Viscous Fluid Flow
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Module - 04
Transient One – dimensional Unidirectional Flows
Lecture - 04
Transient Axisymmetric Poiseuille Flow

Hello everyone. So, in last class, we have solved transient Couette flow, where you have learned when and how we can use separation of variables method to solve a one dimensional transient problem.

So, in last class, you have seen how to choose the sign of the constant depending on the homogeneous and non homogeneous direction. So, obviously in the homogeneous direction, we considered the sign of the constant such a way that, in homogeneous direction we get characteristic value problem.

You have also learned the solution of Hagen Poiseuille flow, which is actually axisymmetric Poiseuille flow and the velocity profile is parabolic. So, that you have already derived the velocity profile for steady Hagen Poiseuille flow. Today we will consider transient axisymmetric Poiseuille flow; that means transient Hagen Poiseuille flow. So, where you have zero initial condition everywhere and suddenly at t is equal to 0 plus, you have a imposed pressure gradient del p by del z.

So, obviously, the evolution of velocity profile will occur and from zero to parabolic profile. So, you can see that at a larger time, where t tends to infinity; obviously you will have a steady state solution. So, for this problem, again we will use separation of variables method and we will convert the partial differential equation to two sets of ordinary differential equation.

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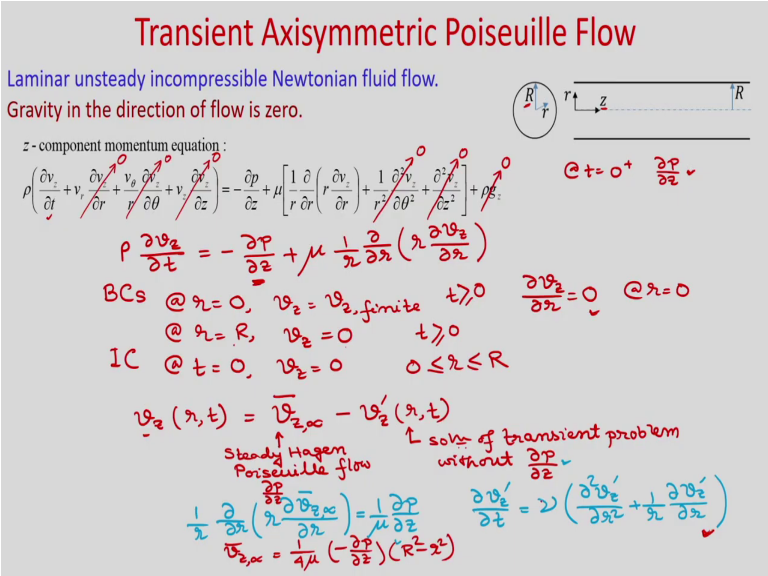

So, now you can see that this is our flow inside a circular pipe, where pipe radius is r and z is the axial direction; r is measured from the central line and we consider laminar unsteady incompressible

Newtonian fluid flow and at t is equal to 0 plus, we have a imposed pressure gradient del p by del z. So, initially this fluid medium is velocity of the fluid medium is zero and suddenly at t 0 plus, as you have a imposed pressure gradient del p by del z; so obviously flow will start and the velocity will get in the positive z direction. And we are also neglecting the gravity in the direction of flow.

So, you can see this is your z component of momentum equation and as it is a unsteady flow; so obviously this first term will remain and for other terms as we have discussed earlier in the left hand side, it will become 0, ok. And also you can see in the right hand side, these terms also will become 0, ok. So, we have already discussed why those are 0.

So, obviously, you will get the governing equation for this unsteady incompressible fluid flow as rho del v z by del t that is the unsteady term and in the right hand side, you will get the pressure gradient term and we have the viscous term.

So, now, you can see this is the partial differential equation and we have the boundary conditions at r is equal to 0; obviously you have a finite velocity v z is finite for t greater than equal to 0. And we have at r is equal to R, v z is equal to 0 ok; because no slip boundary condition, so v z will be 0. So, t greater than equal to 0 and initial condition at t is equal to 0; obviously v z is equal to 0 everywhere, ok.

So, you can see at r is equal to 0; obviously these boundary condition you know that, you will get maximum velocity at the central line. So, this also you can write as del v z by del r is equal to 0, ok. So, at r is equal to 0, you can write del v z by del r is equal to 0. So, if you see the boundary conditions, obviously r direction you have two homogeneous boundary conditions; because this is your homogeneous Neumann del v z by del r is equal to 0 and r is equal to R, v z is equal to 0.

So, these are homogeneous boundary conditions. So, r is the homogeneous direction. So, now, the question is that, can you now use separation of variables method for this problem? So, what we learnt in the last class that, to use separation of variables method; you should have linear and homogeneous governing equation and in homogeneous direction, you should have two homogeneous boundary conditions. So, you can see although r is homogeneous direction, but the governing equation is non homogeneous.

