Viscous Fluid Flow
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Module - 05
Steady, Two-dimensional Rectilinear Flows
Lecture - 03
Flow Through Elliptical Duct

Hello everyone. So, in today's lecture we will find the velocity profile for Poiseuille flow inside elliptical duct with uniform cross section. We will consider infinitely long elliptical duct and pressure gradient is constant with negligible gravitational acceleration and also we will consider fully developed flow.

In this case we have already derived the governing equation. So, we will start with that equation and in this particular case we will find the velocity profile or we will assume the velocity profile such a way that it will satisfy the no slip condition at the wall.

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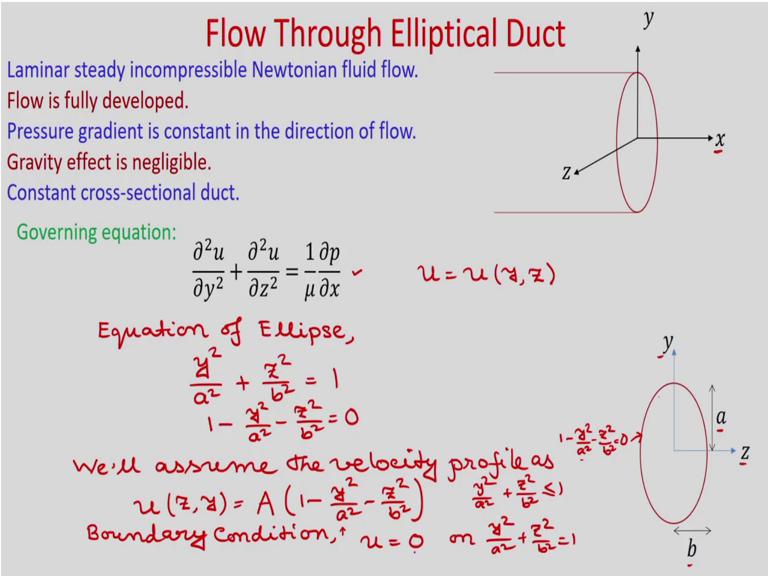

So, consider this elliptical duct where x is the axial direction, this is the y direction and this is the z direction. So, if you take any particular cross section then you will get this is the elliptical duct with uniform cross section where this is the z direction, this is the y direction, this is the major axis and this is the minor axis. So, obviously, for this particular case we know the governing equation is this one where del p by del x is a constant in the direction of flow and u is function of y and z right.

So, now we will try to find the velocity profile from the just assuming a velocity profile. So, let us consider the elliptical this duct for this wall what is the equation? We will consider first the equation of ellipse ok. So, you know for this particular case or particular geometry you can have the equation of ellipse as y square by a square plus z square plus b square is equal to 1 or we can write 1 minus y square by a square minus z square by b square is equal to 0.

So, this is the equation for this wall ok. So, 1 minus y square by a square minus z square by b square is equal to 0. So, we will assume the velocity profile inside this domain as some constant into this equation of ellipse then automatically the no slip condition will be satisfied at the wall for this velocity profile. So, we will assume the velocity profile as u which is function of z and y is equal to some constant into the equation of ellipse 1 minus y square by a square minus z square by b square ok.

So, from here you can see that the velocity profile clearly satisfies no slip condition at the tube walls because this term becomes 0 right at the wall and boundary condition. So, you can see that u will be 0 which is your no slip condition on the wall and this is valid when y square by a square plus z square by b square less than equal to 1 because this is actually inside the domain and when equal to 1 this will become boundary condition. Now, we will determine this constant a just satisfying the governing equation ok.

