Viscous Fluid Flow

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Module - 06
Lubrication Theory
Lecture - 22
Reynolds Equation for Lubrication

Hello everyone. So, in today’s class, first we will simplify the fluid flow governing equation for the application of lubrication and then will derive the Reynolds Equation for Lubrication. As you know that, lubrication makes the relative motion between two surfaces very smooth; it reduces the friction and it minimizes the wear. You know that in the application of tribology like slider bearing.

The relative motion between two surfaces causes excessive wear; therefore, it is very much essential to reduce the normal stresses imposed by applied load and to reduce the shear stresses induced due to the relative motion. So, first let us classify the fluid film lubrication.

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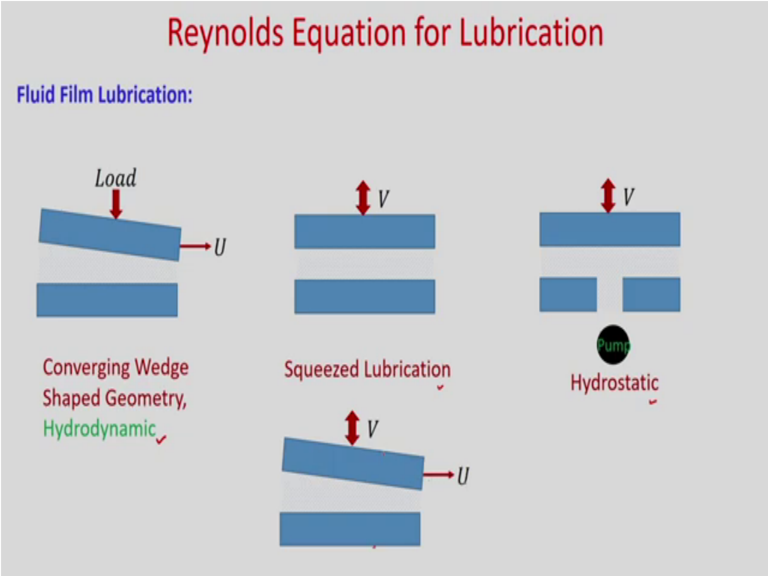

So, you can see that we can classify this fluid film lubrication as hydrodynamic, squeezed film, and hydrostatic. In case of hydrodynamic, you can see it is a converging wedge shaped geometry and there is a applied load and it is having the tangential velocity. So, obviously there will be normal stresses as well as the shear stresses and inside between these two surfaces, there is lubricant.

In squeeze lubrication, there is no tangential stresses, only there is a normal motion of this surfaces and inside you have the lubricant; obviously you can see that in this cases, viscosity of lubricant plays an important role to support the load. In hydrostatic you can see that, external pressure of fluid needs to be supplied to generate hydrostatic lubrication; therefore one pump is used.

So, in this cases you can see that, this lubricants plays an important role and we need to find the governing equations for these lubricants. In addition you can see, we can have both the velocities in normal direction as well as in tangential direction and both the surfaces may move. But if you see that if you write in terms of the relative motion; so obviously you can make the bottom surface stationary. And you make the relative vertical motion as well as the tangential motion accordingly.

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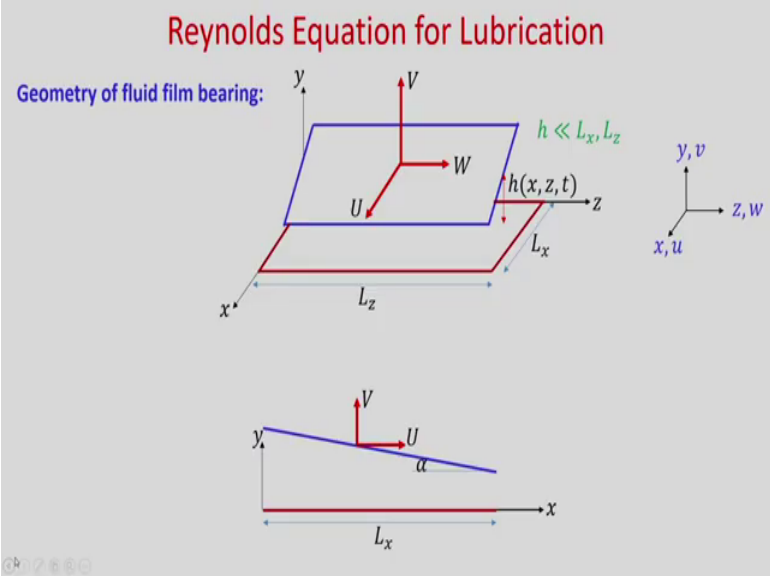

