Viscous Fluid Flow
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Module - 01
Introduction
Lecture – 02
Fluid Kinematics

Hello everyone. So, in today's class, we will discuss the preliminary concepts when the fluid is in motion which is known as Fluid Kinematics. There are two ways of describing a fluid motion: one is Lagrangian approach and the other one is the Eulerian approach.

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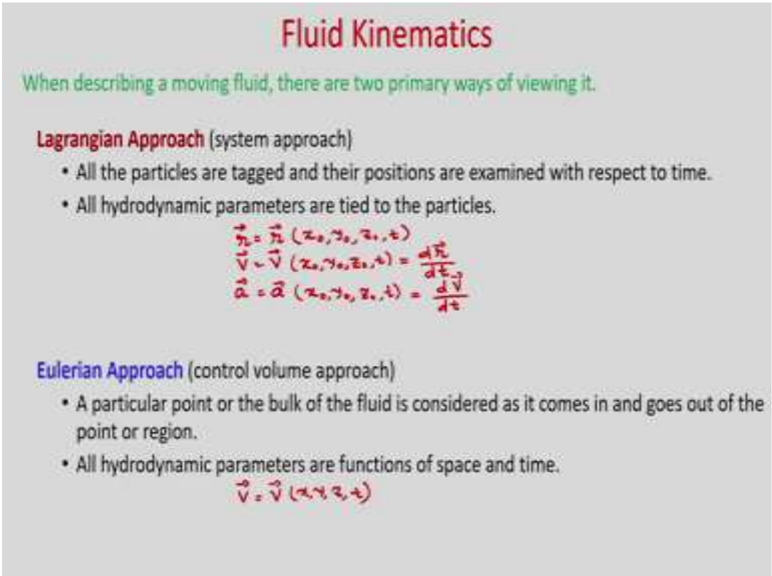

In Lagrangian approach, all the particles are tagged, and their positions are examined with respect to time and all hydrodynamic parameters are tied to the particles. In the Lagrangian approach, if r is a position vector, then r can be represented as a function of the initial position x_o, y_o, z_o and time.

Similarly, the velocity vector is a function of the initial position so that means, x_o, y_o, z_o, t where x_o, y_o, z_o locate the starting point and v also can be represented as dr/dt right. So, you can see in Lagrangian approach, you track each and every particle and acceleration you can also represent as a function of x_o, y_o, z_o and time t and this you can write as dv/dt.

Whereas, in Eulerian approach, a particular point or the bulk of the fluid is considered as it comes in and goes out of the point or region. All hydrodynamic parameters are a function of space and time. So, you can see that any parameter like velocity v is a function of space and time.

So, obviously, you can see that when we will discuss the system approach and control volume approach, in Lagrangian approach is equivalent to the system approach where you have a fixed mass or fixed particles and you are tracking each particle. And in the Eulerian approach, it is a control volume approach so, you are focused on a particular region and you are tracking what is coming in and going out.

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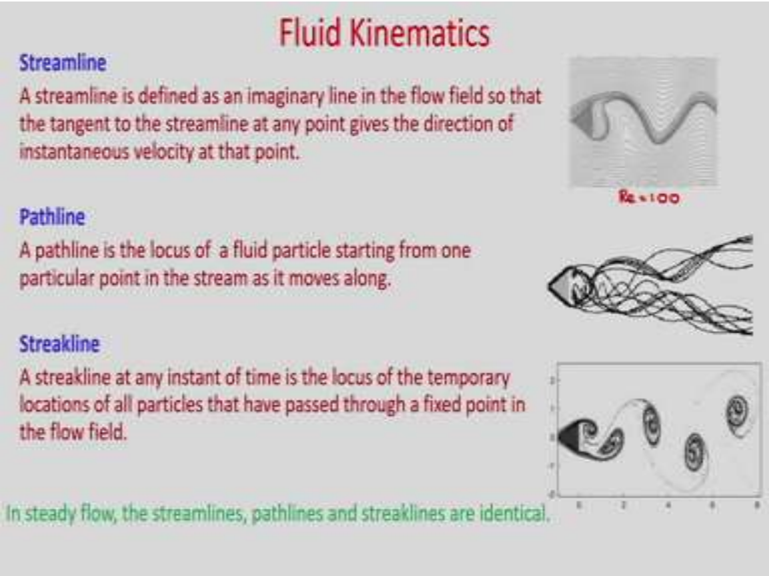

So, when you are considering the Eulerian approach, there must be some visualization technique for seeing the fluid flow. So, for visualizing the fluid flow, first, we will discuss the streamline.

A streamline is defined as an imaginary line in the flow field so that that tangent to the streamline at any point gives the direction of instantaneous velocity at that point. You can see that flow over a triangular cylinder at Reynolds number based on the projected area is 100 and these are some visualization of this fluid flow and it shows the streamline.

Next, we will discuss the pathline. A pathline is the locus of a fluid particle starting from one particular point in the stream as it moves along. So, pathline you can see that if you are actually tracking a particular fluid element and where it moves about in the fluid domain. So, that is the pathline. So, for this particular case, you can see flow over a triangular cylinder at Reynold’s number 100, it shows the pathline and, at Reynold’s number 100, it is an unsteady flow so, instantaneous pathline is shown here.