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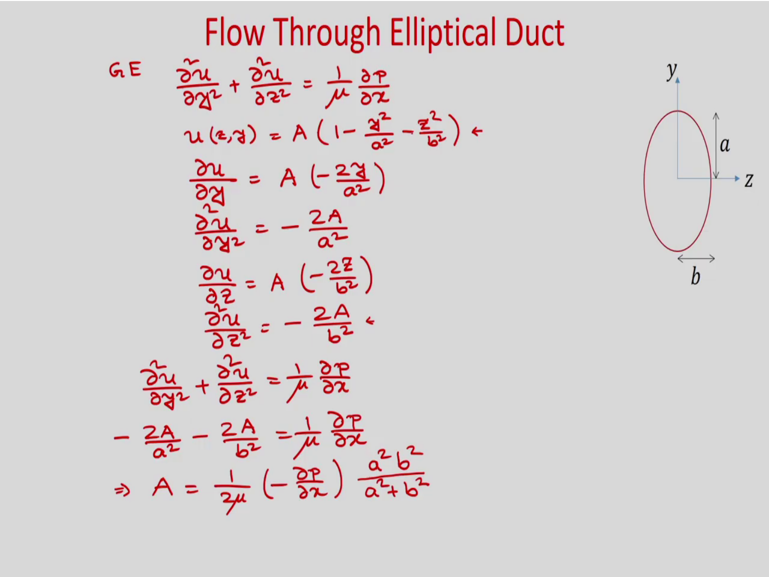

So, we know the governing equation as del 2 u by del y square plus del 2 u by del z square is equal to 1 by mu del p by del x right. So, now, we have assume the velocity profile as A into 1 minus y square by a square minus z square by b square. So, you find the derivative ok. So, you can see from here you can see del u by del y it will be just A minus 2 y by a square and if you write del 2 u by del y square then it will become minus twice A by a square.

Similarly, you can see the derivative del u by del z you can write as A minus 2 z by b square and del 2 u by del z square is equal to minus twice A by b square ok. Now you satisfy this derivatives in the governing equation and let us find the constant a. So, now, let us write in the governing equation del 2 u by del y square plus del 2 u by del z square is equal to 1 by mu del p by del x.

So, del 2 u by del y square is minus 2 A by a square and del 2 u by del z square it is minus 2 A by b square is equal to 1 by mu del p by del x. So, you can see from here it will be A is equal to 1 by twice mu minus del p by del x a square b square by a square plus b square ok. So, now, we know the constant A, we know the velocity profile. So, if you put the value of A here. So, we will get the velocity profile inside the elliptical duct.

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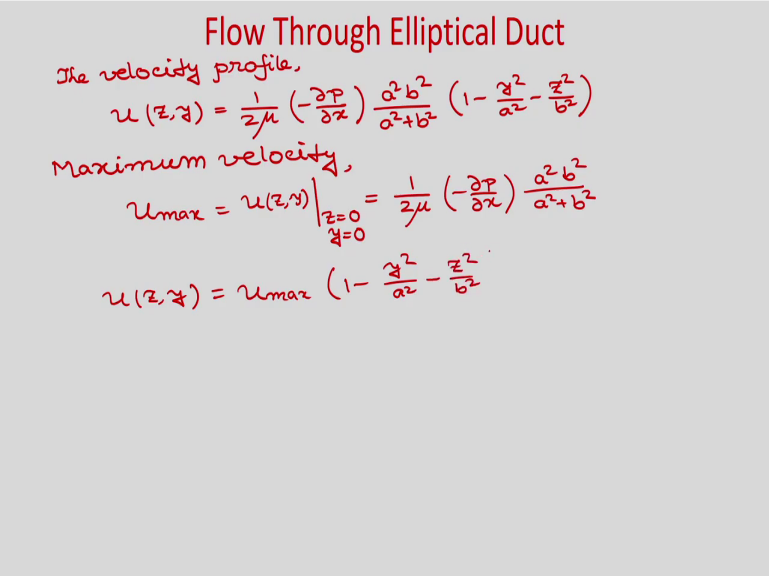

So, the velocity profile ok. So, if you put the value of a here you will get u as 1 by twice mu we will write minus del p by del x because this is the favorable pressure gradient, it will be positive we have a square b square by a square plus b square and 1 minus y square by a square minus z square by b square. So, this is the velocity profile inside the elliptical duct with uniform cross section, now if you want to find the maximum velocity where it is occur ok.

