Viscous Fluid Flow

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Module - 06
Lubrication Theory
Lecture - 01
Creeping Flow Around a Sphere

Hello everyone, so in this module we will study Lubrication Theory, it is well known that two solid bodies can slide over one another easily when there is a thin film of fluid sandwiched between them. The analysis of fluid flow in thin layer is known as lubrication theory. One such example is the motion of fluid flow in thin layer in bearing of a shaft.

Lubrication theory is the major disciplines where the starting point is Creeping Flow approximation. So, what is creeping flow? The creeping flow is the flow where the Reynolds number based on a suitable linear dimension is very very small; that means, Reynolds number will be much much less than 1. So, before going to study this lubrication theory, first let us derive the governing equation for creeping flow and we will study the Stokes flow past a sphere.

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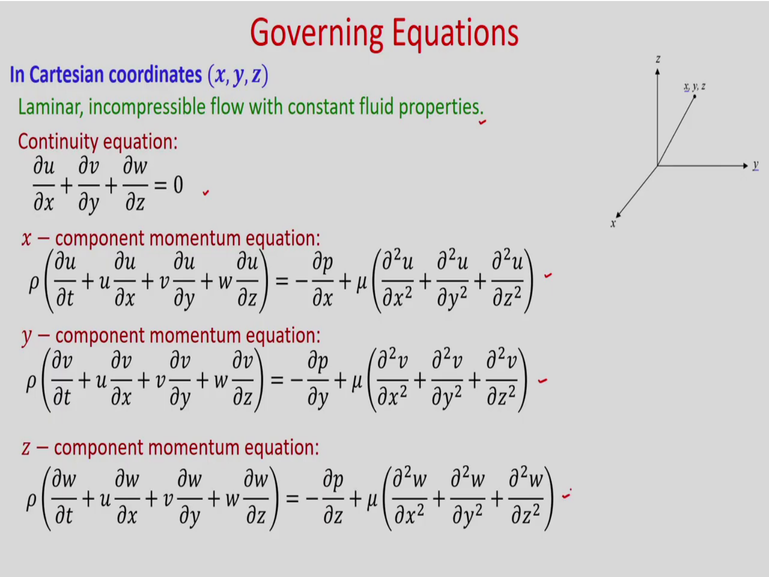

So, you know these are the governing equations in Cartesian coordinate for laminar incompressible fluid flow with constant fluid properties. So, this is the continuity equation, this is the x component momentum equation, this is the y component momentum equation and this is the z component momentum equation.

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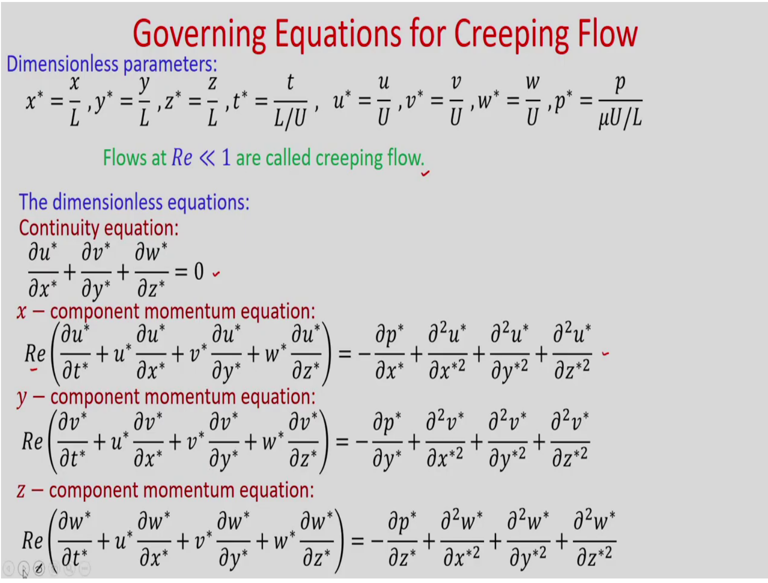

So, the basic assumption of Creeping Flow developed by Stokes is the density associated terms are negligible in the momentum equation. So, we will use this governing equation and we will use the suitable non dimensional parameters and for very very small Reynolds number will simplify the governing equations.

In this case the pressure will be non dimensionalized using the viscous force not the inertia force, because it is a very slow motion flow. So, these are the dimensionless parameter we will use where L is the characteristic length and U is the characteristic velocity.

So, x star is equal to x by L, y star is equal to y by L, z star is equal to z by L, t star is equal to t by L by U and the velocities are non dimensionalized using the characteristic velocity U and the pressure will non dimensionalize it using the viscous force mu U by L.

So, you can see that flows at low Reynolds number are called creeping flow and if you use this dimensionless parameter then we can write these non dimensional equation. So, these are the continuity equation, this is the x momentum equation. So, you can see in the left hand term this Reynolds number will appear.

So, this term is multiplied with the Reynolds number. So, in y and z momentum equations also you will find that a Reynolds number is multiplied with the left hand side terms. So obviously, you can see that if it is a very slow motion flow where Reynolds number is very very small, so if Reynolds number tends to 0 then obviously, you can drop these terms in the left hand side.

