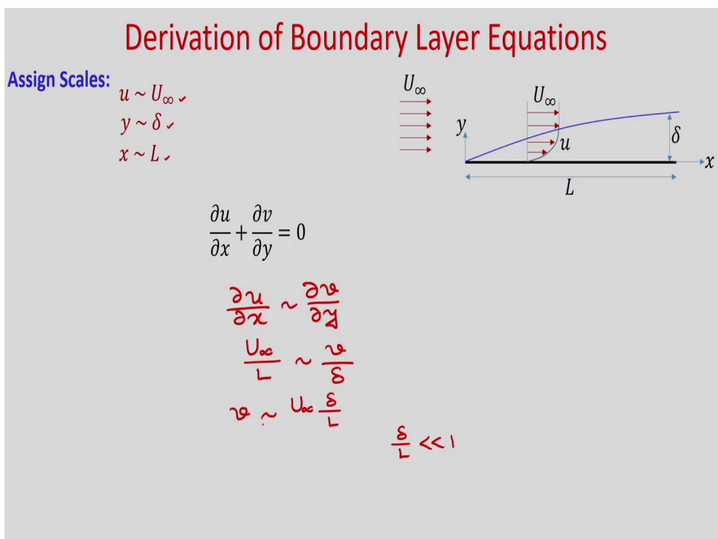

So, we have the scale for u as U infinity, y as delta, and x as L. So, this is the continuity equation. So, if you see that we can write del u by del x it will be order of del v by del y. So, now, assign the scale. So, u is U infinity, x is L order of v by delta. So, you can see that the scale for v as U infinity delta by L; but we have already postulated that delta by L is much much smaller than 1, so obviously v will be very small. So, now, we got the scale for velocity v.

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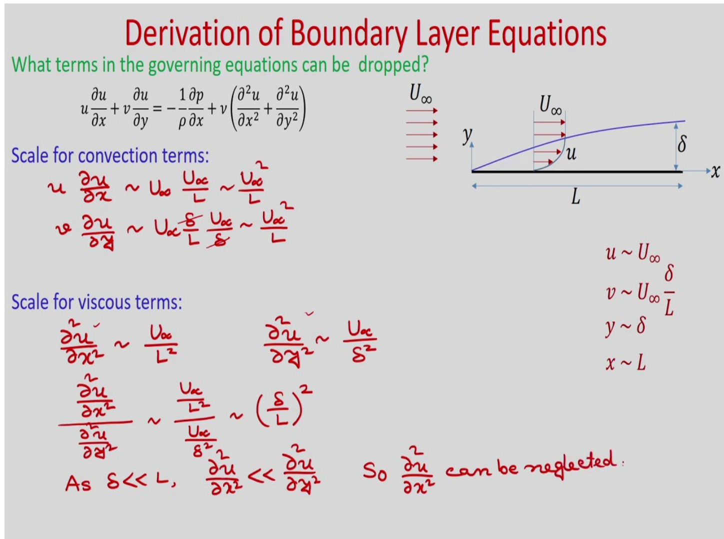

So, the next question is that, what terms in the governing equations can be dropped? So, first let us consider the x component of the momentum equations. So, these are the inertia terms, this is the pressure gradient term and this is the viscous term, where nu is the kinematic viscosity of the fluid and rho is the density of the fluid. And we have the scales as u order of U infinity, v already we have derived v is order of U infinity delta by L, y order of delta, and x order of L.

So, first let us see whether we can drop some terms in the left hand side, which are inertia terms. And each inertia term let us see, what is the order. So, if you consider the first term u del u by del x. So, what is the order of this? So, you can see u U infinity, this is your U infinity by L; so obviously this is the order of U infinity square by L.

The second term we have v del u by del y. So, this is the order of v we know, U infinity delta by l and u U infinity by delta. So, obviously this will cancel and we can write U infinity square by L. So, you can see in the left hand side, these two inertia terms are of same order, right. So, we cannot drop any term in the left hand side.

Now, let us consider the viscous term. So, we have two viscous terms and let us see, what are the order of the each term. So, if you consider the first term del 2 u by del x square, ok. So, what is the order? So, it will be U infinity by L square, ok. And the second term if you see. So, it will be del 2 u by del y square, so order of U infinity by y; order is delta, so delta square, ok.