So, you can see due to the presence of this pressure gradient minus del p by del z, this becomes non homogeneous governing equations. So, directly we cannot use the separation of variables method. So, again we will use the superposition method. So, we will find the solution v z, which is function of r and t as a solution from the steady problem with a imposed pressure gradient, which is your solution of Hagen Poiseuille flow at minus 1 transient velocity neglecting the pressure gradient, ok.

So, now you can see that we will decompose this problem v z, which is function of r and t as v bar z infinity, t infinity. So, this is a steady state solution, minus 1 transient term v z prime which is function of r and t. So, we are decomposing this velocity in such a way that, this is your steady Hagen Poiseuille flow, ok. So, solution from steady Hagen Poiseuille flow, where you have a imposed pressure gradient del p by del z.

But these problem if you put it here that you will get, this is the solution of transient problem without pressure gradient. So, you can see that from here, if you impose here; so you will get two equations ok, I am not going to solve it. But if you put del v z by del t and put it here and this time if you put; then you will you are going to get two sets of equation, ok. So, one is for the steady state problem. So, you will get this problem as 1 by r del of del r r del v z, infinity bar; so this is the steady state velocity profile, del r is equal to 1 by mu del p by del z.

So, this is one problem and you will get the solution and you can find v z infinity and this problem if you substitute, you will get another governing equation that is your del v z prime by del t is equal to nu del 2 v z prime by del r square plus 1 by r del v z prime by del r. So, you can see this is the transient velocity profile v z prime for this axisymmetric Poiseuille flow, but there is no pressure gradient term involved.

So, obviously, you can see this is a homogeneous equation right; but this is your steady state problem with pressure gradient and you know the solution for this problem. So, this is the solution you know, v z bar, infinity is equal to 1 by 4 mu minus del p by del z into R square minus r square. So, you have already solved this problem.

Now, we need to find the solution of v z prime for this problem and this problem you can see this is a homogeneous and linear equation. Now, let us see the boundary conditions; if r direction is homogeneous boundary conditions we have, then we can use separation of variables method.

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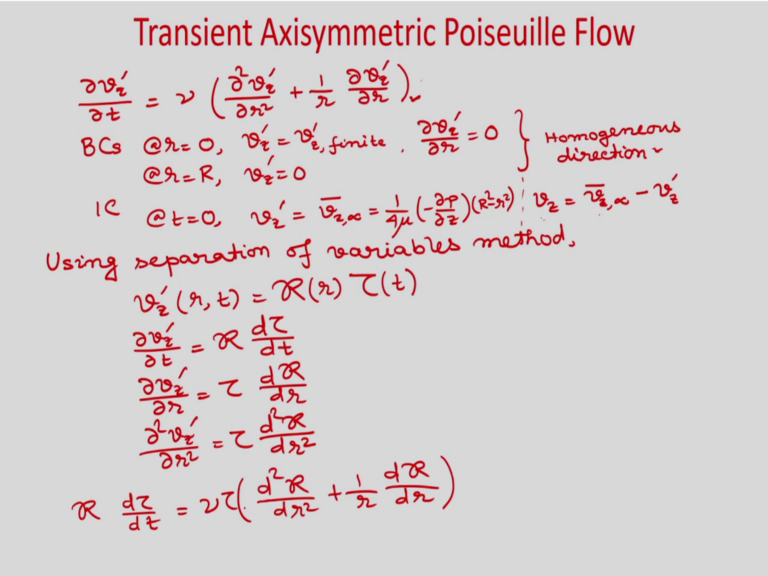

So, now let us write the governing equation. So, we have del v z prime by del t is equal to nu del 2 v z prime by del r square plus 1 by r del v z prime by del r, ok.

So, what are the boundary conditions? If you see the boundary conditions for this problem, you will get at r is equal to 0, again v z prime will be finite ok; at r is equal to R, v z prime will be 0. So, this you will get del v z prime by del r is equal to 0; because maximum or minimum velocity you will get at central line, ok. So, del v z by del r will be 0 at r is equal to 0 and r is equal to R, v z prime is equal to 0. So, you can see it is homogeneous direction, ok. And what is the initial condition?

Now, if you put, because you have v z is equal to v bar z, infinity minus v prime z, right. So, you can see, this is the solution of steady Hagen Poiseuille flow and v z is 0 at r is equal at t is equal to 0. So, at t is equal to 0, v z is equal to 0. So, v z prime you will get v z, infinity bar, ok. And this is the solution you know that, 1 by 4 mu minus del p by del z into R square minus r square. So, this is the initial condition for this problem, ok.

So, you can see now we have linear and homogeneous governing equation and r direction is homogeneous direction, so we can use separation of variables method, ok. So, now, we will find the solution of v z prime as product of two individual solution; R which is function of r only and tau which is function of t only, ok. So, using separation of variables method ok, we will find the solution v z prime, which is function of r and t as product of two solutions; one is R which is function of r, and another solution tau which is function of t only.