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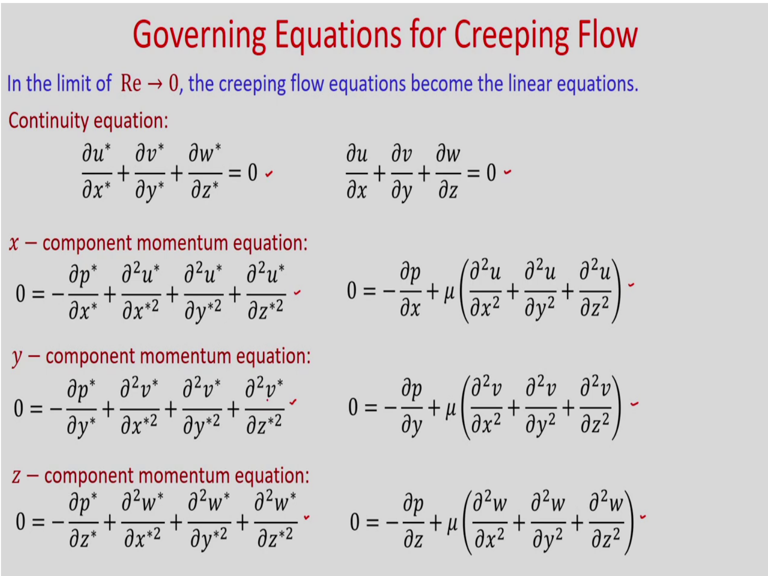

So, in the limit of Reynolds number tends to 0, the creeping flow equations becomes the linear equations, because left hand side terms become 0. So, this is the non dimensional equations, continuity equation and this is the dimensional form of the continuity equation, this is the x component momentum equation dropping the left hand side terms, this is the y and z component momentum equations.

And in the right hand side we have written the corresponding dimensional equations. So, you can see here we have dropped the inertia terms.

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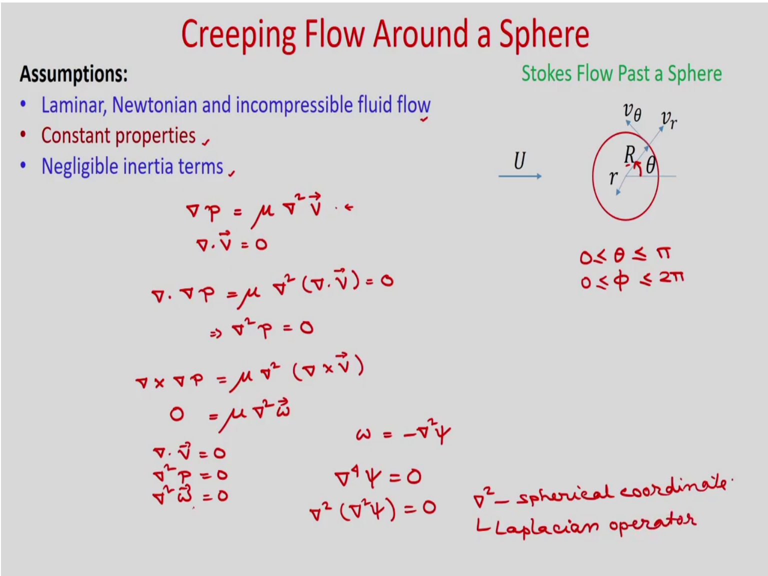

So, now consider the creeping flow over a sphere. So, in this case we will assume Laminar, Newtonian and incompressible fluid flow with constant properties and negligible inertia terms. So, whatever equations we have derived obviously you can see that we can write it as grad p is equal to mu grad square v, where v is the velocity vector.

So, in this particular case you can see that we have the incompressible flows. We have continuity equation divergence of v is equal to 0 and now if you take the gradient in this equation then you can write divergence of grad p is equal to mu grad square divergence of v ok and divergence of v is equal to 0.

So, this term will become 0; that means, you will get grad square p is equal to 0. And if you take curl of this equation then we can write curl of grad p is equal to mu grad square curl of this velocity vector. So, in this particular case you can see curl of a gradient of a scalar. So, this will become 0. So, in right hand side you can write mu grad square and curl of velocity vector is nothing but the vorticity vector ok.

So, now you can see that from here we can write divergence of v is equal to 0, grad square p is equal to 0 and grad square omega is equal to 0. So, if you use one component of this vorticity vector, then we can write omega is equal to minus grad square psi. So, this is the in terms of stream function psi is the stream function. So, we can write omega is equal to minus grad square psi.

So, if you write it and if you put it in this equation then we will get grad to the power 4 psi is equal to 0 ok. So, which is the biharmonic equation and psi is the stream function. So, in this particular case we are considering flow over a sphere of radius R and you can see that if you consider the spherical coordinate then obviously, R is measured along the radius and theta is measured from here.