So, this is the geometry of fluid film bearing. So, we can see that this bottom surface is stationary and we have a this upper surface which is inclined and we have the relative motion as U, V, W in x, y and z direction respectively. So, we have given this relative motion on the upper wall and we have made the bottom surface as stationary.

And obviously, as the upper surface is inclined; so the height from the bottom surface, this h is function of x, z, and time t. And in this case, if the length of the surface in x and z direction are L x and L z respectively. And this height h is very very small compared to L x and L z. So, if you take one x y plane; then you can see in x y plane, it will look like this.

So, we have this tangential velocity U in the x direction and we have the normal velocity V in y direction and length is L x. And if the upper surface is inclined and it is making very small angle alpha and W is also the tangential velocity in z direction. So, in this case you can see, one or both bodies may be moving; but we are making the upper surface moving and giving the relative velocities with respect to the bottom surface, which we have made as stationary.

So, to study this type of fluid film lubrication, we need the governing equation of fluid flow and first let us write the governing equation, which we have derived in Cartesian coordinate.

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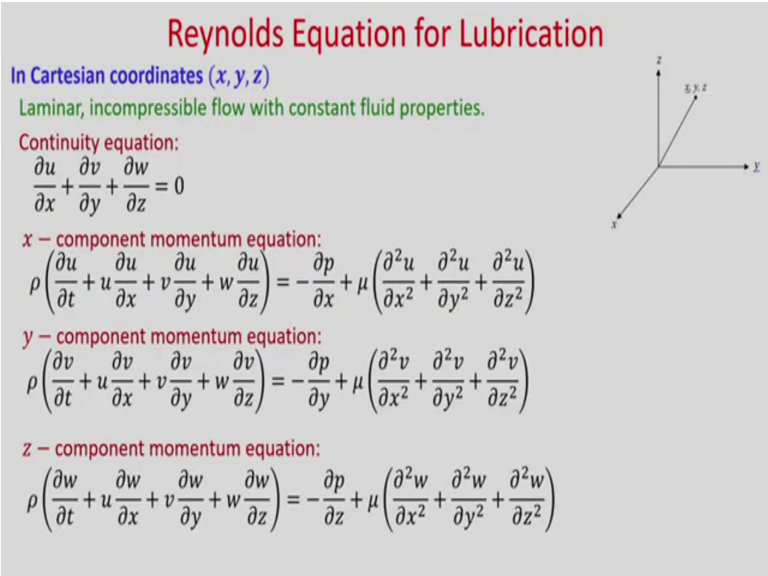

So, we can see for laminar incompressible flow with constant fluid properties, this is the continuity equation, where u, v, w are the velocities in x, y and z directions respectively. And this is the x component momentum equation, where left hand side is the inertia term and right hand side this is the first pressure gradient term. And this is the viscous term, where rho is the density of the fluid and mu is the viscosity of the fluid.

Similarly, this is the y momentum equation and this is the z momentum equation. And you can see these are coupled and non-linear and it is second order partial differential equations. So, now, these equations will simplify for the application of this lubrication theory. So, first let us see that, whether some terms in the governing equation, whether we can drop in the application of this lubrication theory.

You can see that, this is the viscous dominated flow; because viscosity plays an important role. So, first let us write down the non dimensional form of the governing equations with a suitable scale. And in this case as viscous force is the dominating force; so the pressure will non dimensionalize based on the viscous force.