So, maximum velocity obviously, it will occur you can see that it will occur at the origin where y is equal to 0 and z equal to 0. So, from the symmetrical geometry and the boundary condition you can tell that ok. So, if you put z is equal to 0 and y is equal to 0 then you will get the maximum velocity at the center line ok. Let us find the maximum velocity. So, maximum velocity ok. So, you will get u max. So, where it will occur? Obviously, where ever we have taken the origin; that means, u at z is equal to 0 and y is equal to 0 at the centre line.

So, at the center line now if you put here z is equal to 0 y is equal to 0. So, you will get just 1 by twice mu minus del p by del x a square b square by a square plus b square. So, this is the maximum velocity and you can write this velocity profile in terms of u max as u z y is equal to. So, this first part you can see this will be just u max into 1 minus y square by a square minus z square by b square. Now first let us calculate the volumetric flow rate inside this elliptical duct and from there we will find the average velocity.

So, to find the volumetric flow rate inside this elliptical duct, we will consider this one quarter of this domain. So, you can see.

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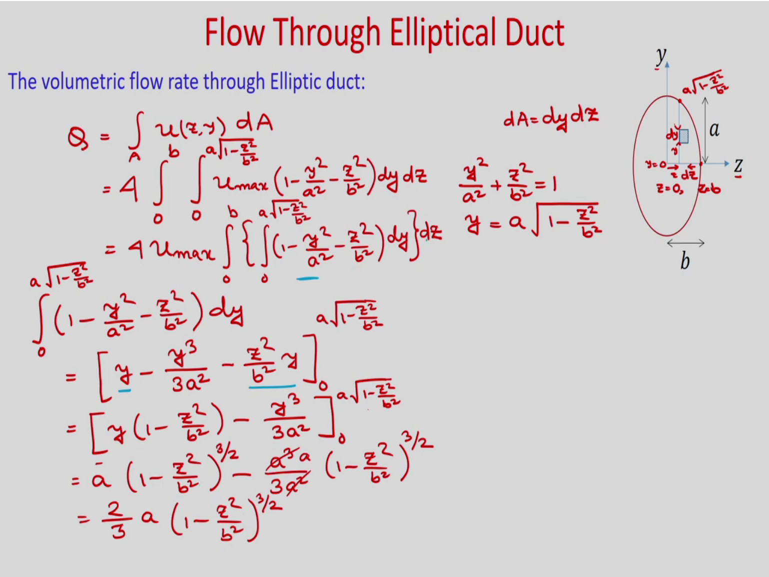

Ah If you see it is actually one fourth of the domain, if we can find the volumetric flow rate and if we multiply with 4, then we will get the total volumetric flow rate inside the elliptical duct ok. So, what we will do? We are considering this right top this part and this is the z this is the y and at a particular distance z let us take one elemental area of a distance d z.

Similarly, at a distance y you take one a small strip of length d y. So, you can see this elemental area is d y d z ok. So, d A is d y into d z ok. So, now, if you integrate this area over this one quarter of this elliptical duct, then you will get the total area and now let us find the volumetric flow rate for this one quarter as Q is equal to; obviously, it is area integral u which is function of z y d A right. So, now, area integral. So, this is the small elemental area we have considered.

So, now you can see that if you vary z is equal to 0 to z is equal to b at this point z is equal to b then at a particular location z you can see y will vary ok. So, how it will vary. So, that we can find from the equation of ellipse. So, you can see here that equation of ellipse is y square by a square plus z square by b square is equal to 1. So, from here you can see y as a root 1 minus z square by b square. So, you can see at a particular z you will have the y as a root 1 minus z square by b square and this y will vary with z ok.

So, now, we will write the integral. So, now, we are multiplying by 4 because whatever volumetric flow rate we are calculating in the one fourth of the domain. So, we are multiplying with 4 now we will integrate 0 to b first in z direction. So, now, in y direction now we will have from y is equal to 0 to y is equal to this ok. So, we will get 0 to a root 1 minus z square by b square ok.