So, now let us compare these two term. So, del 2 u by del x square divided by del 2 u by del y square, ok. So, it will be order of U infinity by L square divided by U infinity by delta square. So, you can see this will be order of delta by L square. So, we have already postulated that delta is much much smaller than L, ok.

So, as delta is much much smaller than L; so obviously you can see del 2 u by del x square will be much much smaller than del 2 u by del y square. So, you can see if you compare these two terms, this term and this term; so we can neglect this term del 2 u by del x square. So, del 2 u by del x square can be neglected.

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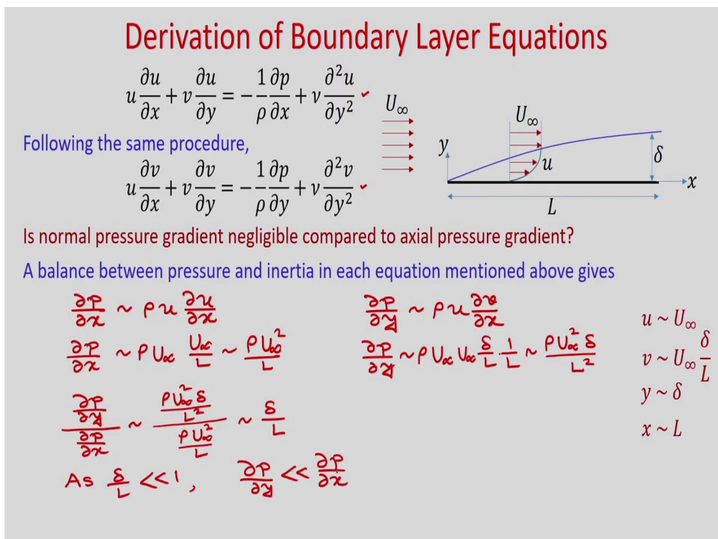

So, now, let us consider the y component of the momentum equation. And if you do the similar analysis, then you will find that inertia terms you cannot neglect; but in the viscous term, one term del 2 v by del x square can be neglected. So, we have already derived this equation and following the same procedure y component of momentum equation, you will get this.

Now, the question is that, is normal pressure gradient negligible compared to the axial pressure gradient? So, axial pressure gradient is del p by del x and normal pressure gradient is del p by del y. Can we neglect del p by del y compare to the del p by del x? So, what we will do?

First we will do a balance between pressure and inertia in each equation mentioned here. So, you can see that, if you write del p by del x ok. So, del p by del x ok, we can write it is order of one inertia term; so rho u del u by del x, right. So, we can write del p by del x order of, so rho u order of U infinity U infinity divided by L. So, it will be rho U infinity square by L.

Now, consider the normal pressure gradient. So, del p by del y, ok. So, del p by del y if you compare with one inertia term; so you can write rho u del v by del x, ok. Both the inertia terms are of same order, so you can compare with any of the inertia terms. So, del p by del y. So, it will be order of rho. So, u U infinity v U infinity delta by L and x is 1 by L.

So, you can see this we can write as rho U infinity square delta by L square, ok. Now, compare these two terms; so we can write del p by del y divided by del p by del x ok order of. So, you can see del p by del y is rho U infinity square delta by L square divided by rho U infinity square divided by L. So, you can see rho infinity square will get cancel, then you will get delta by L, ok. So, as delta by L is much much smaller than 1; so you can write del p by del y will be much much smaller than del p by del x, ok.

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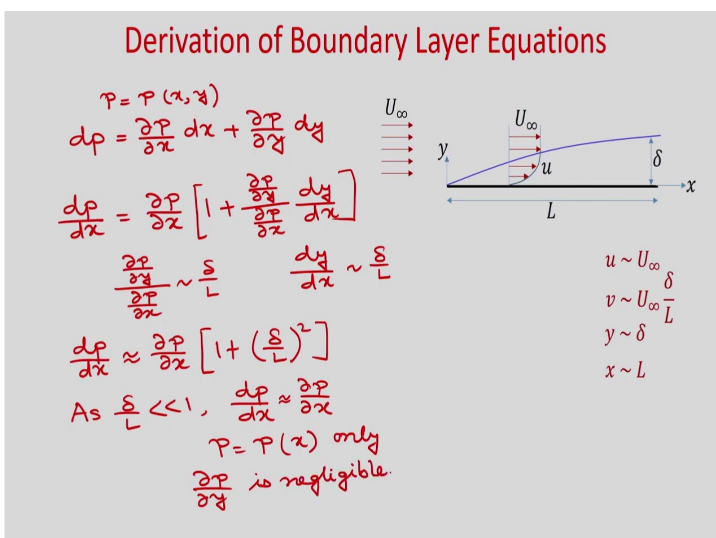