So, now you find del v z prime by del t is equal to R d tau by d t ok; because tau is function of t, so ordinary derivative we are writing. And del v z prime by del r; so obviously you can see it will be tau d R by d r and del 2 v z prime by del r square, so you will get tau d 2 R by d r square, ok. So, now, all these terms you put it here, ok. So, what you are going to get? So, we will get R d tau by d t is equal to nu. So, tau I am taking outside d 2 R by d r square plus 1 by r d R by d r.

So, what we will do now, we take all the tau term in the left hand side and all the R terms and which is function of r in the right hand side. So, we have separated the variables.

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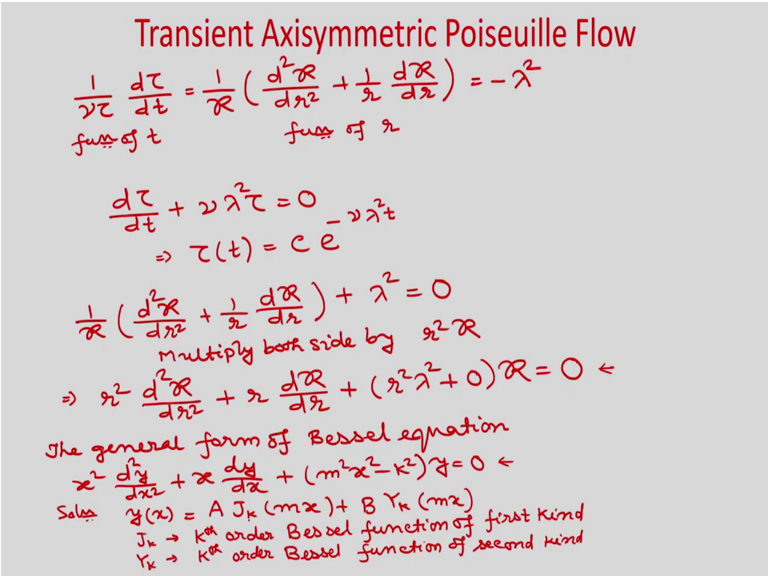

So, if you write in that way. So, we can write it now 1 by nu tau d tau by d t. So, you can see left hand side is function of time only and right hand side 1 by r d 2 R by d r square plus 1 by r d R by d r.

So, you can see this is function of t only and this is the function of r only. So, you can see that, we have separated the variables and left hand side depends only time only and in the right hand side it depends on r only. So, we can write this equal to some constant. Now, what will be the sign of that constant ok? So, we have to choose the sign of that constant such a way that in homogeneous direction, you get a characteristic value problem; that means its solution will be periodic in nature.

So, now we will choose the value as, the sign of the constant as minus, so that we will get a Bessel equation, ok. So, and we know the solution of Bessel equation is periodic, so that I will show here, so we are choosing the constant sign as minus. So, we will use minus lambda square, ok. So, you can see that, this if you write. So, you will get this equal to minus lambda square. So, you will get one ordinary differential equation d tau by d t plus nu lambda square tau is equal to 0 and the solution you will get tau which is function of t only.

Let us say it is c e to the power minus nu lambda square t ok, where c is the integration constant. And another equation you will get, that is your 1 by r d 2 R by d r square plus 1 by r d R by d r plus lambda square is equal to 0.

So, if you multiply both side with r square R, so what you will get? Multiply both side by r square R, ok. So, if you multiply, what you will get? So, you can see, here you will get r square d 2 R by d r square plus r d R by d r and this you will get; you can see r square lambda square plus 0 we are writing into R is equal to 0, ok.

So, now this equation resembles with the Bessel equation; if you remember the Bessel equation, we can write as the general form of Bessel equation x square d 2 y by d x square plus x d y by d x plus m square x square minus k square y is equal to 0. And the solution of this Bessel equation is, y is equal to constant A J k m x plus B Y k m x.

So, you can see these J k is known as Kth ordered Bessel function of first kind and Y k is known as Kth order Bessel function of second kind, ok. Now, you compare these two equations; this equation and this equation.

So, you can see here, this k square is 0, right. So, you can see the order of this Bessel function is 0. So, the solution we can write as A J 0 lambda r plus B Y 0 lambda R, ok. So, you can see, because if you see the these two equations, similarly this k is 0; so the order of these Bessel function is 0, ok.