Now, we have the expression of the u z y as u max u max is constant 1 minus y square by a square minus z square by b square into d y d z ok. First we will integrate with respect to y then we will integrate with respect z ok. So, first let us evaluate this term ok. So, we will can write 4 u max outside the integral 0 to b. So, first we will evaluate this integral. 0 to a root 1 minus z square by b square 1 minus y square by a square minus z square by b square d y then after that we will integrate this term with respect to z ok.

So, first let us evaluate the integral inside the curly bracket. So, we can see. So, 0 to a root 1 minus z square by b square 1 minus y square by a square minus z square by b square d y this first let us evaluate.

So, you can see if you integrate it, it will be y minus y cube by 3 a square minus z square by b square y limit 0 to a root 1 minus z square by b square ok. So, you can see that if we take y common from this two terms then we can write it as y 1 minus z square by b square and we have minus y cube by 3 a square limit 0 to a root 1 minus z square by b square.

So, now, if you put the value of y is equal to 0 so; obviously, it will become 0. So, y is equal to just a root 1 minus z square by b square you just write. So, you will get. So, y if you put.

So, it will be 1 minus z square b square to the power half and here 1 is there. So, 1 plus half will be 3 by 2. So, you can write a 1 minus z square by b square to the power 3 by 2 ok.

And minus so, you can see here y cube. So, it will be a cube by 3 a square and it is 1 minus z square by b square to the power half. So, it will be 3 by 2. So, 1 minus z square by b square to the power 3 by 2. So, this 1 a will be here and you can see from here. So, it will be a by 3 and a; that means, it will be 2 by 3 right. So, it will be 2 by 3 a 1 minus z square by b square to the power 3 by 2. So, now we can see that we have evaluated this term which is 2 by 3 a 1 minus z square by b square to the power 3 by 2. So, let us put it here and again integrate it.

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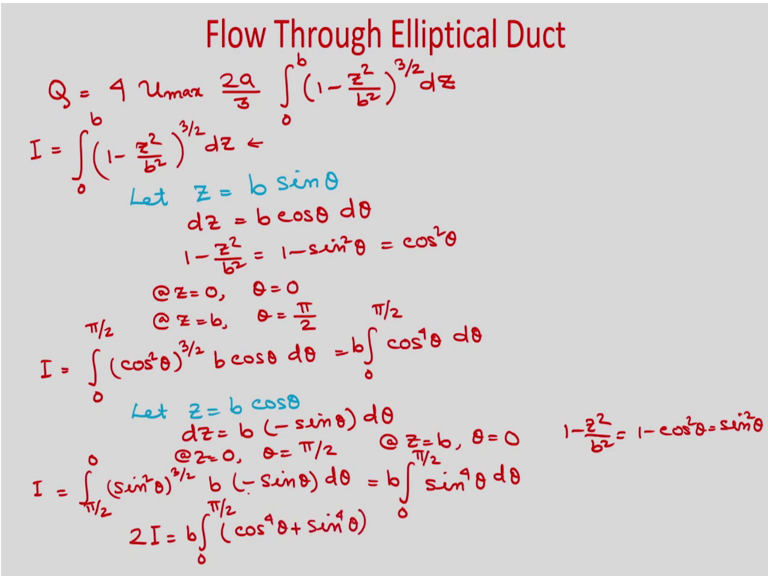

So, now we got Q as 4 u max twice a by 3 integral 0 to b 1 minus z square by b square 3 by 2 d z. Now let us integrate this term ok. So, just we will show the derivation now we need to evaluate this term 0 to b 1 minus z square by b square to the power 3 by 2 d z ok. First we will just take let z is equal to b sin theta ok.

So, if z is equal to b sin theta then we can write d z is equal to b cos theta d theta, now if you put here you can see 1 minus z square by b square will be just 1 minus sin square theta. So, it will be cos square theta. Now let us discuss about the limit ok. So, you can see at z is equal to 0 so, obviously, you can see that theta will be 0 because sin theta is 0 and at z is equal to b ok. So, sin theta is 1. So, if sin theta is 1 then theta is equal to pi by 2 ok.