Next, we will discuss the streakline. A streakline at any instant of time is the locus of the temporary locations of all particles that are passed through a fixed point in the flow field. So, you can see these are the locus of these particles which have come through a particular point and these declines are shown at Reynold’s number 100. So, this is also instantaneous because it is an unsteady flow, but if you consider a steady flow, then the streamlines, pathlines and streaklines are identical.

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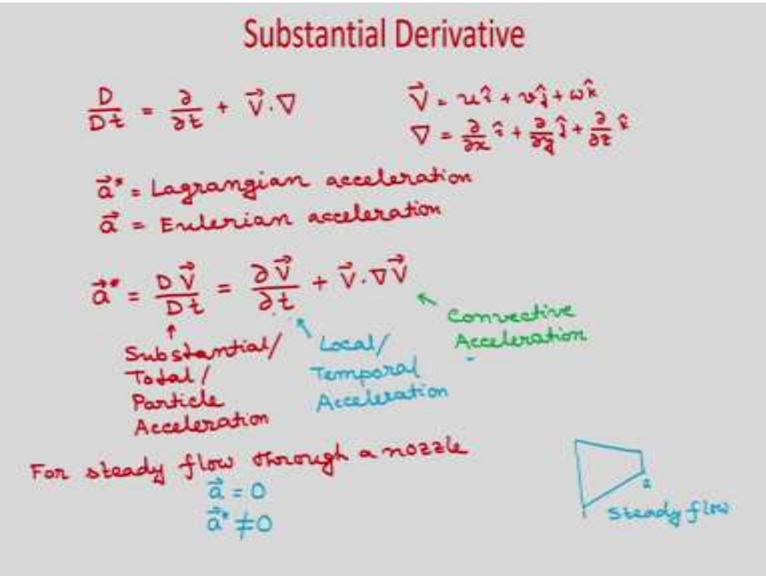

Since Lagrangian and Eulerian variables describe the same flow, there must be some relation between the two. So, that will actually relate through this substantial derivative. So, this relation is expressed through this substantial derivative which is denoted as

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{V}.\nabla$$

So, where V is the velocity vector which is

$$\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$$

and this gradient operator in cartesian coordinate we can represent as

$$\nabla = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial x}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}$$

So, if we want to represent the Lagrangian acceleration, then we can write that Lagrangian acceleration is let us say $\vec{a}^*$ Lagrangian acceleration. So, obviously, it represents the acceleration of a flowing fluid particle and if you represent that $\vec{a}$ as Eulerian acceleration, then you can relate these two through this substantial derivative $\vec{a}^*$ which is your Lagrangian acceleration you can denote as

$$\vec{a}^* = \frac{D\vec{V}}{Dt} = \frac{\partial\vec{V}}{\partial t} + \vec{V}.\nabla\vec{V}$$

So, you can see that obviously, this represents the acceleration in the Lagrangian framework and this $\frac{\partial\vec{V}}{\partial t}$ is the acceleration in the Eulerian framework and we represent this $\frac{D\vec{V}}{Dt}$ as substantial or total or particle derivative in this case it is particle acceleration. This part is known as local or temporal acceleration and the last part is known as convective acceleration.

So, you can see that if velocity is 0, then; obviously, this part will become 0. You can see it relates between this Lagrangian approach and the Eulerian approach so obviously, you can see that if it is a steady flow, $\frac{\partial\vec{V}}{\partial t}$ is 0 right because for a steady flow in the Eulerian approach, you can make it 0; however, this convective part is not 0. So, there will be a substantial acceleration if the flow is steady.

So, you can see that for steady flow through a nozzle so, consider a nozzle so, you can see it is a bearing cross-section so obviously, from 1 to 2, there will be a change in the velocity as the area is decreasing. In this particular case, although it is a steady flow, as there is a spatial change in the velocity so, there will be a convective part right which is not 0 although it is a steady flow.

So, the first part is your local derivative so, that will be 0 and the substantial derivative will be non-zero. So, in this particular case for steady flow through a nozzle, you can see that this acceleration, Eulerian acceleration it will be 0; however, the Lagrangian acceleration is not 0 because it is having the contributions from this convective acceleration as there is a spatial variation of velocity.

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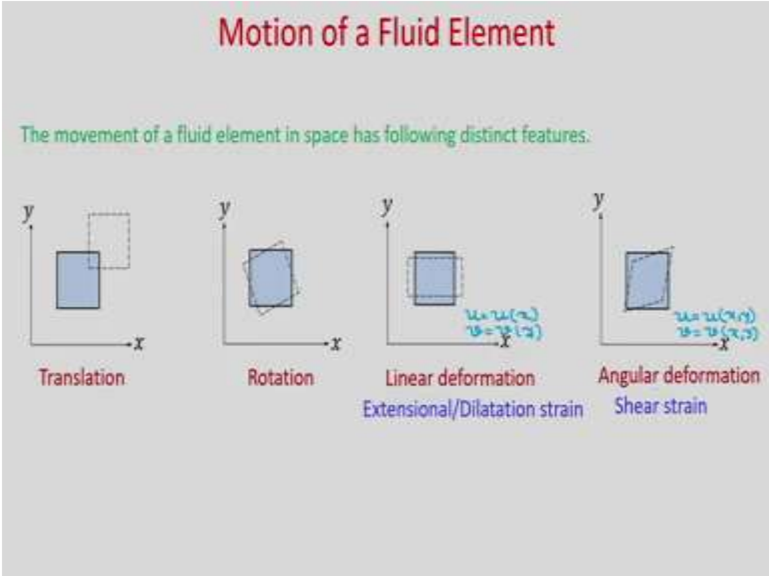

As you are considering fluid is in motion so obviously, when one fluid element if it is moving with the fluid velocity so, obviously, there will be deformation of the fluid element. The movement of a fluid element in space has the following distinct features; one is translation.