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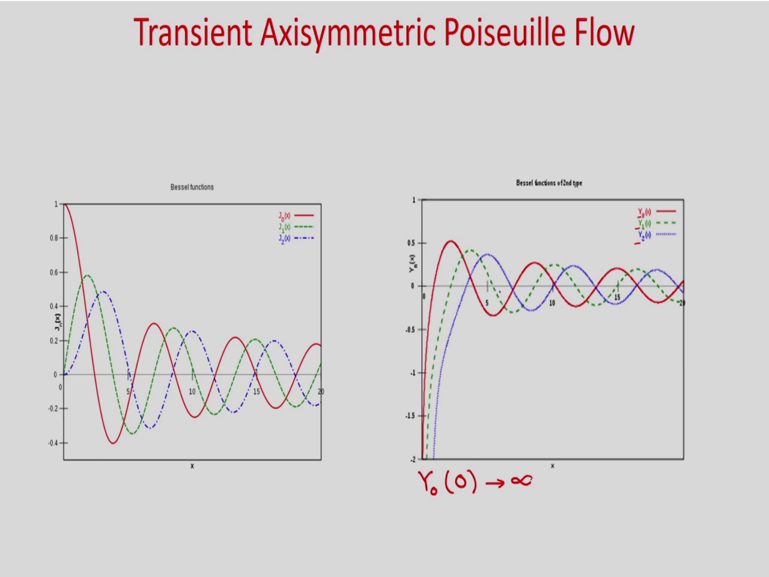

And we can write, now you can see that the solution; if you see the solution of this J ok, J k, so you can see this solution is periodic. So, this is the zeroth order Bessel function of first kind, this is the second order; this is the first order, this is the second order. So, J 0, J 1, J 2 and you can see the solution is a periodic in nature. And similarly if you see the second kind of Bessel function, so that is your Y. So, Y 0 is zeroth order, this is the first order, this is the second order and you can see its solution is also periodic, ok.

So, the in r direction is homogeneous direction; so obviously you can see that you are getting a characteristic value problem and you can see that Y 0, Y 0 at x equal to 0 it is infinity. So, Y 0 at x equal to 0, it is infinity ok; from here you can see this curve.

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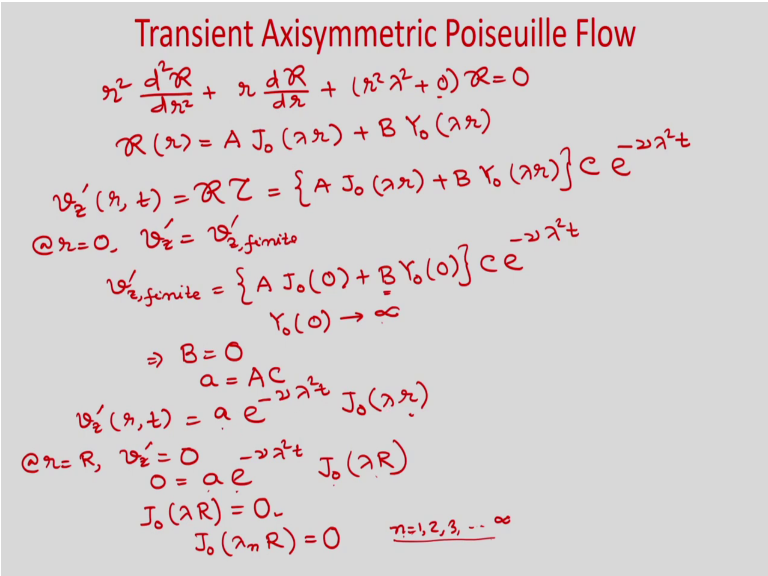

So, now we can write the solution of this equation. So, we have r square d 2 R by d r square plus r d R by d r plus r square lambda square plus 0 R is equal to 0. So, we have written 0 to know the order of this Bessel equation.

So, obviously, you can see the solution you can write r is equal to constant A and it is zeroth order, so it will be J 0. So, this is the Bessel function of first kind lambda r plus B Y 0 lambda r, where Y 0 is the zeroth order Bessel function of second kind, ok.

So, now we have found the solution of r and tau individually. So, now, you can find the actual velocity profile v prime z and apply the boundary condition and find the constants. So, now, we can write v z prime which is function of r, t as product of r and tau, ok. So, you can see this will be A J 0 lambda r plus B Y 0 lambda r into C e to the power minus nu lambda square t.

So, now let us find the constants applying the boundary conditions. So, we know at r is equal to 0, v z prime is equal to v z prime finite, ok. So, this should have finite value. So, if you apply here. So, left hand side will be v z finite value; but in right hand side, the first term you can see A J 0 0 plus B Y 0 0 into c e to the power minus nu lambda square t.

So, we have seen that, what is the value of Y 0 at lambda r is equal to 0. So, you have seen that it is infinity, right. So, if it is infinity, left hand side is finite value. So, B must be 0; because this term has to be 0, because Y 0 tends to infinity right. But left hand side is finite value of v z prime, so we should have B is equal to 0, ok. So, B is equal to 0 and let us write one another constant a as product of A into C. So, you can write v z prime r, t is equal to a e to the power minus nu lambda square t and J 0 lambda r, ok.

So, now you apply the another boundary condition at r is equal to R, r is equal to R, v z prime is equal to 0, ok. So, if it is 0, then the left hand side is 0; we have a e to the power minus nu lambda square t J 0 lambda r, ok. Now, you can see this constant cannot be 0; otherwise you will not get the solution right, this term is not 0. So, J 0 lambda R must be 0, ok. So, J 0 lambda R should be 0.