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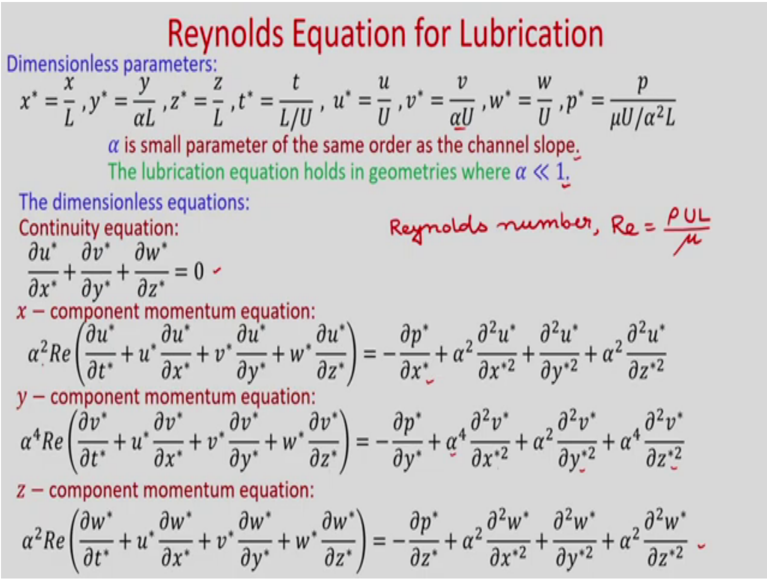

So, we can see we are using this dimension less parameters. So, x star which is your non dimensional x coordinate that is x by L, where L is the characteristic length. And y star you can see that, the height is very small compared to the length in the x and z direction. So we will take this y star as y by alpha into L, where alpha is small parameter of the same order as the channel slope, and the lubrication equation holds in geometries where alpha is much much less than 1.

So, in this case as you know that in y direction, we have the height which is function of x z and t; so we are taking in the denominator alpha into L; z star is equal to z by L; t star is equal to t by L by U, where U is the characteristic velocity; u star is equal to u by U; v star is equal to v by alpha U, because in the y direction the velocity v is very small you can see, so we are taking a very small velocity alpha U.

W star is equal to w by U and p star we are taking p by mu U by alpha square L. So, these we have non dimensionalized using the viscous force; as you know the lubrication flow is viscous dominated flow. So, now if you use these dimensionless parameters and if you put in the dimensional governing equation; then you can write these dimensionless equations.

So, this will be your continuity equation; after rearrangement you can see, you will get this x component momentum equation, where you have alpha square Re, where Re is the Reynolds number, Reynolds number, where Re is rho is the density of the fluid, U is the characteristic velocity, L is the characteristic plane divided by the viscosity of the fluid.

So, in the left hand side, it is multiplied with alpha square Re and in the right hand side, this is the non dimensional pressure gradient. And you can see in the viscous terms, this is alpha square del 2 u by del x square plus del 2 u star by del y star square plus alpha square del 2 u star by del z star square.

So, in this case star represents the dimensionless parameters. So, similarly if you make this y component momentum equation; you will get in the left hand side alpha to the power 4 into Re. And in the right hand side you can see, these are the terms, where in all viscous terms we have these alpha to the power 4 with this term, alpha square is this term with alpha to the power 4 is multiplied with this term.

And z component momentum equation you will get like this. So, if you can see that, for the application of this lubrication. So, the lubrication equation holds in geometries were alpha is much much smaller than 1; so obviously if alpha is tending to 0, so alpha Re will be also tending to 0 or alpha square Re will be tending to 0.

So, you can see the left hand side, the inertia terms you can neglect in the application of this lubrication theory. So, and also in viscous term you can see as alpha is very very small close to 0; so this term and this term also you can drop, ok. And in y and z momentum equations also similarly, the left hand side terms you can drop. And in right hand side for y momentum equation you can see, we have alpha to the power 4 alpha square and alpha to the power 4, so obviously these terms you can drop.

And in the z component of momentum equation, this term and this term you can drop as alpha is very small.