So, from here we can see that, from the order of magnitude analysis that del p by del y is very very small compared to the axial pressure gradient del p by del x. So, now, let us write p as function of x and y and we can find whether we can drop the normal pressure gradient in the governing equation, ok.

So, what we can write? We can write dp is equal to del p by del x dx plus del p by del y dy, ok. So, dp by dx we are dividing by dx. So, we can write del p by del x and you take outside. So, it will be 1 plus del p by del y divided by del p by del x dy by dx, ok. So, now, you can see, already we have seen that del p by del y divided by del p by del x is order of delta by L, and dy by dx this is also order of delta by L.

So, it will be delta by L whole square, right. So, dp by dx will be del p by del x 1 plus delta by L whole square. So, as delta by L is much much smaller than 1, so we can write that dp by dx is del p by del x; that means p is function of x only, p is function of x only and del p by del y is negligible.

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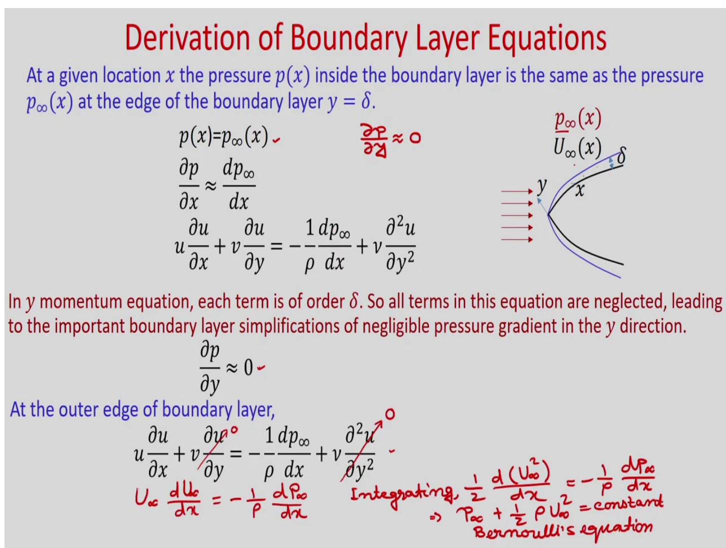

So, you can see that p is function of x only and the normal pressure gradient del p by del y can be neglected, ok. So, now we have the pressure gradient term in the x component of the boundary layer equation whatever till now we have derived, so that we can write as dp by dx, ok. So, at a given location x, if the pressure p x inside the boundary layer is the same as the pressure p infinity x at the edge of the boundary layer y is equal to delta.

So, you can see that we have already derived that del p by del y is equal to 0 right, it is negligible. So, this means that the pressure across the boundary layer does not change and the pressure is impressed on the boundary layer. So, obviously you can see that, at any location inside the boundary layer if that is the pressure p x; then that will be equivalent to the outside pressure p infinity x at this location ok, that means at that x location.

So, that means we can write p x is equal to p infinity x, where p infinity is the pressure outside the boundary layer; as pressure is impressed inside the boundary layer, so in the normal direction pressure will not change. So, obviously at any location x inside the boundary layer; if you consider that pressure is p x, that will be equivalent to the outside pressure p infinity at that location x.

So, that means we can write del p by del x is or is equivalent to dp infinity by dx. So, in the x component of the momentum equation; now we can write instead of del p by del x, we can write dp infinity by dx. And in the y component of momentum equation if you see that, each term the inertia terms, viscous term, and the pressure gradient term; so that will be order of delta, ok.