So, you have a fluid element so, after some time it may translate from one place to another and it will happen when the velocity is constant. There may be fluid rotation. So, you can see it is a kind of solid body rotation. So, it is having the initial position like this, but after some time, just it rotates as a solid body.

We can have a rate of deformation. So, there are two types of deformation; one is linear deformation, and another is angular deformation. So, linear deformation also is known as extensional or dilatation strain. In this case, you can see that this is the fluid element so, if velocity is a function of one space coordinates so, let us say u is the function of x only or v is a function of y only, then only linear deformation will take place.

But if the velocity is a function of both x and y, we are considering a two-dimensional fluid element so, in this case, u and v are a function of x and y. So, in this particular case, there will be an angular deformation and that is known as shear strain. So, you can see when one element is in motion so, it will have some extensional strain as well as shear strain due to angular deformation.

First let us consider linear deformation that means, the velocity u is a function of x only or v is a function of y only. So, first let us consider one fluid element where v is 0 and u is a function of x only to simplify the problem.

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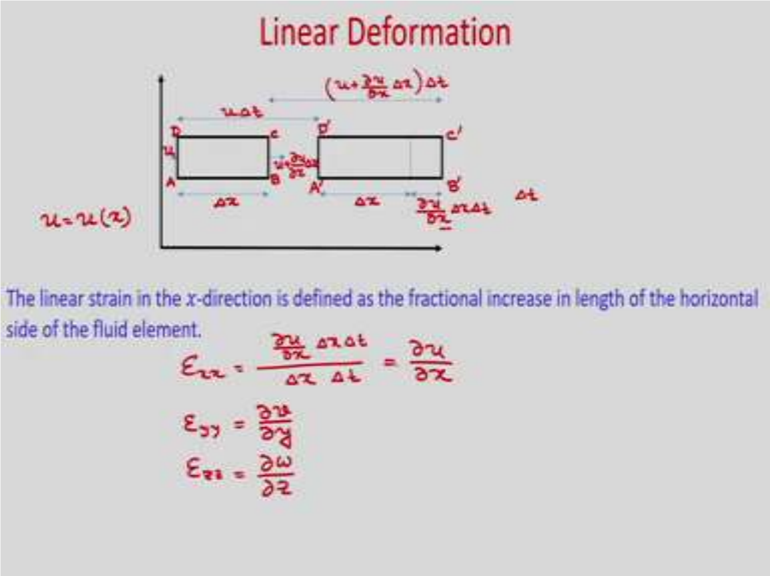

Let us consider this fluid element A, B, C and D and in this case, u is a function of x only right. So, u is velocity in the x direction. So, obviously, you can see that we have the velocity u here. So, if the distance is Δx, this fluid element in the x direction, then obviously, at this particular point using Taylor series, you can say that velocity will be $u + \frac{\partial u}{\partial x}\Delta x$. So, if you expand this using Taylor series, $u(x + \Delta x)$ locations then you will get $u + \frac{\partial u}{\partial x}\Delta x$.

So, you can see now this point, this A or D you can see obviously, due to the linear deformation, it will travel due to the velocity u and this element has come here after a small-time step delta t ok. So, obviously, you can see that as you have a velocity u so, at Δt it will travel Δt . So, and this is the location at Δt time B', C' and D'.

So, similarly, you can see that you have the velocity $u + \frac{\partial u}{\partial x}\Delta x$. so, at time Δt it will move to this place B', C' so, this distance from C to C', it will be $\left(u + \frac{\partial u}{\partial x}\Delta x\right)\Delta t$ and this was the original length of this dotted line it is shown so, it is Δx. So, you can see the difference will be just $\frac{\partial u}{\partial x}\Delta x\Delta t$ due to this linear deformation, there is a change in the size in the x direction is d$\frac{\partial u}{\partial x}\Delta x\Delta t$. So, in this case, you can see the linear strain in the x direction is defined as the fractional increase in length of the horizontal side of the fluid element.

So, in this particular case, the linear strain now we can write as the or the change in the length that is

$$\varepsilon_{xx} = \frac{\frac{\partial u}{\partial x}\Delta x\Delta t}{\Delta x\Delta t} = \frac{\partial u}{\partial x}$$

So, you can see that if you consider the linear deformation in the x direction, then it is represented by the velocity gradient $\frac{\partial u}{\partial x}$.