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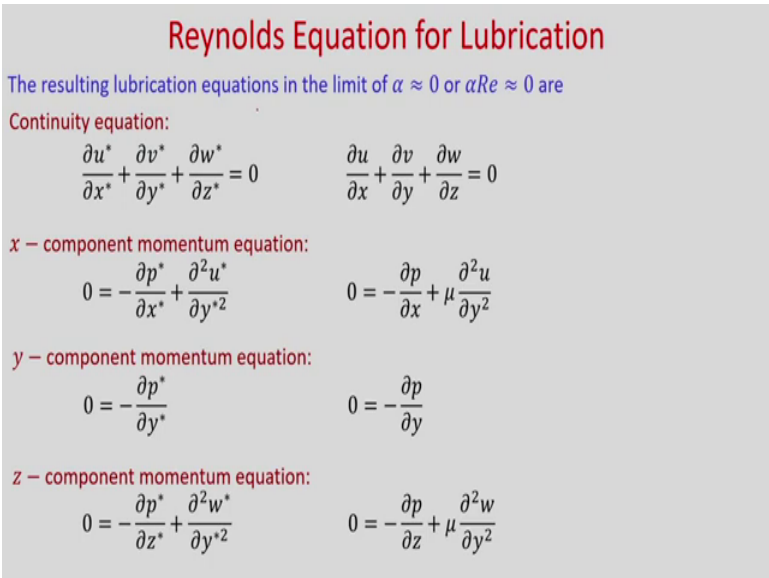

So, the resulting lubrication equations in the limit of alpha tending to 0 or alpha Re or alpha square Re tending to 0. So, are this is the dimensionless continuity equation and if you write in terms of dimensional continuity equation, so it will look like this. So, it is remaining same, x component momentum equation dropping the other terms; so you can get this equation, ok. So, in dimensional form, this is the equation.

And from y component momentum equation, you will get del p by del y is equal to 0; that means in the y direction, you can have the pressure uniform. So, that del p by del y will be 0. And z component momentum equation, you will get like this. So, now, we have derived the simplified governing equation in the application of lubrication theory. Reynolds use this equations and estimated the equation for pressure in the application of this lubrication theory.

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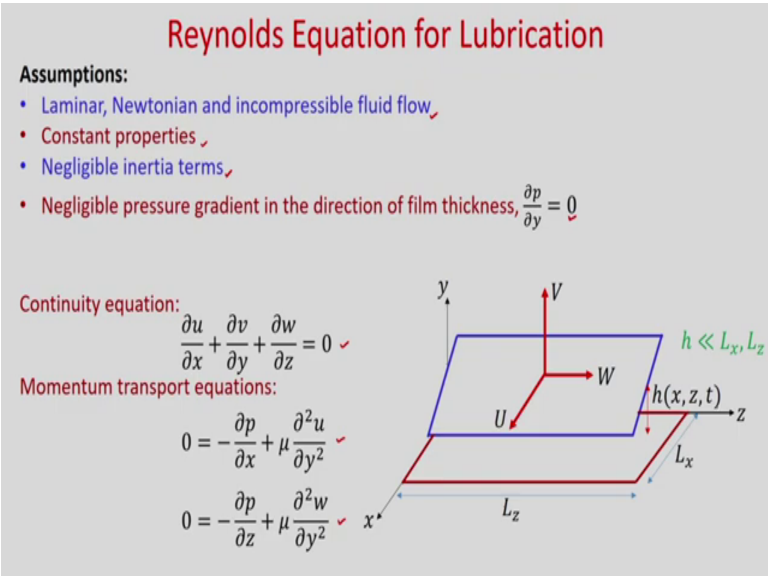

So, now, we will derive with the following assumptions; laminar, Newtonian and incompressible fluid flow with constant properties. Now, we are neglecting the inertia terms, so that we have already derived those equations and obviously, negligible pressure gradient in the direction of film thickness. So, del p by del y is equal to 0.

So, we derived this equations, this is the continuity equation and these are the momentum transport equations. And this is the geometry and the relative motion of the upper surface with respect to the bottom surface, which we have made stationary. And the velocities in x, y, z directions are U, V and W respectively and h which is the height is function of x, z and time t.

So, now, we have the governing equations. So, first let us integrate these equations with proper boundary conditions and find the velocity distribution in terms of the pressure gradient.