Now, if you remember that, we have seen the solution is periodic and at different values of lambda R, J 0 becomes 0. So, for different values of lambda, you will get J 0 lambda R as 0. So, we can write that J 0 for different values of lambda n, you will get this as 0, ok. So, you will get different solution for different values of lambda ok, for n 1, 2, 3 to infinity ok. Because you have seen the solution is periodic; so obviously at different values of lambda r, you will get these J 0 lambda R as 0, so obviously J 0 lambda n R will be 0 for different values of n.

So, you will get a different solution. So, these will get a different solution for different n values; that means different lambda n values and you can actually sum it up, because you have the linear governing equation.

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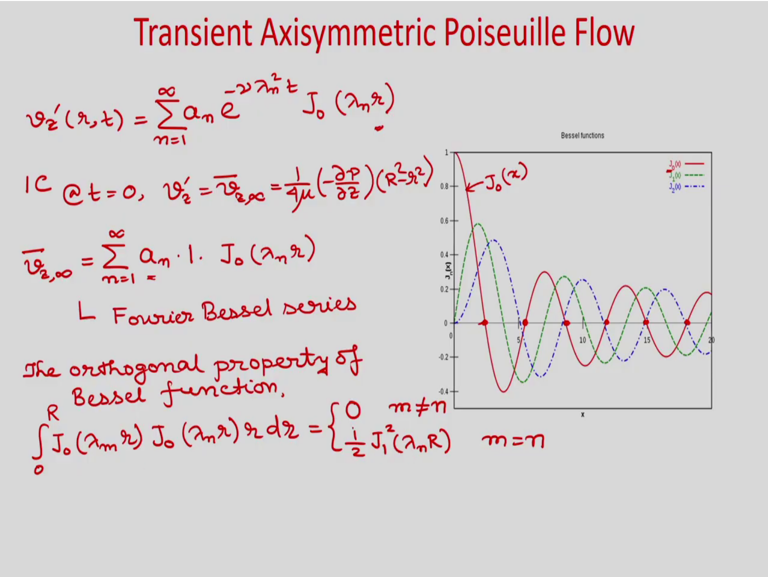

So, you can see here. So, this is your J 0 ok, J 0. So, J 0 lambda R; so that means this you can see here it is becoming 0, here it is becoming 0, here it is becoming 0, here it is becoming 0. So, you can see at different values of x, it is becoming zero; that means at a different lambda n R value, this J 0 will become 0, because this is the J 0, right.

So, you can see that will get a different solution; so for different values of lambda n, so we can write the solution v z prime r, t at summation of all the solutions for different values of lambda n, because it is a linear governing equation. So, you can write this sum of all the solutions. So, we will get a n e to the power minus nu lambda n square t J 0 rho lambda n r.

So, this we have written the sum of all the solutions, ok. Now, you apply the initial condition ok; at t is equal to 0, v z prime is equal to v bar z, infinity, which is your 1 by 4 mu minus del p by del z R square minus r square. So, now, if you apply this initial condition, then from there we have to find the another constant a n right; because a n is unknown.

So, if you can find the constant n, then you will be able to find the velocity profile v prime z, ok. So, now, if you put it here, so you will get left hand side v bar z, infinity is equal to summation of n is equal to 1 to infinity a n. So, this at t is equal to 0, this value will become 1 and we have J 0 lambda n r. So, you can see this is a Fourier Bessel series, ok.

So, now we have to find the constant a n from here. So, as we discussed the orthogonality property earlier; so for these Bessel function also we will discuss about the orthogonal property.

So, the orthogonal property of Bessel function ok; so these J 0 lambda m r and J 0 lambda n r will be orthogonal to each other and you will get integral 0 to R J 0 lambda m r J 0 lambda n r r d r in this case is equal to. So, you will get 0 for m not equal to n and this you will get half J 1 square lambda n R for m is equal to n.

So, for m is equal to n, this you will get half J 1 square lambda n R, where J 1 is the first order Bessel function of first kind J 1, ok. So, you will get half J 1 square lambda n R. So, now, if we use this property, then you can find the value of this constant n, ok.