So, now if we say that the integral is I, then you can write I is equal to integral 0 to pi by 2. So, in terms of theta we are writing the limit cos square theta to the power 3 by 2 and d z is b cos theta d theta ok. So, this you can write as 0 to pi by 2 and b is constant you can write outside the integral and here it be cos cube theta and cos theta. So, it will be cos 4 theta d theta and again let us write that z is equal to b cos theta ok.

So, if z is equal to b cos theta then we can find d z is equal to b minus sin theta d theta and if you see the limit at z is equal to 0 so; obviously, cos theta is equal to 0 and cos theta is equal to 0 means theta will be pi by 2 and at z is equal to b obviously, cos theta is 1 so, theta will be 0. So, this integral now we can write this integral if you put these values again we can write that I is equal to pi by 2 to 0 and 1 minus z square by b square. So, it will become 1 minus cos square theta so; that means, it is sin square theta ok.

So, we will get sin square theta to the power 3 by 2 and this will get b minus sin theta d theta ok. So, here one minus sin is there and integral pi by 2 0 there. So, now, we will just change the limit from 0 to pi by 2. So, this minus sin will not be there. So, we will get now sin to the power 4 theta, d theta and b will be there outside the integral. So, what we will do now just we will add these two. So, if we add these two then we will get 2 I is equal to 0 to pi by 2 and we have cos 4 theta plus sin 4 theta d theta.

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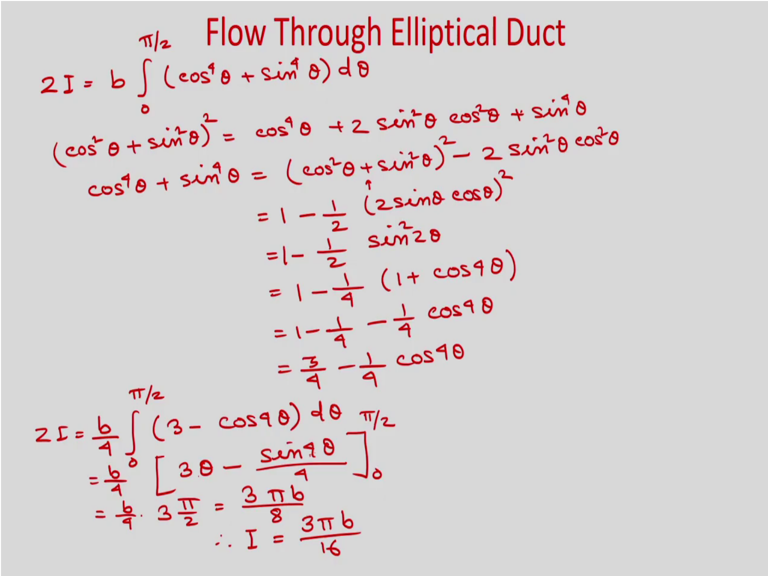

Now, let us find the integral. So, 2 I we can write as b 0 to pi by 2 cos 4 theta plus sin to the power 4 theta d theta. So, this term you can see we can write as cos square theta plus sin square theta whole square we can write as cos to the power 4 theta plus 2 sin square theta cos square theta plus sin to the power 4 theta ok. So, you can see from here we can write cos to the power 4 theta plus sin to the power 4 theta is equal to. So, cos square theta plus sin square theta whole square minus 2 sin square theta cos square theta.

What is the value of this term? Right hand side first term. So, it is 1 right. So, we will get 1 and what we will do here? We will write just half 2 sin theta cos theta square. So, now, we know that 2 sin theta cos theta is sin 2 theta. So, we can write 1 minus half sin square 2 theta and now we can again write that 1 minus 1 by 4 2 sin square theta ok. Now 2 sin square theta is 1 plus cos 4 theta ok.

So, it will be 1 minus 1 by 4 minus 1 by 4 cos 4 theta. So, it will be 3 by 4 minus 1 by 4 cos 4 theta ok. So, now, let us put it here and integrate it. So, we will get 2 I is equal to b by 4 integral 0 to pi by 2.