Similarly, if you consider that v is a function of y and w is a function of z, then similarly you can write the linear deformation in y and z direction as

$$\varepsilon_{yy} = \frac{\partial v}{\partial y}$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z}$$

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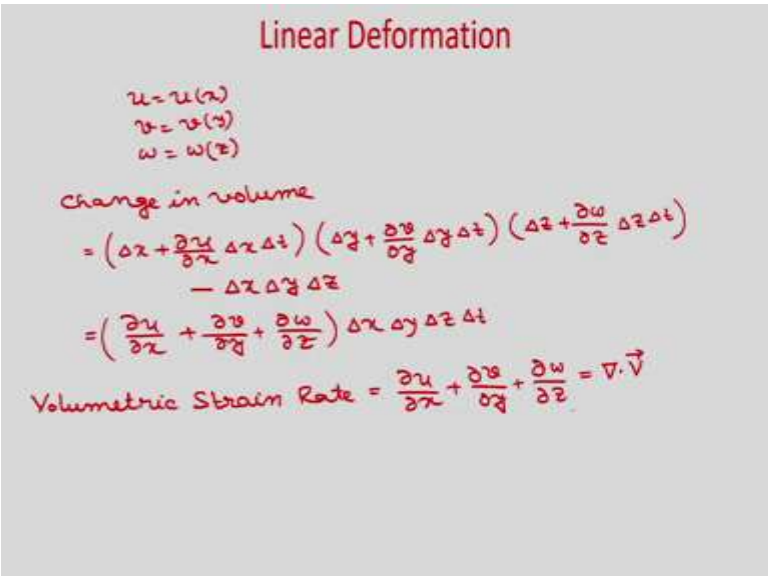

So, you can see that if you consider; if you consider that u is a function of x, v is a function of y and w is a function of z, then the change in the volume you can consider so obviously, you can see that your after deformation there will be a length $\left(\Delta x+\frac{\partial u}{\partial x}\Delta x\Delta t\right)$ so, this is in the x direction, in the y direction it will be $\left(\Delta y+\frac{\partial v}{\partial y}\Delta y\Delta t\right)$ and in the z direction $\left(\Delta z+\frac{\partial w}{\partial y}\Delta z\Delta t\right)$. So, this is the volume after deformation minus the original volume so, that is $-\Delta x\Delta y\Delta z$.

So, now, if you neglect the higher-order term, then we can write change in volume as

$$=\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\Delta x\Delta y\Delta z\Delta t$$

So, now, if you write the volumetric strain rate; volumetric strain rate, then we can write it as just you can see the change in volume divided by the original length per unit time so, it will be

$$=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=\nabla.\vec{V}$$

So, obviously, you can see that if you consider incompressible fluid flow so, the volumetric strain rate will be 0 and divergence of v ($\nabla.\vec{V}$) will be 0.

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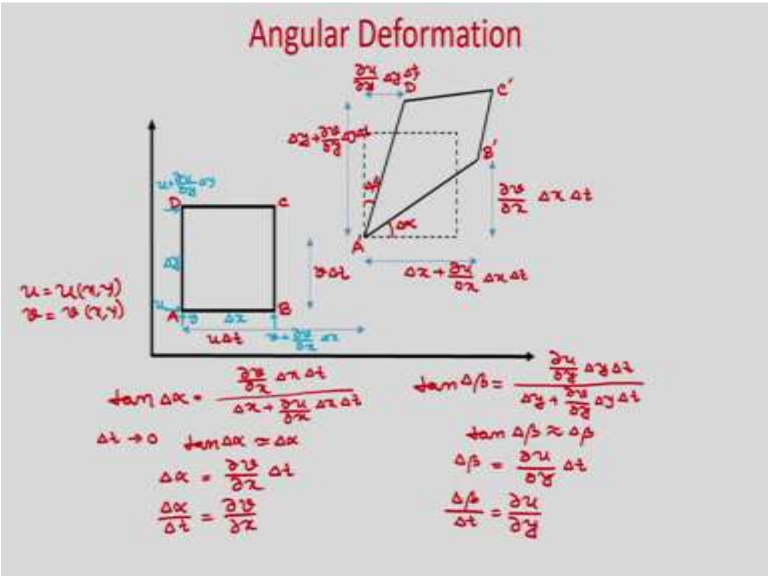

Now, let us consider two-dimensional fluid element and velocities u, v both are the function of x and y, then obviously, we will get the angular deformation. So, you consider first this initial position of the fluid element at time t so, A, B, C, D and after time $t+\Delta t$ obviously, as velocity u is a function of x and y, v is a function of x and y, then it will have some angular deformation, and this is the position.

So, it is A', B', C', D'. So, after Δt time, it has had it has angular deformation and the fluid element looks like this A', B', C', D'.

So, you can see there is a change in the angle so, we can say that this angle is $\Delta\alpha$ and this angle is $\Delta\beta$. So, similarly, you can see that if you have velocities here so, as u is a function of x and y so, there will be a change in the velocities. So, in this fluid element, if velocity is u here, here it is u and it is v so obviously, in this position, if you can see that the length is Δx and this length is Δy, then here the velocity will be $u+\frac{\partial u}{\partial y}\Delta y$.

So, there is a change in the velocity, and due to that this deformation is different in x and y direction. So, similarly if it is v then; obviously, you will get here $v+\frac{\partial v}{\partial x}\Delta x$. So, now, if you consider that there is a change in the position from time t to $t+\Delta t$, then obviously, you can see that this length because it is having the velocity u so, A to A' at a time Δt so, it will be $u\Delta t$.