So, each term in the equation will be order of delta; so obviously we can neglect the other terms, so that your it will become del p by del y is equal to 0. So, in y momentum equation each term is of order delta. So, all terms in this equation are neglected; leading to the important boundary layer simplifications of negligible pressure gradient in the y direction, so del p by del y will be 0, ok.

So, now you can consider that at the outer edge of the boundary layer we can simplify this equation, right. So, if you consider outside of this boundary layer, which is your inviscid region. So, now, in that region, obviously you can see that velocity will be U infinity ok and U infinity we can write dU infinity by dx. So, outside this boundary layer, v will be 0 ok or you can write del u by del y is 0; because there will be no change of U infinity right, U infinity function of x.

So, there will be no change of velocity in the y direction. So, this will be 0, ok. And this you can write 1 by rho dp infinity by dx. And as del u by del y is 0 outside this boundary layer, so del 2 u by del y square also will be 0, ok. So, now, you can see that you got this 1 by rho dp infinity by dx, you can write U infinity dU infinity by dx, where U infinity is function of x, ok.

So, now if you integrate this equation, what we will get? Integrating what we will get? So, you can see that it will be, we can write half dU infinity square by dx is equal to minus 1 by rho dp infinitely by dx. So, if you integrate it, you will get p infinity is it plus half rho U infinity square is equal to constant, ok.

So, obviously you can see, this is the well-known Bernoulli's equation, ok. So, in the inviscid region, we have this equation which is known as Bernoulli's equation. So, now, we can substitute this the axial pressure gradient term with the, with this ok, if U infinity is function of x.

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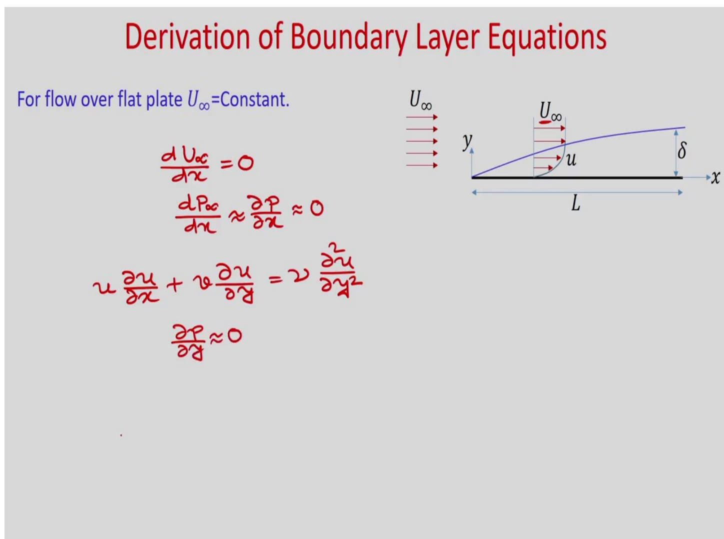

Now, let us consider a special case where we are considering flow over a flat plate. So, if it is external flow and we are considering flow over flat plate. So, the free stream velocity U infinity which is the velocity outside the boundary layer that remains constant; that means U infinity is constant.

So, obviously you can see that, gradient with respect to the axial direction dU infinity by dx will be 0. Hence the pressure gradient dp by dx will be 0. So, if you consider for flow over flat plate, where U infinity is constant; that means dU infinity by dx is equal 2 0, ok. So, outside this boundary layer this free stream velocity will remain same ok, it is not function of x. So, it is constant.

So, obviously if dU infinity by dx is 0; so obviously you can see that dp infinity by dx which is equivalent to del p by del x, so it will be also 0. So, for flow over flat plate, we can write this x component of momentum equation as u del u by del x plus v del u by del y is equal to nu del 2 u by del y square, ok. And we have del p by del y as 0.