So, we will get 3 minus cos 4 theta d theta it will be b by 4. So, we can see it will be 3 theta minus sin 4 theta by 4 0 to pi by 2. So, we can see here that for theta is equal to 0; obviously, first term will become 0, sin 0 will 0 and if you put theta is equal to pi by 2. So, this will become sin 2 pi and sin 2 pi value is 0.

So, second term will be 0. So, we can write b by 4 3 pi by 2 ok. So, this will be just 3 pi b by 8. So, now, we have found the integral I which actually we wanted to find. So, that is 3 pi b by 16. So, now, we have found the integral I. So, now, if we put this value in the expression of volumetric flow rate, then we can find the volumetric flow rate of inside elliptical duct.

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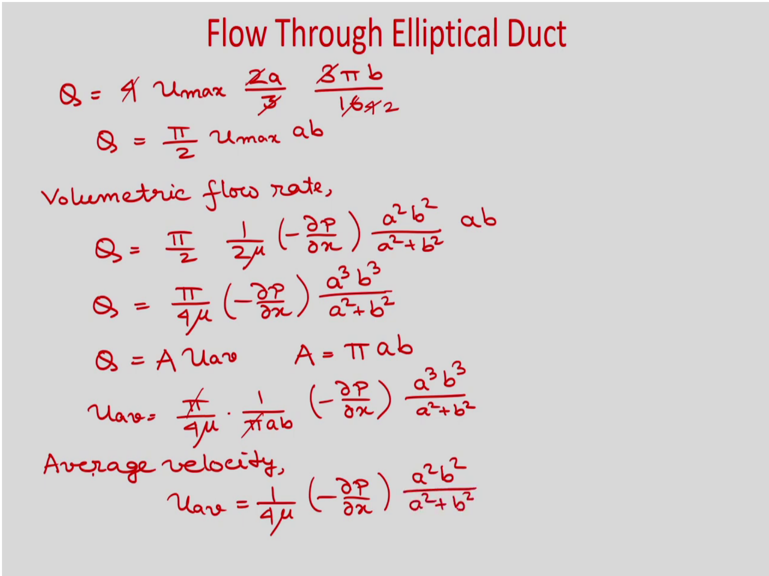

Q will be 4 u max twice a by 3 and that integral we have evaluated as 3 pi b by 16 ok. So, you can see. So, 4 it will become 4 this 2 it will become 2 this 3 3 will get cancelled. So, Q we will get as pi by 2 u max a b ok. So, now, if we put the value of u max, then we can write the volumetric flow rate as Q is equal to. So, u max is 1 by twice mu minus del p by del x a square b square by a square plus b square and another a b is there. So, Q will be pi by 4 mu minus del p by del x a cube b cube by a square plus b square ok.

So, you can see this term minus del p by del x is positive right so; obviously, Q is positive and now let us find what is the value of average velocity. So, for this elliptical duct we know the area of an ellipse is pi a b.

So, you can find the u average. So, Q is equal to A into u average and A is pi a b ok. So, you can see now u average you can find. So, this is the Q if you divide by pi a b. So, you can see pi by 4 mu if you divide by a. So, it will be 1 by pi a b minus del p by del x a cube b cube by a square plus b square ok.

So, this pi pi will get cancelled this a b. So, it will become a square b square. So, you can write average velocity u average as 1 by 4 mu minus del p by del x a square b square by a square plus b square. Now let us consider two special cases. So, you can see for elliptical duct if you put a is equal to b is equal to R then; obviously, it becomes circular duct. So, we will consider that as a first case.