And, this position you can see that this A' to D' this vertical distance will be $\Delta y+\frac{\partial v}{\partial y}\Delta y\Delta t$ and this will be $\Delta x+\frac{\partial u}{\partial x}\Delta x\Delta t$. And this distance if you see it will be $\frac{\partial v}{\partial x}\Delta x\Delta t$ and as this velocity is v so, it has moved to here so, this will be $v\Delta t$ and this distance will be $\frac{\partial u}{\partial y}\Delta y\Delta t$.

So, now we can represent this $\Delta\alpha$ and $\Delta\beta$ in terms of the velocity gradients. So, if you see from here, if you write $\tan\Delta\alpha$ will be

$$\tan\Delta\alpha=\frac{\frac{\partial v}{\partial x}\Delta x\Delta t}{\Delta x+\frac{\partial u}{\partial x}\Delta x\Delta t}$$

$$\tan\Delta\beta=\frac{\frac{\partial u}{\partial y}\Delta y\Delta t}{\Delta y+\frac{\partial v}{\partial y}\Delta y\Delta t}$$

So, now, if you consider that Δt is very small, then obviously, you can write $\tan\Delta\alpha$ is equal to $\Delta\alpha$ and as Δt tends to 0 so, this you can write $\Delta\alpha$ is equal to $\frac{\partial v}{\partial x}\Delta t$. And similarly, as Δt tends to 0, $\tan\Delta\beta$ will be equal to $\Delta\beta$ and $\Delta\beta$ you can write as $\frac{\partial u}{\partial y}\Delta t$.

So, now, you can see that we can represent this delta alpha whatever this change in the angle divided by the time in terms of the velocity gradient similarly delta beta by delta t we can represent in terms of velocity gradient as del u by del y.

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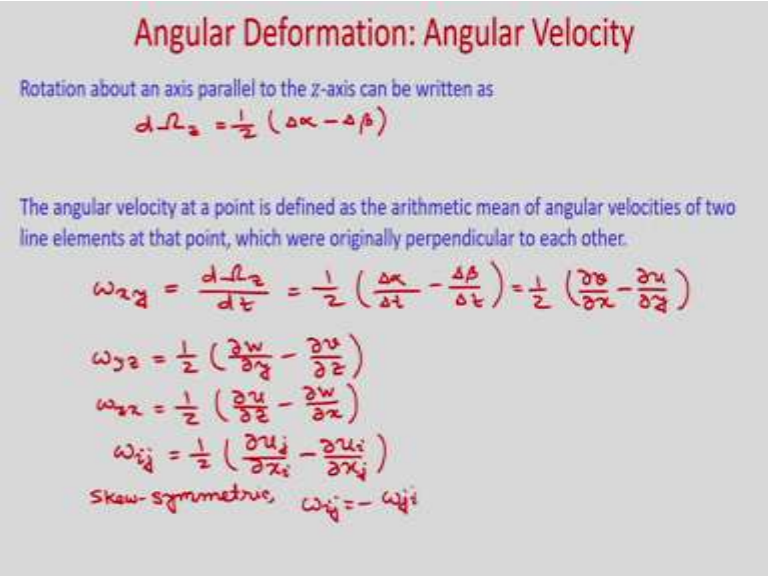

So, now, if we consider that rotation about an axis parallel to the z-axis that we can write as

$$d\Omega_z = \frac{1}{2}(\Delta\alpha - \Delta\beta)$$

And, the angular velocity at a point is defined as the arithmetic mean of angular velocities of two-line elements at that point that were originally perpendicular to each other.

So, in that case, we can define the angular velocity

$$\omega_{xy} = \frac{d\Omega_z}{dt} = \frac{1}{2}\left(\frac{\Delta\alpha}{\Delta t} - \frac{\Delta\beta}{\Delta t}\right) = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

So, now, we have represented the angular velocity in terms of the velocity gradient. And similarly the angular velocity in x and y axis we can write as

$$\omega_{yz} = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

$$\omega_{zx} = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

So, you can see all this, you can write in terms of second order tensor that is

$$\omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_i} - \frac{\partial u_i}{\partial x_j}\right)$$

And obviously, from here, you can see that this angular velocity tensor is a skew symmetric that means, it is a skew symmetric tensor as

$$\omega_{ij} = -\omega_{ji}$$

So, now, let us discuss the shear strain rate due to these deformations of the fluid element, we have already evaluated the change of angle per unit time in terms of the velocity gradient.

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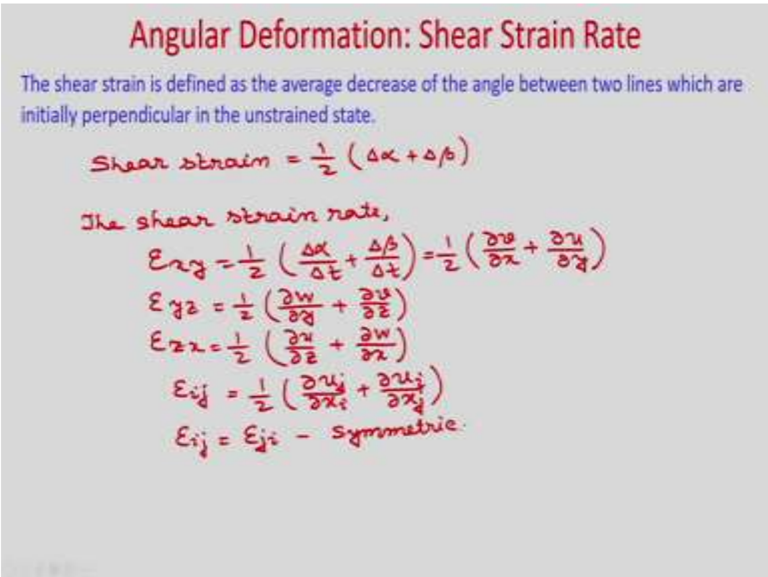