So obviously, v r is the radial velocity and with theta is the tangential velocity and obviously we will be assuming axisymmetric flow. So, in this case you can see in the phi direction it is axisymmetric. So, in this particular case obviously your theta will vary between 0 and pi and phi will vary between 0 to 2 pi ok and it is axisymmetric flow. So, for this case now these nabla square which is your, we can write as nabla square nabla square psi is equal to 0, in this case nabla square will write in spherical coordinate.

So in Cartesian coordinate it is easy, but in nabla square or which is your Laplacian operator. So, these we will write in spherical coordinate. Now, if you consider the spherical coordinate. And we are considering flow over a sphere then you know how to write the radial velocity v r and tangential velocity v theta in terms of the stream function.

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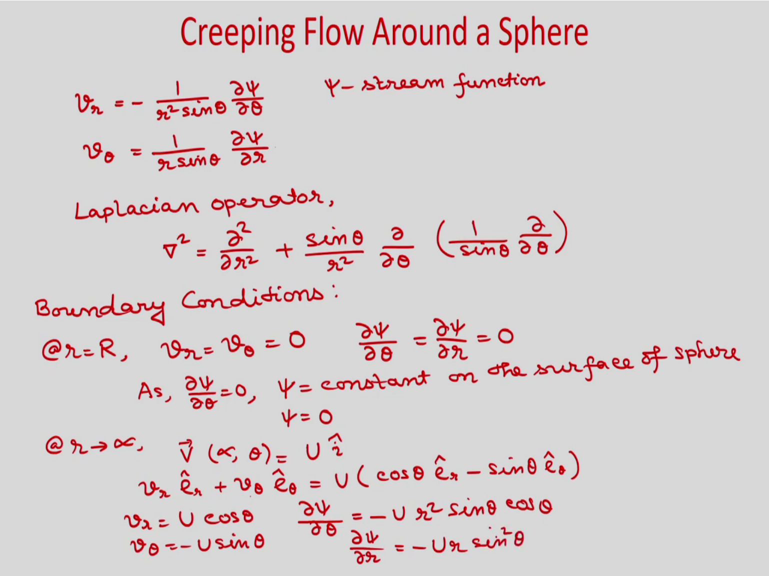

So, considering the axisymmetric flow we can write radial velocity v r as minus 1 by r square sin theta del psi by del theta, where psi is the stream function ok and v theta is 1 by r sin theta del psi by del r. In this case the Laplacian operator in spherical coordinate we will write as del 2 by del r square plus sin theta by r square del of del theta 1 by sin theta del of del theta ok.

So, now let us write the boundary conditions. So, what are the boundary conditions? So obviously, you can see on the surface of the sphere at r is equal to R, v r and v theta will be 0. So, in this case you can see from here you can write del psi by del theta is equal to del psi by del r will be 0 ok. And you can see that del psi by del theta is equal to 0, that means psi does not change in the along the surface.

So, you can say that psi is equal to constant on the surface of sphere ok. As del psi by del theta is equal to 0 psi is constant on the surface of sphere. So obviously, you can choose any value of psi on the sphere, so for convenience let us chose psi is equal to 0 on the sphere ok, so, psi is equal to 0. And far away from the sphere, so where r tends to infinity obviously you will have the free stream velocity U.

So, you can write, so here v infinity theta is equal to U i. So, now, if you write in terms of the spherical coordinate, so, you can write v r e r these are the unit normal in r direction v theta e theta is equal to U. So, this i this unit normal i is the unit vector in the x direction, so obviously you can write in spherical coordinate as cos theta e r minus sin theta e theta.

So obviously, now you can write that v r is equal to U cos theta. So, del psi by del theta you can write as v r is this one. So, del psi by del theta you can write as minus U r square sin theta cos theta and similarly v theta you can write as minus U sin theta. So, v theta is equal to minus U sin theta and in terms of psi you can write as del psi by del r is equal to minus U r sin square theta.

So, now we know the gradient of this stream function as r tends to infinity. So, first let us find what is the psi at r tends to infinity.

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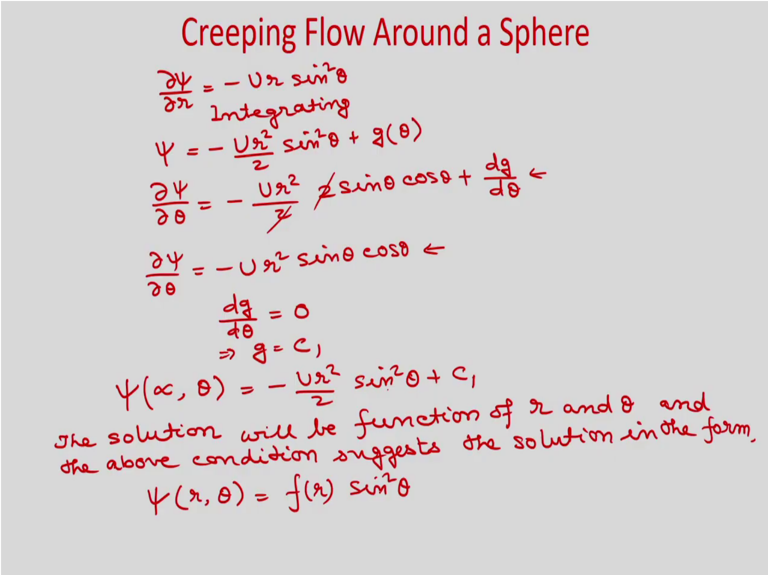

So, you know del psi by del r is equal to minus U r sin square theta ok. So, integrating you will get psi is equal to minus U r square by 2 sin square theta plus some function which is function of theta ok. And now, if you take the derivative with respect to theta then you can write del psi by del theta is equal to minus U r square by 2 2 sin theta cos theta plus d g by d theta ok.