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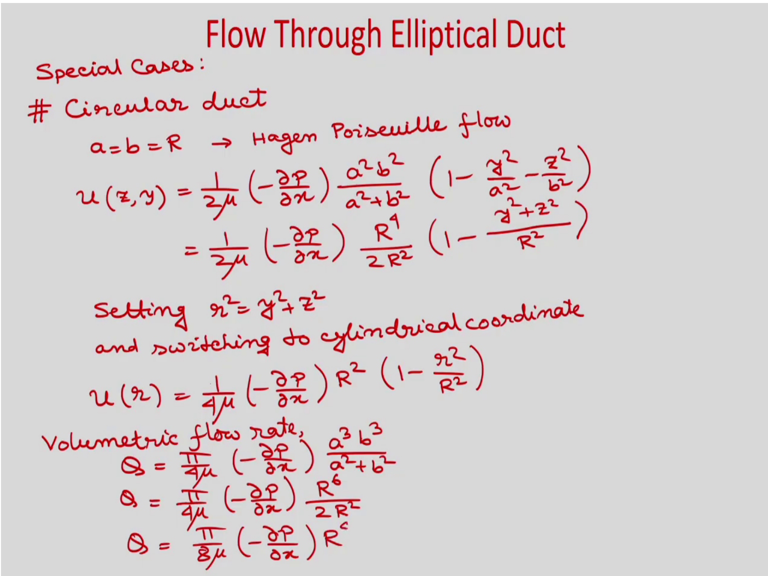

So, special cases. So, we will consider circular duct ok. So, if you put a is equal to b is equal to R then it will become just Hagen Poiseuille flow right Hagen Poiseuille flow ok. So, now, if you put a is equal to b is equal to R then we will get the Hagen Poiseuille flow. So, we have the expression of velocity u z y as 1 by twice mu minus del p by del x a square b square by a square plus b square 1 minus y square by a square minus z square by b square ok. So, now, let us put the value of a b as R.

So, you will get 1 by twice mu minus del p by del x. So, it will be R square R square. So, R to the power 4 here R square plus R square 2 R square and you will get 1 minus. So, in the denominator if you write R square. So, it will become y square plus z square.

So, now, if we switch to cylindrical coordinate and if you put small r square is equal to y square plus z square then we can see the Hagen Poiseuille flow this velocity profile. So, now, setting r square is equal to y square plus z square and switching to cylindrical coordinate. So, we can put here. So, you can see u will be function of r only.

So, it will become 1 by 4 mu minus del p by del x and you can see here R square and 1 minus r square by R square. So, you can see that this is the same expression what we derived for Hagen Poiseuille flow and now if you calculate the volumetric flow rate. So, we have Q is equal to pi by 4 mu minus del p by del x a cube b cube by a square plus b square. So, if you put a b as r then you can write Q is equal to pi by 4 mu minus del p by del x. So, it you can see you it will be R cube R cube. So, R to the power 6 divided by 2 R square

And final you can write Q as. So, pi by 8 mu minus del p by del x R to the power 4. So, it is the same expression what we derived for Hagen Poiseuille flow. So, now, let us derive the special case where you consider the plane Poiseuille flow ok. So, in case of plane Poiseuille flow we will consider b is equal to h and a is much much greater than h where 2 h is the distance between the 2 parallel plates.

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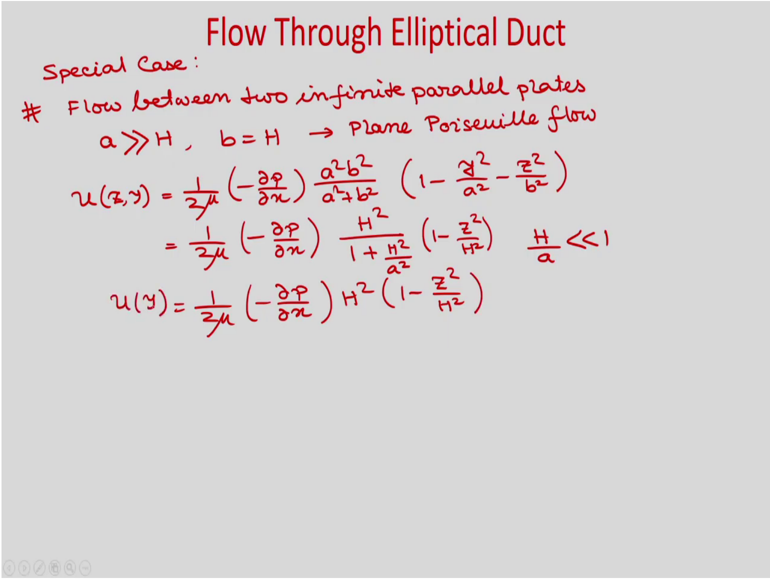

So, special case. So, this is the second special case where we are considering flow between two infinite parallel plates.