So, now the shear strain is defined as the average decrease of the angle between two lines that are initially perpendicular in the unstained state. So, obviously, you can see that shear strain we can define as $\frac{1}{2}(\Delta\alpha + \Delta\beta)$.

So, now, we can write the shear strain rate so, the shear strain per unit time so, that we can write as

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\Delta\alpha}{\Delta t} + \frac{\Delta\beta}{\Delta t}\right) = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

So, similarly the other shear strain also can write as

$$\varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)$$

Similarly,

$$\varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

So, obviously, from here, you can see that shear strain rates are symmetric. So, in terms of tensor if we write then

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)$$

and you can see that ε_{ij} is equal to ε_{ji} right so, this is asymmetric tensor ok.

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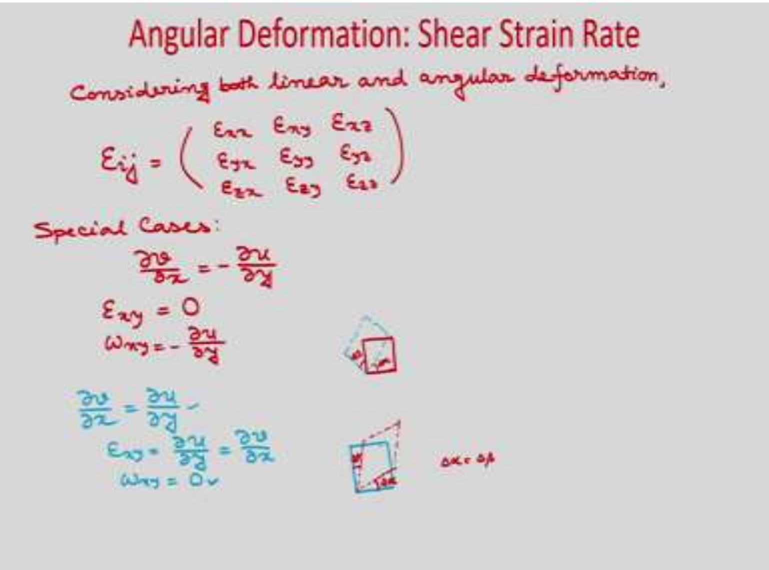

So, now, if you consider both the linear deformation as well as angular deformation and if you want to write the shear strain rate in terms of ε_{ij}, then we can write as so, considering both linear and angular deformation ok, we can write now ε_{ij} as second order tensor. So, you can see it will be

$$\varepsilon_{ij} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix}$$

So, it is a second order tensor.

Now, let us consider two special cases ok. So, special cases; now, if you see that if $\frac{\partial v}{\partial x}$ is equal to $-\frac{\partial u}{\partial y}$. So, in that case, you can see that angular deformation ε_{xy} obviously, from the expression you can see it will be 0 and ε_{xy} will be - $\frac{\partial u}{\partial y}$. So, what is happening here? So, you can see that if ε_{xy} is equal to 0 so, there will be no deformation so, it will rotate like a solid body right.

So, if you have initially one fluid element like this, then after time Δt obviously, it will rotate like a solid body like this ok. The other special case is if $\frac{\partial v}{\partial x}$ is equal to $\frac{\partial u}{\partial y}$. So, in this particular case, you can see ε_{xy} will be equal to $\frac{\partial u}{\partial y}$ is equal to $\frac{\partial v}{\partial x}$ and ω_{xy} will be 0 so, that means, the fluid element has an angular deformation rate, but no rotation about z axis.

So, it will look like if you have one fluid element initially like this so, if you have the same angular deformation so if $\Delta\alpha$ is $\Delta\beta$ in the opposite direction if it is same so, it will look like this. So, this angle $\Delta\alpha$ and $\Delta\beta$ will be same in the opposite direction and here also this will be same $\Delta\alpha$, $\Delta\beta$; so, in the same direction $\Delta\alpha$, $\Delta\beta$.

So, obviously, you can see that if you have the same angular deformation like this $\Delta\alpha$ is equal to $\Delta\beta$, then there will be no rotation of this fluid element; however, it will undergo an angular deformation. Now, you can see that when we are considering the Lagrangian approach, then obviously, we need to track each, and every particle and it is very difficult while studying the fluid mechanics' problems.

So, in this particular case, if we consider the Eulerian approach, then it will be convenient for us because we will be focusing on a region or domain and we will see what is coming in and going out. So, but we need to have the relation between these two approaches so, as you discuss that Lagrangian approach is known as the system approach and the Eulerian approach is known as the control volume approach.