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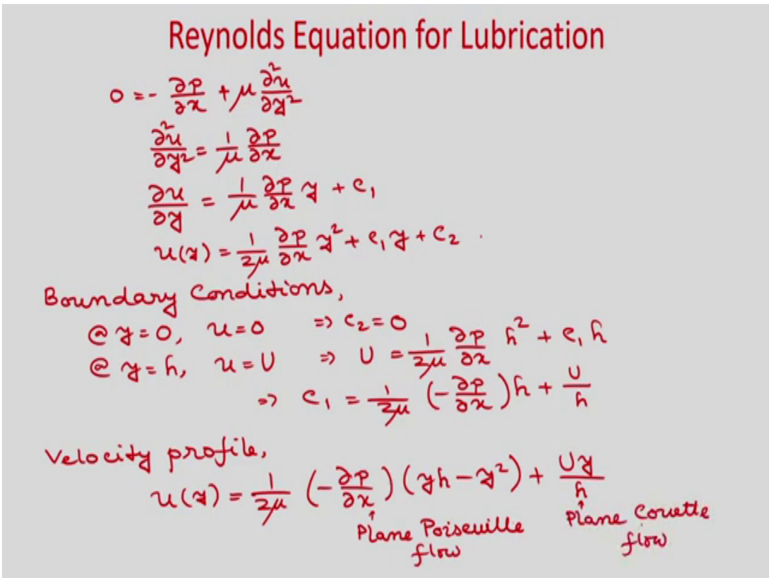

So, first let us consider the x component momentum equation, which is your 0 is equal to minus del p by del x plus mu del 2 u by del y square. And you can see that this equation, you can integrate and you can write del 2 u by del y square is equal to 1 by mu del p by del x; and you can write del u by del y is equal to 1 by mu del p by del x y plus c 1. So, at a particular time, obviously you can assume the pressure gradient del p by del x as constant.

So, you can integrate and write this equation. Again if you integrate, then you will get the velocity profile u as 1 by twice mu del p by del x y square plus c 1 y plus c 2. So, c 1 and c 2 are the integration constants. And we can find these with the boundary conditions. So, what are the boundary conditions?

So, you can see at y is equal to 0, the bottom surface is stationary. So, u is equal to 0. So, if you put here u is equal to 0 at y is equal to 0, that will give c 2 is equal to 0. And at y is equal to h, we have the velocity u is equal to capital U. So, it will be just U is equal to 1 by twice mu del p by del x h square plus c 1 into h. So, that will give c 1 is equal to 1 by twice mu minus del p by del x h plus U by h.

So, if you put this constants c 1, c 2 in this equation; so we will get the velocity profile u y is equal to 1 by twice mu minus del p by del x y h minus y square plus U y by h. So, you have already derived this velocity profile when we solved the exact solution for combined Poiseuille and Couette flow.

So, you can see this is the super position of two solutions. So, this is the solution from plane Poiseuille flow and this is the solution from plane Couette flow. So, this is solution from plane Poiseuille flow and this is solution from plane Couette flow.

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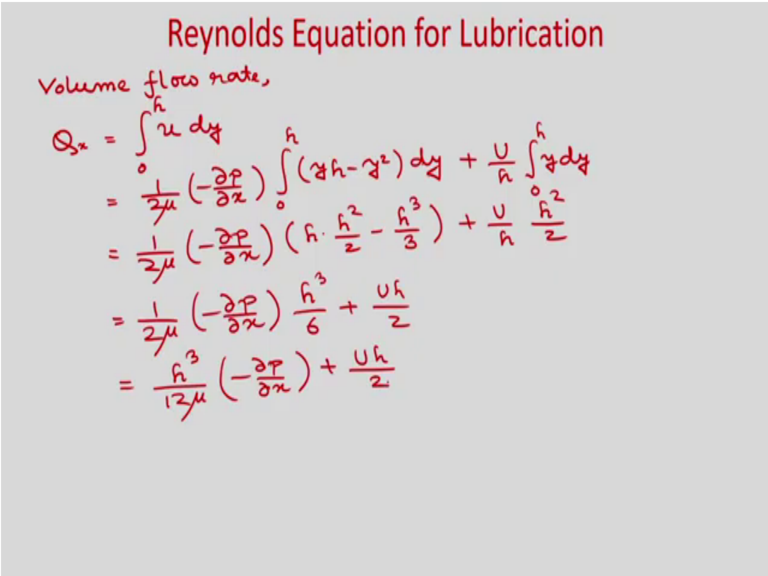

So, now first let us find the volume flow rate at a particular location x. So, volume flow rate, let us say Q x at a particular location x, we are finding this volume flow rate; so integral 0 to h u d y, ok. So, it will be 1 by twice mu minus del p by del x, which is constant; you can take outside the integral, integral 0 to h y h minus y square d y plus U by h integral 0 to h y d y.