So, and already we have found that del psi by del theta in last slide if you see that del psi by del theta we have written minus u r square sin theta cos theta. So, minus U r square sin theta cos theta. So, you can see from these two expressions that d g by d theta must be 0 ok.

So that means g is equal to some constant ok. So obviously, you can see that you can write psi at r tends to infinity theta is equal to minus u r square by 2 sin square theta plus some constant c 1. So, this is the stream function at r tends to infinity. So obviously, you can see that we can assume that the stream function in the fluid domain it will be as function of r, f r and sin square theta.

So, you can write that the solution will be function of r and theta and the above condition suggest the solution in the form. So, if you separate the variables then you can see that it is it will be function of r and sin square theta right. So, we can seek the solution psi which is function of r and theta as f r ok and sin square theta, because you can see this is the expression of psi as r tends to infinity.

So, it will be function of r and sin square theta, so we are seeking the solution using separation of variables method as psi r theta is equal to f r which is the function of r which is we need to find out and sin square theta. So, now we know the biharmonic equation for this stream function.

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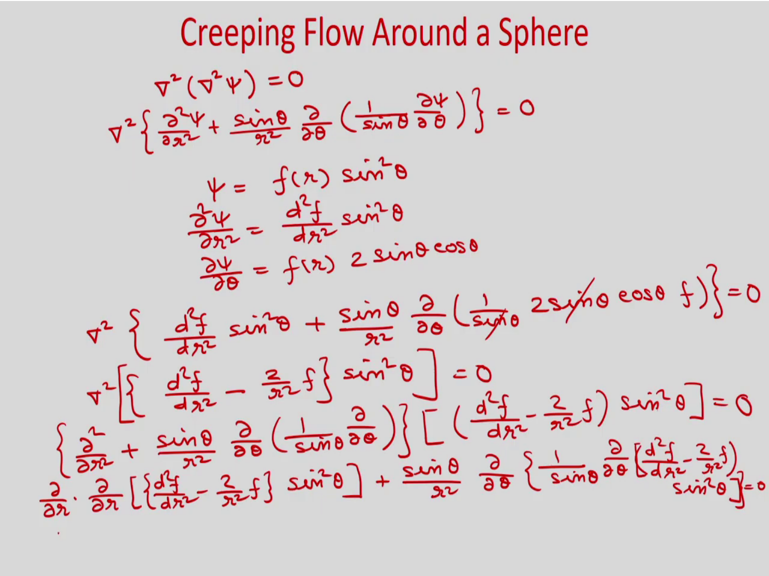

So, nabla square this is the Laplacian operator nabla square psi is equal to 0 ok. So now, you can see that this Laplacian operator in spherical coordinate already we have written. So, you can write Laplacian operator these we can write del 2 by del r square plus sin theta by r square del of del theta 1 by sin theta del of del theta ok.

So, now we can see that you can write this Laplacian operator psi. So, this we can write psi psi is equal to 0 ok. So now, psi we know that psi is equal to we have assumed as f r sin square theta, so obviously we can write del 2 psi by del r square as d 2 f by d r square sin square theta and del psi by del theta is f r 2 sin theta cos theta.

So, now if you put it here, so you can write d 2 f by d r square sin square theta plus sin theta by r square del of del theta. So, it will be 1 by sin theta and del psi by del theta is 2 sin theta cos theta f is equal to 0. So obviously, this sin theta sin theta will get canceled and if you write del of del theta cos theta, so it will be minus sin theta.

So, you can write d 2 f by d r square minus, so it will be minus sin theta. So, that sin square theta if you take outside then you can write d 2 r d 2 f by d r square 2 by r square f you can write sin square theta is equal to 0 ok. So, now next this nabla square you put ok. So, you can see that you can write now del 2 by del r square plus sin theta by r square del of del theta 1 by sin theta del of del theta of this quantity right; d 2 f by d r square minus 2 by r square f sin square theta is equal to 0 ok.

So, carefully you just do this simplification. So, you can see that this you can write as so, this is not function of so, this is the function of r so obviously you can write. So, these first will write del of del r and del of del r of d 2 f by d r square minus 2 by r square f sin square theta ok and next this we will write sin theta by r square del of del theta.

So, 1 by sin theta and del of del theta of d 2 f by d r square minus 2 by r square f sin square theta is equal to 0 ok.