So, if you consider one particular control volume and what is happening, what is coming in, what is going out that if we note, then it will be easy to study the fluid mechanics' problems.

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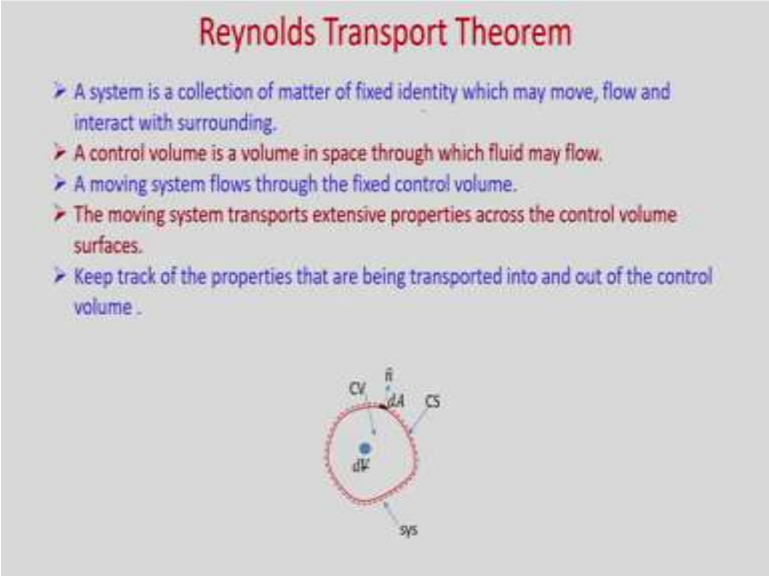

So, we can see that a system is a collection of matter of fixed identity which may move, flow and interact with surroundings. So, you can see that it will have the same mass inside a system. A control volume is a volume in space through which fluid may flow. So, obviously, it is convenient for fluid mechanics study to use the control volume approach.

A moving system flows through the fixed control volume. So, if you consider that this is a system, and this is a control volume and this control volume surface is known as a control surface. And in this case, if a moving system flows through this fixed control volume and the moving system transports extensive properties across the control volume surface and now, we need to keep track of these properties that had been transported into and out of the control volume.

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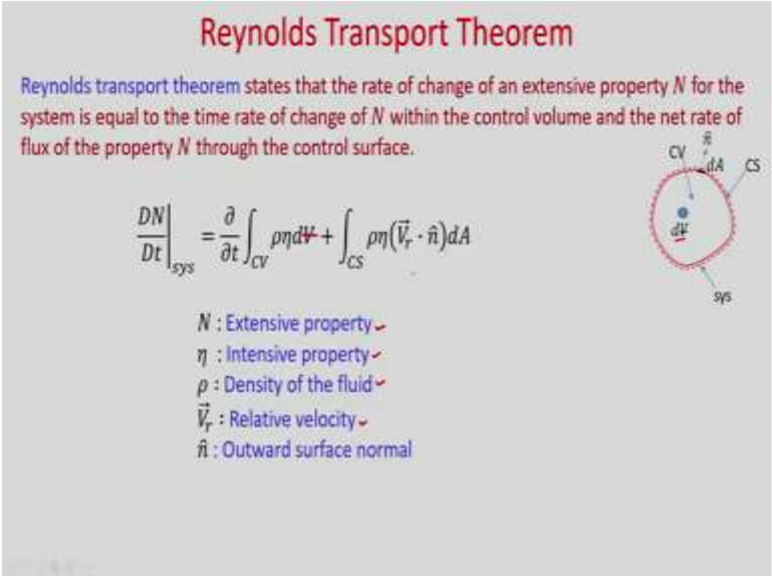

So, this we can relate this system approach and control volume approach through this Reynold's transport theorem. So, what does it state? Reynold's transport theorem states that the rate of change of an extensive property N for the system is equal to the time rate of change of N within the control volume and the net rate of the flux of the property N through the control surface.

So, we are considering this as a system and this is the control volume and in the control volume, what is the surface through which the flow will come in and go out. And, if we consider one elemental volume that is dV and one elemental surface dA and n is the outward normal unit normal.

So, now, you can see that through this Reynolds transport theorem, we can write

$$\frac{DN}{Dt}\Big|_{sys} = \frac{\partial}{\partial t}\int_{CV}\rho\eta d\forall + \int_{CS}\rho\eta(\overrightarrow{V_r}.\hat{n})dA$$

So, in this case, N is any extensive property, it has the intensive property, ρ is the fluid density, V_r is the relative velocity and $\hat{n}$ is the outward surface normal?

So, now using this Reynold's transport theorem, we will derive the continuity equation. So, we will first consider the conservation of mass. So, if you consider the conservation of mass, then N will be equal to the mass of the fluid. So, that is the extensive property and intensive property η will be just 1 in this particular case.