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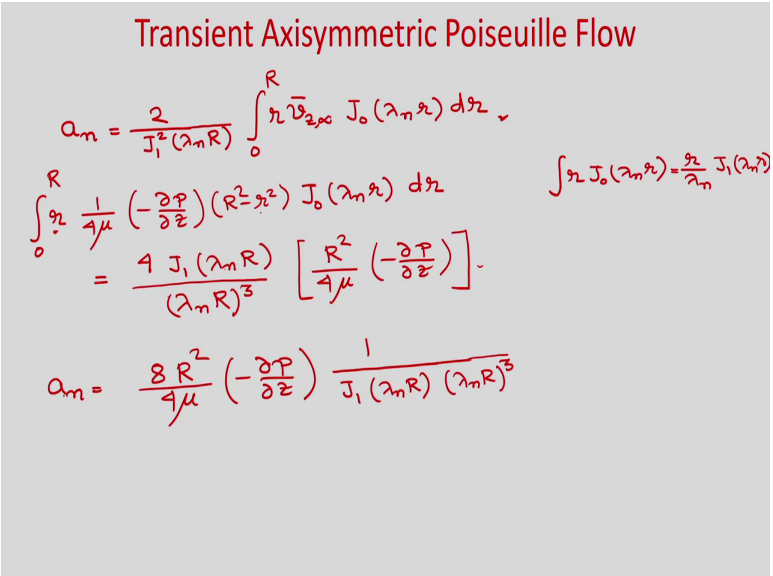

So, you can see that, we can find a n as 2 by J 1 square lambda n R, which is your this from orthogonal property, you are getting half J 1 square lambda n r for m naught equal to n and you will get 0 integral 0 to R r v z, infinity bar J 0 lambda n r d r, ok.

So, now we have to find this integral, ok. So, if you put the value of v z, infinity bar. So, you will get 0 to R r is there; this you will get 1 by 4 mu minus del p by del z R square minus r square. So, this is the velocity profile, then we have J 0 lambda n r d r, ok. So, you can see one r is here, r square and J 0 lambda n r.

So, you can actually see some mathematics book and find how to integrate this Bessel function; one of the integral I will just write, integral r J 0 lambda n r ok; you will get r by lambda n J 1 lambda n r. So, you can see this zeroth order this Bessel function of first kind becomes first order Bessel function of first kind, ok.

So, there are some integral integration rules. So, you can see for these Bessel functions what are the integrals and finally, you will get the integral of this quantity as. So, you will get 4 J 1 lambda n R divided by lambda n R cube and we will have R square by 4 mu minus del p by del z, ok. After the integration, you will get this; so you can see a n, a n will become. So, this will be, now you can see in the denominator, you have J 1 square lambda n R and in the numerator you have J 1 lambda n R. So, one will get cancelled 2 into 4 will become 8.

So, you will get 8 R square by 4 mu minus del p by del z, ok. So, in denominator you will have one J 1 lambda n r. So, 1 by J 1 lambda n R and we have lambda n R cube, so lambda n R cube, ok. So, now, if you put this constant a n; then you will get the final velocity profile.

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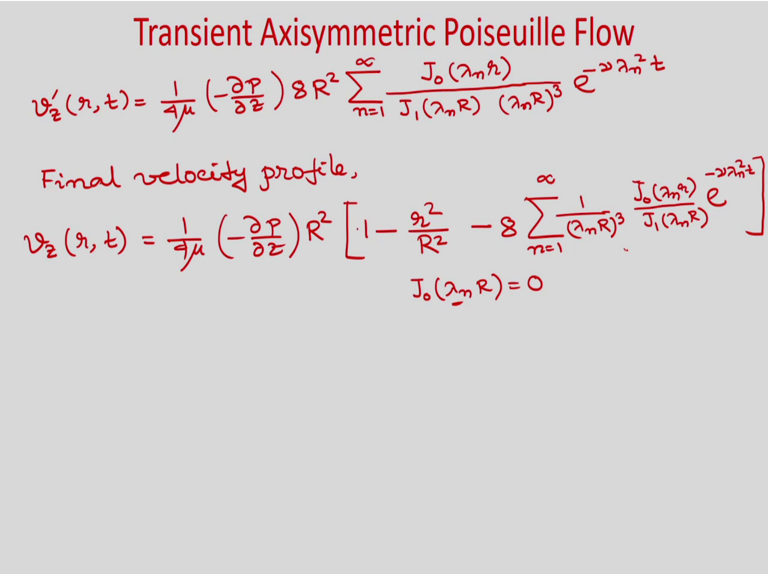

So, v z prime r, t; so you will get as 1 by 4 mu minus del p by del z, ok. Then we will get 8 R square; after putting the value of n, we are writing this expression.

Summation of n is equal to 1 to infinity J 0 lambda n r divided by J 1 lambda n R; then we have lambda n R cube and e to the power minus nu lambda n square t. So, we have found the solution of this transient part v prime which is function of r and t. Now you put it in the final expression of v z. So, in that you will get one party as solution from the steady Hagen Poiseuille flow and minus this solution ok, which is your transient solution.

So, we can write the final solution as. So, final velocity profile, transient velocity profile, because it varies with time. So, you will get v z which is function of r and t. So, one steady part if you take common 1 by 4 mu minus del p by del z R square; then you will get 1 minus r square by R square and from here you will get minus 8 summation of n is equal to 1 to infinity, ok. Then we have 1 by lambda n R cube J 0 lambda n r divided by J 1 lambda n R and e to the power minus nu lambda n square t ok. This lambda n, you have to find for this problem that, J 0 lambda n R is equal to 0, ok.