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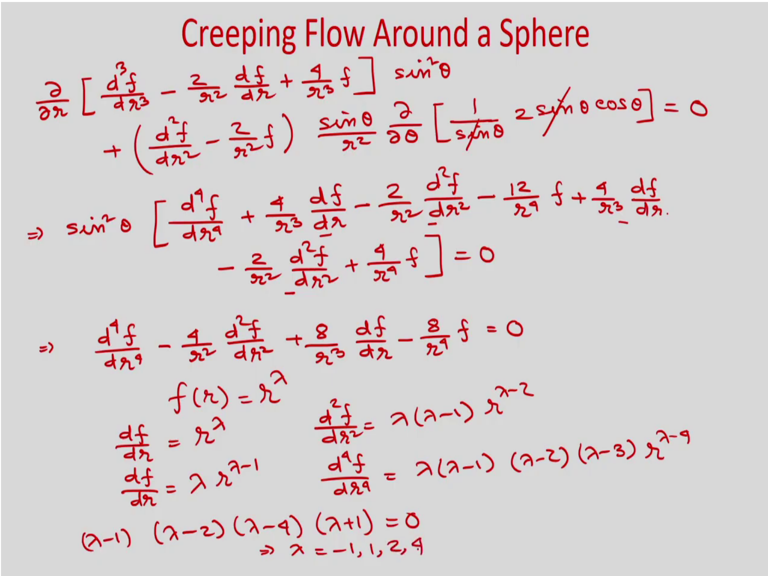

So, you can see if you simplify this one, so, you will get del of del r. So, first one you will get d cube f by d r cube minus 2 by r square d f by d r plus 4 by r cube f ok. And here you will get say you can take this outside d 2 f by d r square minus 2 by r square f because this is not function of theta. So, you will get sin theta by r square del of del theta 1 by sin theta. So, del of del theta 2 sin square theta you will get 2 sin theta cos theta is equal to 0.

So, this sin theta sin theta will get canceled. Again if you take the derivative with respect to r, so, here one sin square theta will be there. So, now you can write sin square theta. So, d 4 f by d r 4 plus now here you can see it will be 4 by r cube d f by d r minus 2 by r square d 2 f by d r square right, because there are 1 by r square and d f by d r.

So, 2 times we have used it, so then you will get minus 12 by r to the power 4 f. So, next you can see that del of del r 4 by r cube f. So, one term you have written as minus 12 by r to the power 4 f and another term will remain 4 by r cube d f by d r. And in the in this term you can see that it will be del del theta cos theta, so that means, it will be minus sin theta and this will be sin square theta.

So, sin square theta let us take outside then you will get inside 2 by r square into this. So, you can write minus 2 by r square d 2 f by d r square and plus 4 r to the power 4 f is equal to 0. So, obviously sin square theta not is equal to 0. So, these term if you simplify you will get d 4 f by d r 4, so you can see here this d 2 f by d r square this is also d 2 f by d r square.

So, we will get minus 4 by r square d 2 f by d r square and the term associated with d f by d r. So, this one, this one so it will be 8 by r cube d f by d r and this will be plus 4 by r to the power 4 f and here minus 12 r to the power 4 f. So, we will get minus 8 by r to the power 4 f is equal to 0. So, you can see that this equation is homogenous in r and is known as to have the solution in the form f r is equal to r to the power lambda. So, you can see that if you put it here. So, you will get.

If you put and simplify it obviously, you can see d f by d r will be just r to the power lambda, d f by d r you will get lambda r to the power lambda minus 1, d 2 f by d r square you will get lambda lambda minus 1 r to the power lambda minus 2 and d 4 f by d r 4 you will get lambda into lambda minus 1 lambda minus 2 lambda minus 3 r to the power lambda minus 4.

So, if you put it here and if you simplify you will get lambda minus 1 lambda minus 2 lambda minus 4 lambda plus 1 is equal to 0 ok. So obviously, you can see that lambda will be minus 1, 1, 2 and 4.

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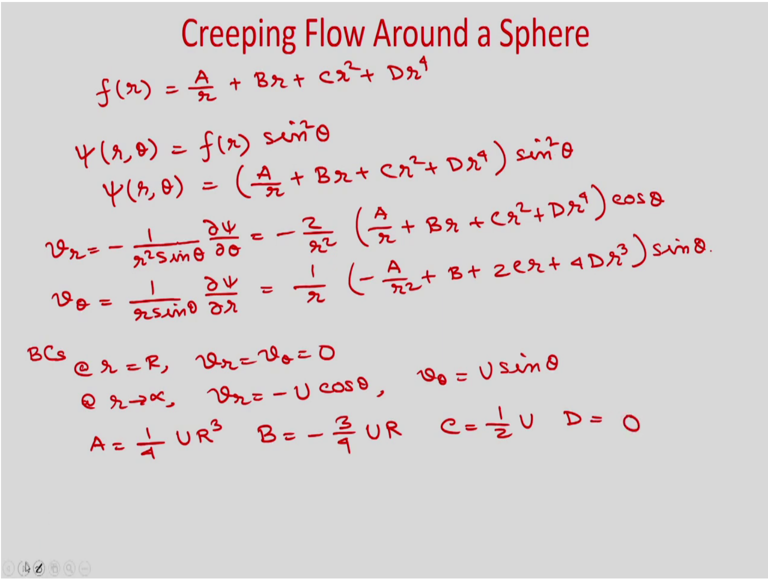

So, now we can write the f r as a function of r as f which is function of r as A by r plus B r plus C r square plus D r to the power 4 Now, we need to find this constant A, B, C, D and we have the boundary conditions. So, first let us write the stream function psi, so psi we have written as f r sin square theta. So, we can write psi as A by r plus B r plus C r square plus D r to the power 4 sin square theta ok.