So, 1 by twice mu minus del p by del x. So, if you integrate this, so you will get y square by 2 and if you put the limits 0 to h, so you will get h square by 2. So, you will get one h and h square by 2. And it will be y cube by 3, so minus h cube by 3 plus U by h and it will be y square by 2, so h square by 2. So, we can write it as 1 by twice mu minus del p by del x. So, it will be h cube by 2 minus h cube by 3. So, it will be h cube by 6 plus U h by 2. So, you can write as h cube by 12 mu minus del p by del x plus U h by 2.

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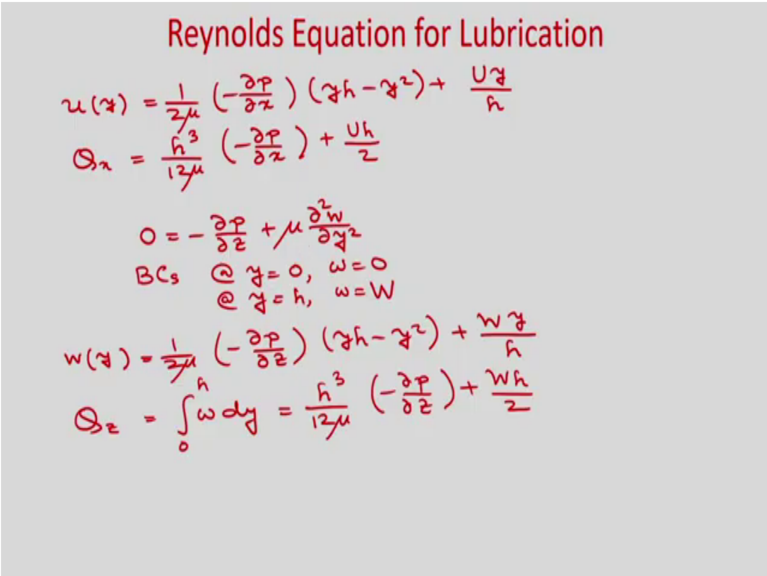

So, now similarly you can derive the velocity profile w as function of y and the volume flow rate Q z at a particular location z. So, we have derived this velocity profile u as 1 by twice mu minus del p by del x y h minus y square plus U y by h. And Q x as h cube by 12 mu minus del p by del x plus U h by 2. So, similarly we can write, we have the other governing equation as 0 is equal to minus del p by del z plus mu del 2 w by del y square.

So, now, we have the boundary conditions at y is equal to 0, w is equal to 0 and y is equal to h, w is equal to capital W. So, now, if you integrate and apply the boundary conditions; similarly you will get the velocity profile w y as 1 by twice mu minus del p by del z y h minus y square plus W y by h. And similarly you can write the volume flow rate at any location z as integral 0 to h w d y is equal to h cube by 12 mu minus del p by del z plus W h by 2.

So, in this derivation, obviously you have seen the fluid viscosity is assumed uniform across the film thickness, which is we already made the assumption.