So, at which values it is becoming 0, from there you can find the value of lambda n. So, this is the final velocity profile, ok. So, obviously at steady state where t is infinity; so at larger time, the second term will become 0. So, you will get the steady velocity profile for this Hagen Poiseuille flow.

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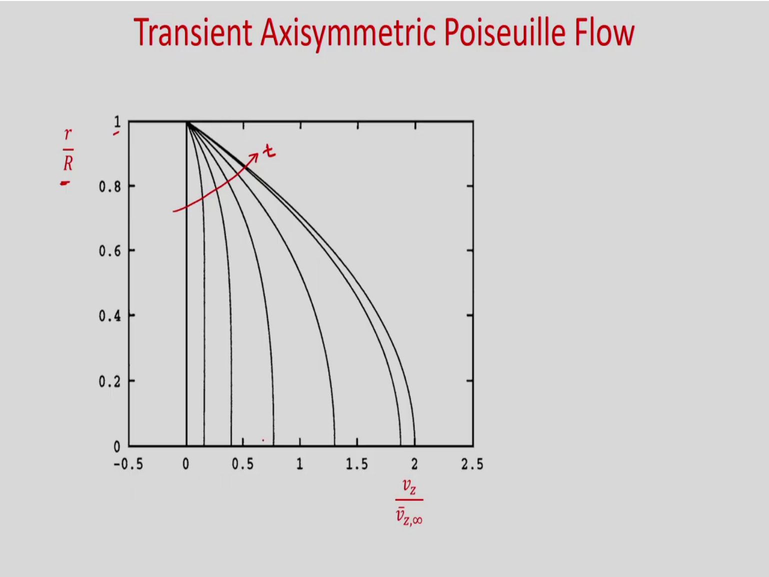

So, now, let us see the evolution of this velocity profile. So, in y axis, we have plotted r by R and obviously it will vary then 0 to 1; at r is equal to R it will become 1 and here we are plotting v z, which is your transient velocity profile and v bar z infinity, which is your steady state Hagen Poiseuille flow velocity profile, ok.

So, if you plot, then obviously at t is equal to 0, you will get 0 profile and as t increases, you can see how this evolution of this velocity profile happening, ok. So, in this case this t is increasing, ok. And obviously, you can see that at steady state, the maximum velocity will become 2 into average velocity and you can see this is becoming 2, ok. So, this is the evolution of this velocity profile with time for this transient axisymmetric Poiseuille flow.

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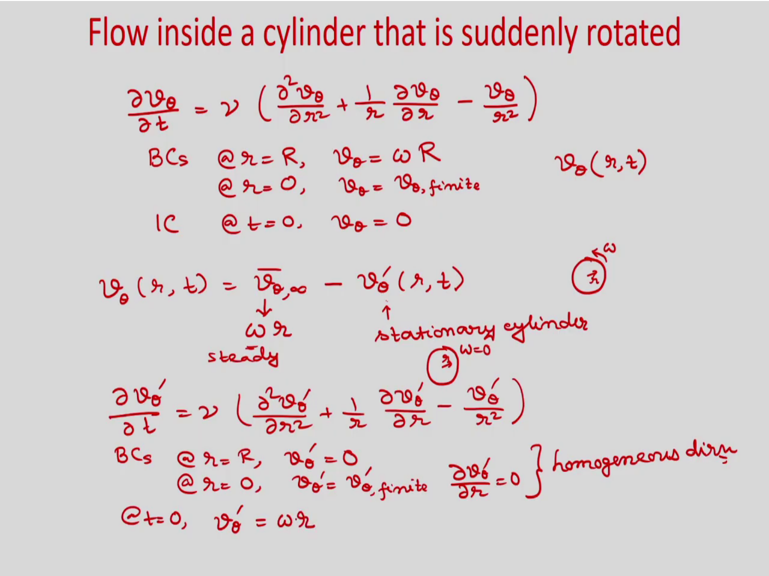

So, now let us consider another problem, where you have a axisymmetric flow and inside a cylinder and this cylinder has started rotating suddenly, ok. So, obviously for steady state you know the solution; if omega is the angular velocity, then omega r is the velocity profile, steady velocity profile, ok.

So, that we have already solved. But in this case obviously, you are getting the circumferential velocity and the governing equation you will get for this problem as del v theta by del t is equal to nu del 2 v theta by del r square plus 1 by r del v theta by del r minus v theta by r square.

So; obviously, you can see this equation is your non homogeneous. So, with boundary conditions, you have at r is equal to R, v theta is omega R ok; at r is equal to 0, v theta is finite and initial condition at t is equal to 0, you have v theta is equal to 0 and obviously, v theta is function of r and t, ok. So, now, this problem actually you can find, means superposition of two velocity profile; one is v theta which is function of r and t ok, one is solution is omega R, which is your v theta bar this is your at t tends to infinity.