So, now let us write the boundary condition in terms of psi. So, we know that v r is equal to minus 1 by r square sin theta del psi by del theta. So, we can write minus 2 by r square A by r plus B r plus C r square plus D r to the power 4 cos theta ok, because del psi by del theta it will be 2 sin theta cos theta and this sin theta will get cancelled and you will get this expression.

Similarly, you can write v theta as 1 by r sin theta del psi by del r. So, you can write 1 by r minus A by r square plus B plus twice C r plus 4 D r cube into sin theta ok. So, we have already written the boundary conditions right at r is equal to R, v r v theta is equal to 0 and at r tends to infinity v r is equal to minus U cos theta and v theta is equal to U sin theta.

So, we know that there are 4 unknowns and we have 4 boundary conditions ok. Invoking these boundary condition find the value of A, B, C, D. So, if you find it you will get A as 1 by 4 U R cube, B is equal to minus 3 by 4 U R, C is equal to half U and D is equal to 0 ok.

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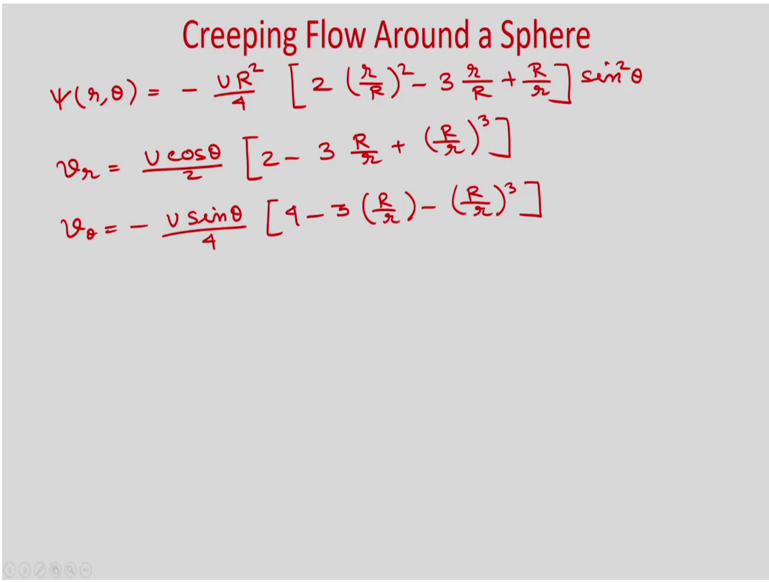

So, you can write the stream function psi as function of r theta as minus U R square by 4 2 r by R square minus 3 r by R plus R by r. And corresponding v r you can write as U cos theta by 2 2 minus here sin square theta will be there 2 minus 3 R by r plus R by r cube and v theta as minus U sin theta by 4 4 minus 3 R by r minus R by r cube ok. So, we have found the velocities v r, v theta for the creeping flow over a sphere. So, let us plot the stream line.

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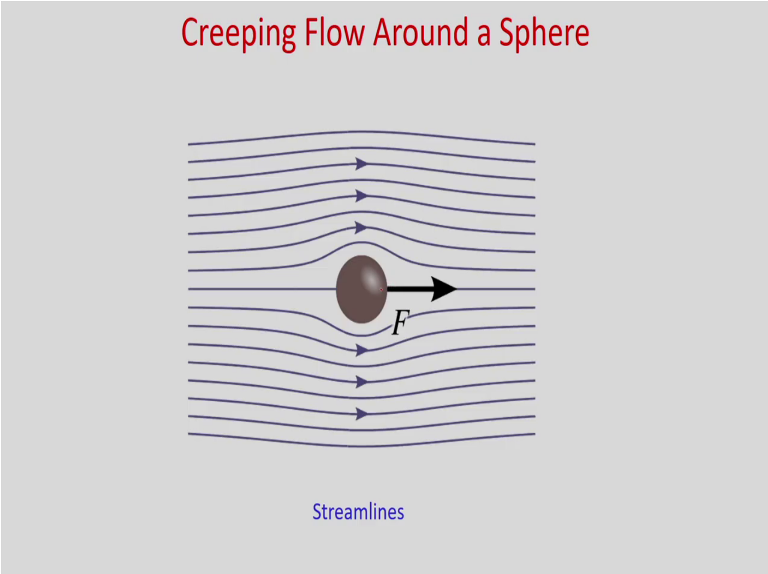

You can see that flow over the streamlines flow over a sphere, so obviously you can see that streamlines and velocities are entirely independent of fluid viscosity and the streamlines possess perfect fore and aft symmetry ok and there is no wake beyond this sphere ok. So, this is known as Stokes flow past a sphere ok. So, this is a creeping flow around a sphere where Reynolds number is very very small.