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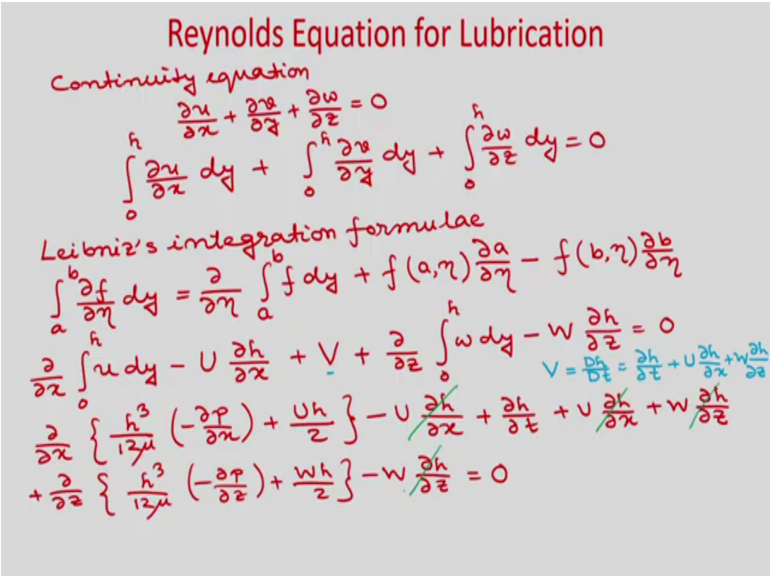

Now, let us write the continuity equation and derive the Reynolds equation. So, the continuity equation we have del u by del x plus del v by del y plus del w by del z is equal to 0, ok. So, now, we will integrate it between 0 to h. So, we can write integral 0 to h del u by del x d y plus integral 0 to h del v by del y d y plus integral 0 to h del w by del z d y is equal to 0. So, now, we will use this Leibniz integration formulae to express this integral and we will take the derivative del of del x outside the integral.

So, you know this Leibniz integration formulae. So, if you have integral a to b del f by del eta d y; so you can write it as del of del eta outside the integral a to b f d y plus f a, eta del a by del eta minus f b, eta del b by del eta, where a and b are the integration limits. So, now, you can see that the first one, obviously what we can write using this Leibniz integration formulae.

So, we can write it as using these, we can see that del of del x integral; so it will be 0 to h u d y, right. And now we have this a is 0. So, this term will become 0 and b is h, so this term will become. So, it will be u at y is equal to h. So, what is that? So, that is nothing, but the tangential velocity in x direction U and we have del h by del x, ok.

This term now we can see that, you will can write it as integral 0 to h del v and you will get v at h minus v at 0. So, at v at y is equal to 0; obviously it is 0 and v at y is equal to h, it is capital V. So, we will write capital V, which is the normal velocity; plus now these we can write as del of del z integral 0 to h w d y.

And now this term at, obviously a is 0 here; so this term will become 0 and this term will become at f, means in this case w at y is equal to h. So, that is minus W del h by del z is equal to 0, ok. So, now you can see this term integral 0 to h u d y it is nothing, but the volume flow rate at a particular location x and similarly integral 0 to h w d y is the volume flow rate at a particular location z.

So, this we have already derived; so you can write that term here. So, you can write del of del x. So, h cube by 12 mu minus del p by del x plus U h by 2; then we have minus U del h by del x. Now, let us write the velocity v in terms of the height h. So, obviously you can see, if we use the material derivative; then v is the velocity which is nothing, but the in terms of the Lagrangian frame del D h by D t.

And this we can write as V which is nothing, but D h by D t and that we can write as del h by del t and we have velocity, so there will be special change. So, U del h by del x plus W del h by del z. So, this velocity V now we will write these terms. So, we can write plus del h by del t plus U del h by del x plus Q del h by del z. And this term, so we have already evaluated the volume flow rate.

So, you can write plus del of del z h cube by 12 mu minus del p by del z plus W h by 2 and we have minus W del h by del z is equal to 0. So, now, you can see these terms will get cancels. So, u minus U del h by del x, it is plus U del h by del x and this is plus W del h by del z and this is minus W del h by del z.