So, obviously, this solution you know; this will be omega into r ok, omega into r. And minus one steady state solution, so that is your v theta prime r, t. So, now, we are decomposing this problem; here you can see that here the cylinder is rotating ok, cylinder is rotating with a velocity omega and you are finding the velocity profile. And this is the steady state problem ok, steady problem and these problem it is a transient problem, where this is a stationary cylinder, ok. So, here omega is 0, ok. So, this is a stationary cylinder.

So, for this you are going to find the velocity profile; then in the r direction, you will get homogeneous direction, but in this case you will not get homogeneous condition, ok. So, for this problem obviously you can see that, we have decomposed into two problem, so that in r direction you will get homogeneous boundary condition. So, for this now if you write the governing equation, you are going to get del v theta prime by del t is equal to nu del 2 v theta prime by del r square plus 1 by r delta v theta prime by del r minus v theta prime by r square.

So, this is the problem and here boundary conditions you will get at r is equal to R, v theta prime is 0 and r is equal to 0, you will get v theta prime as v theta finite. And obviously, you will get maximum and minimum velocity at r equal to 0; so obviously del v theta prime by del r is equal to 0. So, you can see, this is your homogeneous direction and at t is equal to 0, now you will get v theta prime is equal to omega r, ok.

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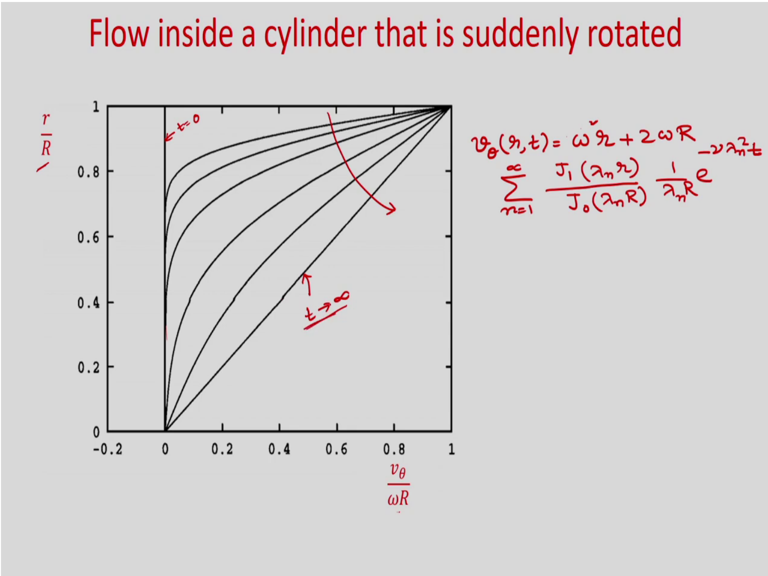

So, this problem if you solve, then finally you will get the velocity profile as v theta r, t is equal to omega r. This is the solution from the steady problem, where the cylinder is rotating with constant velocity omega plus 2 omega R summation n is equal to 1 to infinity J 1 lambda n r divided by J 0 lambda n R and you will get 1 by lambda n R e to the power minus nu lambda n square t, ok.

So, this is the solution from the transient problem and this is the steady problem. Now, if you see the evolution of this velocity profile; y axis r by R, and x axis is v theta by omega R ok, v theta by omega R. So, obviously you can see that, at t is equal to 0; you will get 0 velocity profile, ok. So, this is at t is equal to 0 and gradually it is evolving with time and at t tends to infinity at larger time; you will get a steady solution and obviously, you can see it is a linear profile, because it is omega into R, ok.

So, from central line to the r is equal to R, you will get a linear velocity profile and this is the linear velocity profile at larger time, ok. So, in today's class, we have considered transient axisymmetric Poiseuille flow. In this case, we have a imposed pressure gradient at t is equal to 0 plus. So, obviously you can see that the governing equation is linear, but non homogeneous.

So, as the governing equation is non-homogeneous, directly you cannot use separation of variables method. So, we decomposed this problem into two different problems; one is a steady solution from the axisymmetric Poiseuille flow and minus one transient term.

And if you can see that this transient velocity profile v prime z which you are actually going to solve; so this is a governing equation is linear and homogeneous. And also we have seen that r direction is homogeneous direction; so we used separation of variables method and we found the solution of this transient velocity profile. And finally, we have found the velocity profile for this axisymmetric Poiseuille flow.

Then we considered another case, where this cylinder has started rotating at t is equal to 0 plus; so obviously from 0 velocity to you will get gradual evolution of this velocity profile with time. So, that we have decomposed into two problems; because you have seen that r direction, you do not get the homogeneous direction. So, we have decomposed one problem as a steady problem, where the cylinder is rotating with a angular velocity omega and you know the solution for that problem as omega into r.

And another solution we have transient velocity profile and for that, you have seen that as you decomposed in that way; so we got the r direction as homogeneous direction. And we used the separation of variables method and finally, we have shown the velocity profile.

Thank you.