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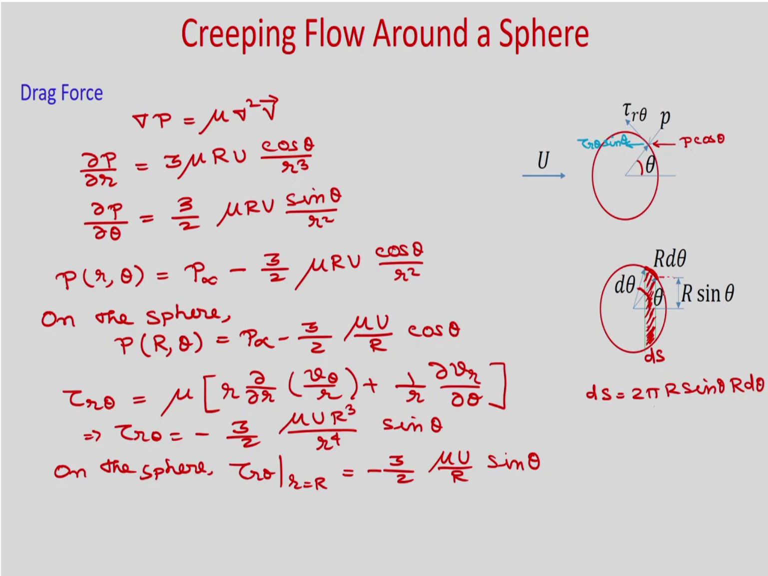

So, when we consider the fluid flow over a sphere we need to know that that force acting on the body in the direction of the fluid flow. So, in this particular case you can see that the pressure will act normal to the surface in this direction and this is the shear stress tau r theta will act.

So, we need to find what is p and what is the shear stress and the component if this is the theta, then obviously in the negative direction of this flow will be p cos theta and the tau r theta component in the opposite direction of the flow will be tau r theta sin theta. So, we know the that grad p will be just mu grad square v and v in this spherical coordinate we have v r and v theta.

So, in this case you can find from this expression invoking the Laplacian operator in spherical coordinate and the velocity components v r and v theta here you can find del p by del r is equal to 3 mu RU cos theta by r cube. And similarly del p by del theta you can find 3 by 2 mu RU sin theta by r square.

So, any one of this you integrate and find the pressure distribution P which is function of r and theta as p infinity minus 3 by 2 mu RU cos theta by r square, where p infinity is the pressure at infinity ok. So, far away from the sphere you have the piston pressure as p infinity.

So obviously, on the sphere we have p R theta is equal to p infinity minus, so it will be R square. So, you will get 3 by 2 mu U by R cos theta ok. So, this is the pressures distribution on the sphere. So, what is the now drag force acting on this? So, first let us find what is tau r theta.

So, tau r theta which is the viscous shear stress that is mu r del of del r in spherical coordinate we are writing 1 by r del v r by del theta ok. So, we know v theta, v r; you simplify it. So, you will get tau r theta as minus 3 by 2 mu UR cube by r to the power 4 sin theta. So obviously, on the sphere tau r theta at r is equal to R you will get minus 3 by 2 mu U by R sin theta.

So obviously, due to this normal pressure and the viscous shear stress this sphere will experience one drag force in the direction of the flow, so that we need to find. So, for this we will consider one elemental area. So, on the spherical surface ok so this is the theta; so obviously if this is the angle theta then these distance will be just if R is the radius of the sphere then it will be R sin theta and d theta is the small elemental angle d theta.

So, this is the d theta, so obviously this will be R d theta ok, so this will be R d theta. So, if you consider this surface as d s. So, what will be d s? So, you can see that d s will be so you can see that this is your R sin theta right. So, this will be twice pi R sin theta ok twice pi R sin theta. So, this is the periphery here and into R d theta.

So, that is the surface area d s, where we have shown here. So, d s will be twice pi R sin theta into this length R d theta ok. So, now, we know that p cos theta is acting in the negative flow direction and also tau R theta is also acting tau R theta sin theta in the negative flow directions.

So, you can now integrate over this surface where theta will vary from 0 to pi ok then whole surface will cover, because it is axisymmetric that is why you have taken in this way. So, now first let us consider the drag due to the pressure which is known as form drag or pressure drag.