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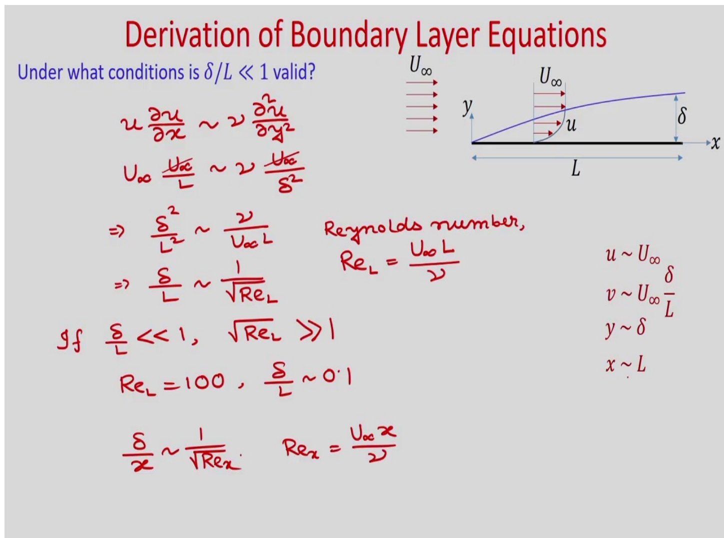

So, now we postulated that delta by L is much much smaller than 1. So, under what condition this is valid? Ok. So, for this now let us compare one inertia term with the viscous term. So, we have inertia term u del u by del x ok; each inertia term is of same order, so you can write any inertia term we can take. So, u del u by del x. So, we will compare with the viscous term. So, that will be nu del 2 u by del y square.

So, obviously we can compare, because these terms are already in the governing equations. And those cannot be drop through, all terms should be of the same order; we are comparing the inertia term with the viscous term, ok. So, now, we put the scale. So, we know the scale of u, v, y, and x. So, U infinity U infinity by L of order of nu, where nu is the kinematic viscosity of the fluid, U infinity by delta square, ok.

So, from here you can write delta square by l square of order of. So, 1 U infinity if you cancel, then you will get nu by U infinity L; because one L we have written here, so it will be nu by U infinity by L. So, now, if we define the Reynolds number, ok Reynolds number is the ratio of inertia force to the viscous force. So, Reynolds number Re based on L; then we can write the free stream velocity U infinity characteristics length L, which is length of the plate divided by the kinematic viscosity nu, ok.

So, you can see that we can write delta by L is order of 1 by root Re L, ok. So, if delta by L is much much smaller than 1; so obviously, root Re L should be much much greater than 1, ok. So, if it is so, you can just see. If Reynolds number is 100, then what will be the delta by L? So, delta by L will be 1 by root 100. So, it will be 0.1, ok.

So, you can see that when Reynolds number is high at least greater than Reynolds number greater than 100; then obviously this delta by L much much smaller than 1 is valid. So, you can see that this boundary layer approximation is valid at high Reynolds number.

So, we have shown here that, when Reynolds number is greater than 100; then obviously we can have the this approximation delta by L much much smaller than 1 valid. So, similarly, so at any length x if you define delta by x. So, you can write 1 by root Re x right, where Re x the Reynolds number based on the length x. So, U infinity x by nu, ok. So, you can write delta by x is order of 1 by root Re x.

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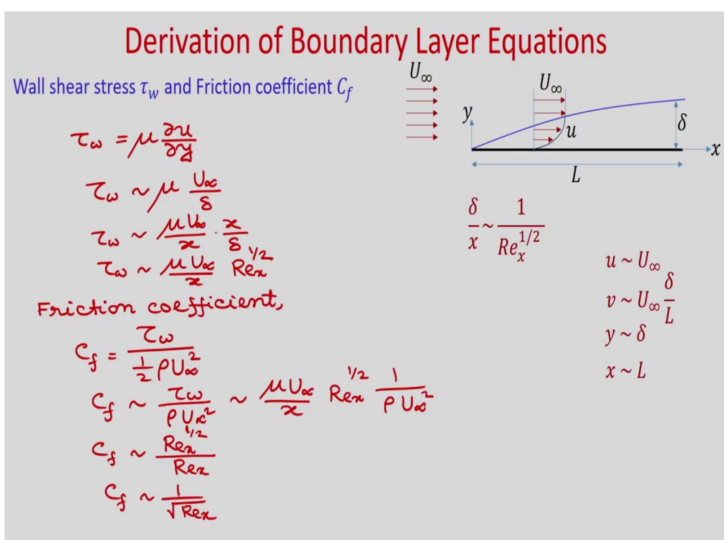

So, when we consider the fluid flow over a surface; then obviously we are interested to know what is the shear stress acting on the surface and what is the friction coefficient which is the measure of non-dimensional shear stress, so that we want to measure, ok.