Here we will consider a is much much greater than H and b is equal to H. So, it will give plane Poiseuille flow ok. So, if we have the expression u which is function of z y as 1 by twice mu minus del p by del x a square b square by a square plus b square 1 minus y square by a square minus z square by b square ok. So, you can see a is much much greater than H. So, ah; obviously, this term y square by a square will become very very small.

So, now, y square by a square will become very very small. So, if you put it then we will get 1 by twice mu minus del p by del x. So, if you put here b square as H square divided by. So, you divide by H square. So, it will be 1 plus b square means H square a square by H square.

So, you can see H by a is much much smaller than 1 much much smaller than 1 so, obviously, you can write this as H square and we have 1 minus ok.

So, we can have z square divided by H square because y square by a square will be very small. So, now, this will be very small. So, you can write 1 by twice mu minus del p by del x H square into 1 minus z square by H square ok. And now u is function of y only.

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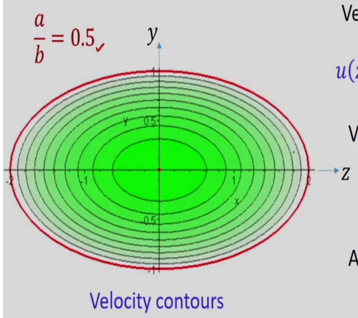
Flow Through Elliptical Duct



Velocity distribution:

$$u(z,y) = \frac{1}{2\mu}\left(-\frac{\partial p}{\partial x}\right)\frac{a^2b^2}{a^2+b^2}\left(1-\frac{y^2}{a^2}-\frac{z^2}{b^2}\right)$$

Volumetric flow rate:

$$Q = \frac{\pi}{4\mu}\left(-\frac{\partial p}{\partial x}\right)\frac{a^3b^3}{a^2+b^2}$$

Average velocity:

$$u_{av} = \frac{1}{4\mu}\left(-\frac{\partial p}{\partial x}\right)\frac{a^2b^2}{a^2+b^2}$$

Maximum velocity:

$$u_{max} = \frac{1}{2\mu}\left(-\frac{\partial p}{\partial x}\right)\frac{a^2b^2}{a^2+b^2} \quad \text{at } (0,0)$$

John Lekner, Viscous flow through pipes of various cross-sections, European Journal of Physics, Vol. 28, pp. 521-527, 2007.

So, we have found the velocity profile for plane Poiseuille flow and it is same as we derived for the flow between two infinite parallel plates. So, now, let us summarize what we have done in today's class. So, we considered one elliptical duct with uniform cross section and we have found the equation for the wall first and; obviously, we assume the velocity profile such a way that it will satisfy the no slip condition at the wall.

So, from the we have written velocity profile as constant into the equation of the wall. So, from there we found the velocity profile and we found the constant a satisfying the governing equation. So, after that we found the volumetric flow rate and once we found the volumetric flow rate we could find the average velocity and we know the maximum velocity will occur at the central line and we have put y is equal to 0 and z is equal to 0 in the expression of velocity profile and we found the maximum velocity.

So, you can see that this is the velocity distribution whatever we have found and this is the volumetric flow rate and average velocity just we have divided by pi a b which is your area. So, then we got the average velocity and setting y is equal to 0 z is equal to 0 we found the maximum velocity. So, you can see here for the case a by b is equal to 0.5 we are showing the velocity contour.

So, in this case you can see that maximum velocity obviously, will occur at the centre line and these are the lines with constant velocity magnitude . So, you can see velocity contours follow the wall ok. So, it is the profile is similar like the wall of this duct.

Thank you.