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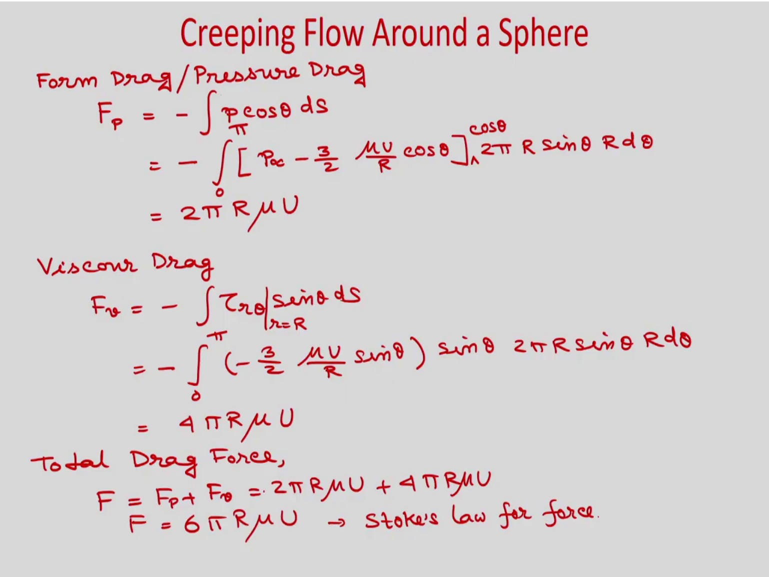

So, will calculate the form drag or pressure drag, so what we will do? So, this is your F p is equal to; so we have to integrate these minus p cos theta over the surface d s ok and d s we have already calculated right. So, you can put that it will be minus integral 0 to pi p we have already calculated p infinity minus 3 by 2 mu U by R cos theta and d s is twice pi r sin theta into R d theta ok. So, if you integrate it this F p will get as twice pi R mu U.

And similarly, viscous drag F v, if you calculate so it will be minus integral tau r theta sin theta d s ok. So, now if you put it this tau r theta at r is equal to R. So, you will get minus it is r is equal to R on the surface we are integrating. So, minus 0 to pi so you will get minus 3 by 2 mu U by R sin theta into sin theta here one cos theta will be there cos theta and we will have d s. So, d s will be twice pi R sin theta into R d theta ok.

So, you integrate it, finally you will get this as 4 pi R mu U. So, total drag force in the direction of flow we will get F as F p plus F v ok. So, it will be twice pi R mu U plus 4 pi R mu U. So, you will get F is equal to 6 pi R mu U ok, so this is known as Stokes law of force ok. So, you can see that one-third of the total force is your coming from pressure drag and two-third of the total force is coming from the viscous drag. So, now, let us see how the pressure varies over the surface.

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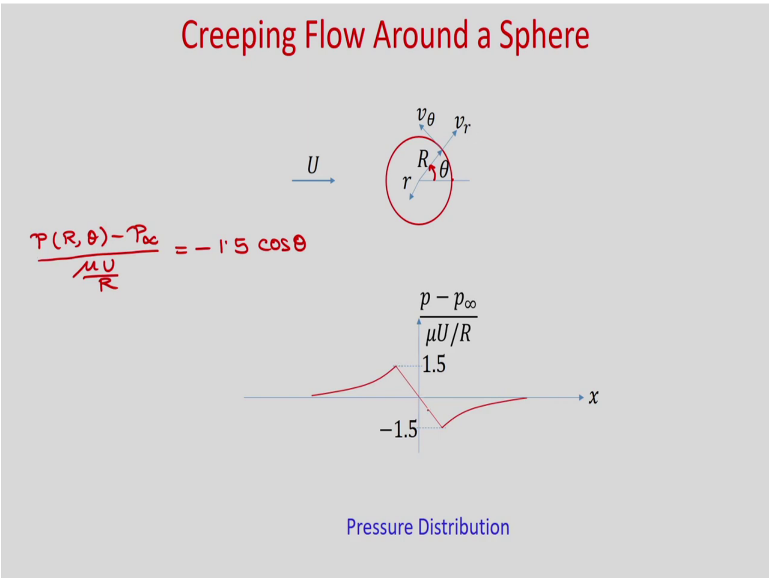

So, theta we are taking from here and theta is varying from 0 to pi and we know the distribution the pressure distribution on the sphere surface, if you write p R theta minus P infinity divided by mu U by R. So, this we can write as minus 1.5 cos theta ok. So, you can see at theta is equal to 0 that means at this point what will be the pressure?

So, cos theta 0 is 1, so it will be minus 1.5 and at theta is equal to pi. So, cos pi is minus 1, so it will be 1.5. So, you can see that pressure obviously you can see over the surface it will vary from 1.5 to minus 1.5 ok and the y axis is just p minus p infinity divided by mu U by R which is the non dimensional pressure distribution and this is the axial direction x. So, the pressure varies like this.

So, in today's class we have started with the lubrication theory as a starting point we are discussed about the creeping flow. And for this purpose we have considered the Stokes flow past a sphere which is the example of this creeping flow. And for this particular case we have simplified the governing equation.

When we have the very very slow motion, that means Reynolds number tends to 0 and we have dropped the inertia terms from the Navier Stoke equations and from there we have written the equation in terms of psi which is the stream function. We considered creeping flow past a sphere. In this case we have used the Laplacian operator in spherical coordinate and from there we have found the stream function distribution for this flow over a sphere and from there we have calculated the pressure distribution, then we calculated the pressure drag and the viscous drag.

So, pressure drag is coming from the normal pressure and it is one-third of the total force is coming from this pressure drag and two-third of the total drag is coming from the viscous drag.

Thank you.