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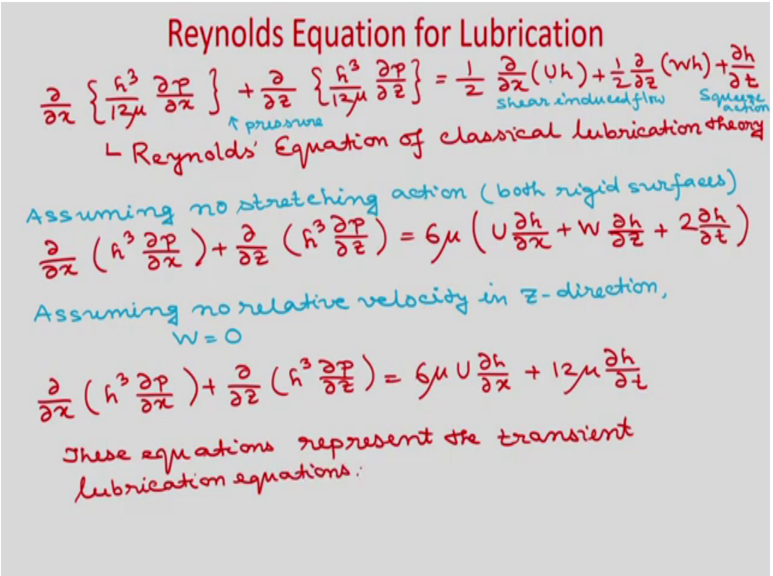

So, now after simplification, we can write the equation as del of del x h cube by 12 mu del p by del x plus del of del z h cube by 12 mu del p by del z is equal to half del of del x of U h plus half del of del z w h plus del h by del t. So, this is the Reynolds equation for this classical lubrication theory and you can see that, we can determine the pressure distribution inside this slider bearing solving this equation with proper boundary conditions.

So, if you can see left hand side denotes the pressure terms, ok. So, you can see these terms. So, this is known as Reynolds equation of classical lubrication theory. So, in general, we have derived this equation and you can see that, obviously these terms in the left hand side are pressure terms, ok.

So, these are pressure terms, ok. And so, these are obviously, hydro dynamic pressure and you can see this first two terms in the right hand side are the shear induced flow by the surface sliding with velocity U. So, these are shear induced flow and you can see that del h by del t.

So, in the normal direction, what is the change of height with respect to time; so it is giving the action of squeeze, so it is squeeze action, ok. So, right hand side terms are treated as the source term when you are solving for the pressure to find the pressure distribution.

So, you can see that in the right hand side two terms, we have written as del of del x U h and del of del z w h. So, if there are no stretching terms like say in the rubber; you can see that obviously at different places, it may have different velocities, velocities U, w may not be constant. But now if you neglect the stretching action; then obviously w and U you can take outside the derivative.

So, you can see that, assuming no stretching action, ok. So, we are assuming that both are rigid surface ok; we are assuming that both are rigid surface. So, this u and w you can take outside the derivative. So, we can write as del of del x, 12 mu you can take in the right hand side. So, you can write h cube del p by del x plus del of del z h cube del p by del z is equal to 6 mu U del h by del x plus W del h by del z plus 2 del h by del t.

Now, if you assume that there is no relative velocity in z direction; then we can simplify this equation as. So, assuming no relative velocity in z direction; so you can make that W is equal to 0. So, you can simplify this equation as del of del x h cube del p by del x plus del of del z h cube del p by del z is equal to 6 mu U del h by del x. So, only we have this U as tangential velocity in the x direction plus 12 mu del h by del t, ok.

So, obviously you can see that these equations represents the transient lubrication equations. So, these equations represent the transient lubrication equations. So, in today’s class, we first wrote the simplified fluid flow equation for lubrication theory. And then we considered slider bearing and where we have the relative velocity at the upper surface as U, V, W in x, y, z direction respectively, where U and W are the tangential velocity and V is the normal velocity, which gives the stretching action in the lubricant.

And we have made the bottom surface as stationary. So, in this case in the simplified governing equations, which we derived for this application; we integrated the equation with proper boundary condition. We found the velocity profile U as function of y and velocity W as function of y. And from there we have derived the volume flow rate at a particular x location and in y z plane and volume flow rate at a particular z location in x y plane.

Then we considered the continuity equation and we integrated this equation between 0 to h, where h is the height of the lubricant or the height between the two surfaces and we derived the Reynolds equation for this classical lubrication theory. If you solve this equation, then you will be able to find the pressure distribution inside the lubricant. Then we simplified for no stretching action keeping this U and W outside the derivative.

And further we simplified that, if there is no tangential velocity in the z direction, W is equal to 0; we have written the Reynolds equation which is the tangent lubrication equations.

Thank you.