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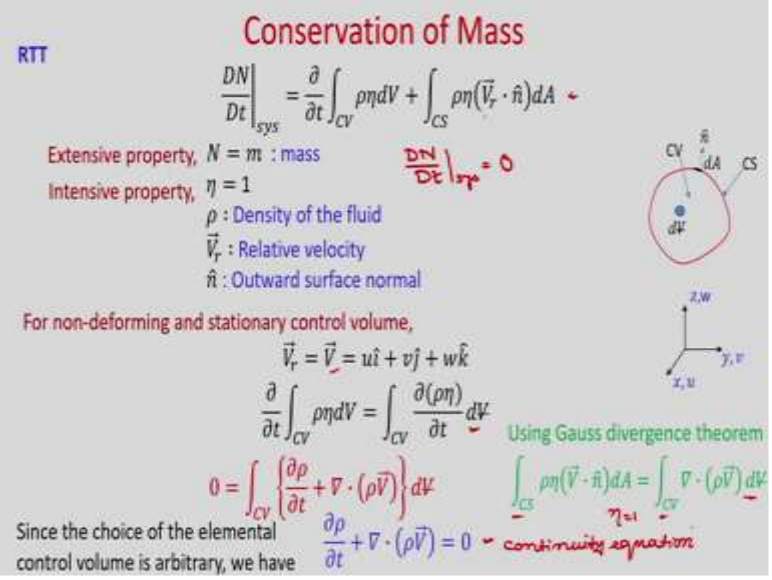

So, you can see that if we consider the conservation of mass, then obviously, N will be m and η will be 1 and as m is the mass is fixed in the system, then obviously, DN/Dt system will be 0 right. So, you can see that for non-deforming and stationary control volume, V_r will be the V and for non-deforming control volume, this first integral we can write as

$$\int_{CV} \rho\eta d\forall = \int_{CV} \frac{\partial(\rho\eta)}{\partial t} d\forall$$

So, now, this second part if you consider η is equal to 1 and V_r is equal to V, then using Gauss divergence theorem we can write

$$\int_{CS} \rho\eta(\vec{V}.\hat{n})dA = \int_{CV} \nabla(\rho.\vec{V})d\forall$$

So, in this particular case, now if you consider this equation, then left-hand side DN/Dt system will be 0 and this is your volume integral and we are considering non-deforming control volume, then we can write $\frac{\partial(\rho)}{\partial\eta}$ as η is equal to 1 plus this last term in this equation, we are converting to volume integral putting η is equal to 1, then you can write $\nabla(\rho.\vec{V})d\forall$.

So, since the choice of the elemental control volume is arbitrary so, we can have

$$\frac{\partial(\rho)}{\partial t} + \nabla(\rho.\vec{V}) = 0$$

So, this is the continuity equation in general.

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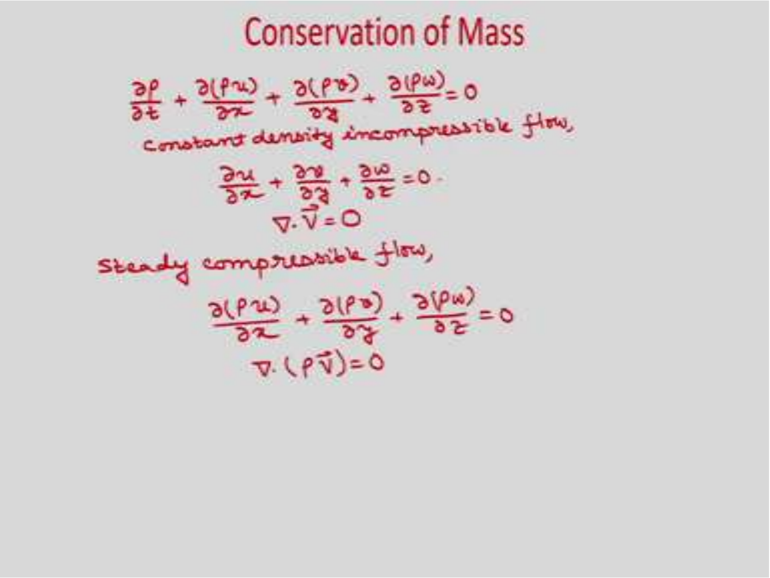

So, we have

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0$$

Now, if you consider incompressible flow where you have a constant density incompressible flow; so, for constant density, you can write it as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

So, from the continuity equation for a constant density incompressible flow, we have written this divergence of V is equal to 0. That means, the volumetric strain rate will be 0 in case of constant density incompressible flow.

If you consider steady incompressible flow; so, for steady compressible flow; for steady compressible flow obviously, this time derivative will be 0 and you can write

$$\frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0$$

that means, the divergence of ρV will be 0.

So, this is for steady incompressible flow, but you can see this equation is valid for both steady and unsteady because we have assumed constant density incompressible flow. So, the first term becomes 0. So, it is applicable for both steady and unsteady incompressible flow.

So, in today's class, we discuss the preliminary concepts when the fluid is in motion. First, we discussed the Lagrangian and Eulerian approach, we have considered that the deformation of the fluid when it moves with a fluid element. In this case, we have considered that fluid element undergoes rotation, translation, angular deformation and linear deformation.

When you considered the angular deformation, we have represented the shear strain in terms of the velocity gradient as well as the angular velocity in terms of the velocity gradient. And we have shown that the angular velocity tensor is skew-symmetric, and the shear strain rate tensor is symmetric.

Next, we have discussed the system approach and the control volume approach and we have related these system approach and control volume approach through Reynold's transport theorem. Next, we used the conservation of mass law and we have derived the continuity equation from Reynold's transport theorem.

Thank you.