So, if we see that the shear stress tau w ok, we can write for this flow over flat plate as mu del u by del y, right.

So, tau w will be order of mu u is U infinity, y is delta. So, tau w will be mu U infinity by x and x by delta, ok. So, delta by x already we have found 1 by root Re x, right. So, you can see that, we can write tau w as order of mu U infinity by x Re x to the power half, ok.

Now, the friction coefficient, so that is the non-dimensional representation of the shear stress. So, you can write c f is equal to tau w by half rho U infinity square. So, for this particular case, now tau w already we have seen. So, c f will be order of tau w by rho infinity square. So, tau w already you have seen that mu U infinity by x Re x to the power half and we have 1 by rho U infinity square.

So, you can see U infinity U infinity, one infinity will be there and mu by rho, it will be nu. So, you will get c f as order of Re x to the power half and we will have Re x ok in the denominator. So, you can see that c f will be 1 by root Re x, ok. So, you can see that delta by x is ordered upon by root Re x and similarly the friction coefficient c f is also order of 1 by root Re x.

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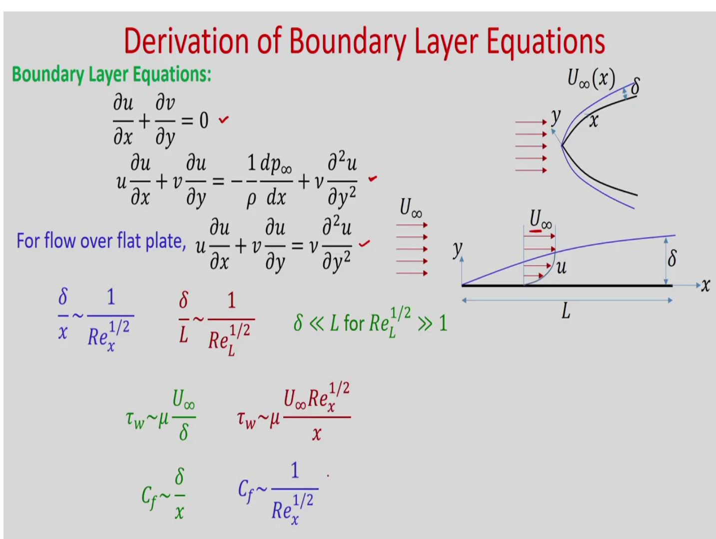

So, now let us summarize what we have studied in today's class. So, first we introduced the boundary layer. So, this is the thin layer near to the solid surface when fluid flow is taking place over a surface. German scientist Prandtl first introduced the concept of boundary layer and we used the order of magnitude analysis, which is known as scale analysis and we have dropped few terms in the Navier Stoke equations.

So, what is the condition when we can have the assumption of boundary layer flow? The first is that, the geometry should be cylinder and there should not be any flow separation; and second thing is that it should be high Reynolds number flow. And we have shown that the boundary layer thickness is very very small compared to the characteristics length. So, delta by L is much much smaller than 1 when Reynolds number is very high.

So, finally, we have shown that the pressure gradient term in the normal direction is 0 and we have the boundary layer equations in general for any curved surface as; this is the continuity equation del u by del x plus del v by del y is equal to 0, and this is the boundary layer momentum equation u del u by del x plus v del u by del y is equal to minus 1 by rho dp infinity by dx plus nu del 2 u by del y square and the normal pressure gradient is 0.

And for a special case when we consider flow over flat plate, then the free stream velocity U infinity is constant. So, this we can simplify dropping the term dp infinity by dx. We have also shown that delta by x is order of 1 by root Re x and for any plate length L, delta by L we can write order of 1 by root Re L. And delta is much less than L, when root Re L is much much greater than 1.

And tau w we have shown that it is order of mu infinity by delta and c f is order of delta by x and c f is also ordered of 1 by root Re x. So, in later class when we considered the flow over flat plate, we will find the value of this skin friction coefficient c f and the boundary layer thickness delta.

Thank you.