Viscous Fluid Flow

Prof. Amaresh Dalal
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

Module - 04
Transient One-Dimensional Unidirectional Flows
Lecture - 02
Flow due to an oscillating plate

Hello everyone. So, in last class, we discussed about the Stokes first problem. So, in Stokes first problem, you remember that in a stationary fluid medium one plate suddenly set into a motion at a particular direction with a constant velocity. Then, we found the tangent velocity distribution and also we calculated the shear stress distribution.

In today's class, we will consider oscillating plate, where the plate is oscillating with time. And we will consider different flow situation, first we will consider in semi-infinite Newtonian liquid set into a motion by an oscillating plate.

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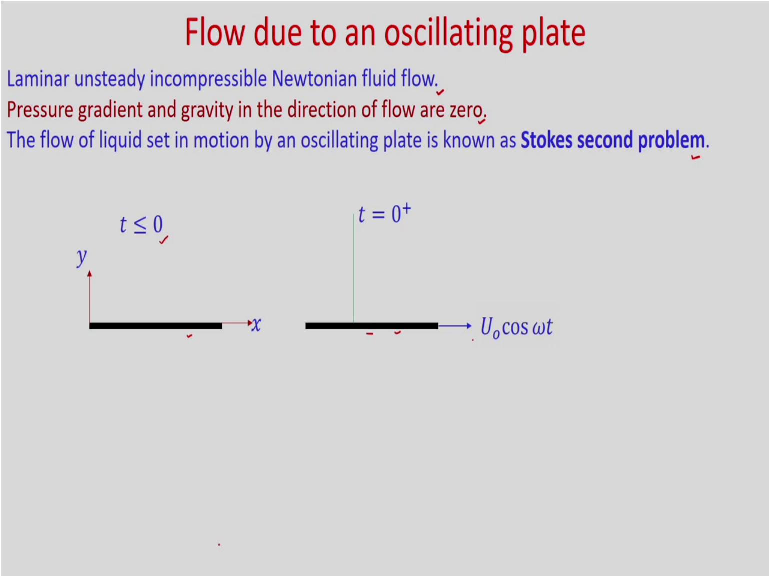

So, let us consider laminar unsteady incompressible Newtonian fluid flow, where pressure gradient and gravity in the direction of flow are 0. And this flow situation where the flow of liquid set in motion by an oscillating plate is known as Stokes second problem.

So, you can see this is the plate, y is measured normal to the plate from the plate, and for t less than equal to 0, this fluid medium and the plate are stationary. As t greater than 0 suddenly this plate is moving with velocity U cos omega t; that means, it is oscillating. Now, due to oscillation of this plate the disturbances will propagate away from this plate. Now, this flow situation is known as Stokes second problem and for this problem we will use the same governing equation which we derived in the last class.

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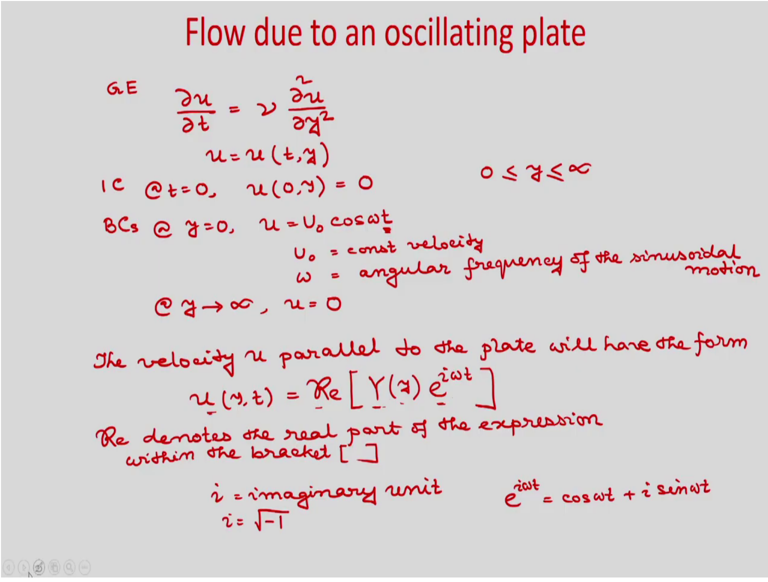

So, if you remember that for this we have written the governing equation as del u by del t is equal to nu del 2 u by del y square, where u is function of t and y, ok. So, for this particular problem, if we discuss about the initial condition and boundary conditions, then you will find that initial condition, obviously at t is equal to 0, u is equal to 0 and the boundary conditions.

Now, at y is equal to 0, obviously it will have the same velocity as the plate; that means, U naught cos omega t, whereas, away from the plate if y tends to infinity, obviously the disturbances will not reach and the velocity will remain 0.

So, in this situation the initial condition at t is equal to 0. So, u 0, y will be 0 for the fluid medium and boundary conditions at y is equal to 0, obviously it will have the same velocity of the plate and that is U naught cos omega t, where U naught is constant and omega is the angular frequency, ok.

So, U naught is constant velocity, and omega is angular frequency of the sinusoidal motion, ok. And at y tends to infinity, obviously we will have u is equal to 0, ok. And you can see here that since the period of oscillation of the plate introduces a timescale, no similarity solution exist to this problem.

So, you can see as the plate is oscillating with U naught cos omega t. So, it is expected that the fluid will also have the oscillation of same frequency omega with a phase lag. So, we will seek a solution of this fluid medium as; so, the velocity u parallel to the flow, parallel to the plate will have the form u y, t is equal to real part of Y which is function of y and e to the power i omega t.

So, you can see we have written in two separate variables Y which is function of y only and this is the function of t, ok. And this Re denotes the real part of the expression within the bracket, ok. And where i is the imaginary unit, where i is equal to root minus 1, ok. And you know that e to the power i omega t also we can write in terms of cos and sin as cos omega t plus i sin omega t, ok.

So, obviously the solution we are seeking u as real part of these two solutions y which is function of y only and this is the function of t time, ok. So, now let us take the derivatives of this u whatever we have assumed the velocity profile and put this time derivative and the second derivative with respect to y in the governing equation. And we will apply the boundary condition to get the function y.

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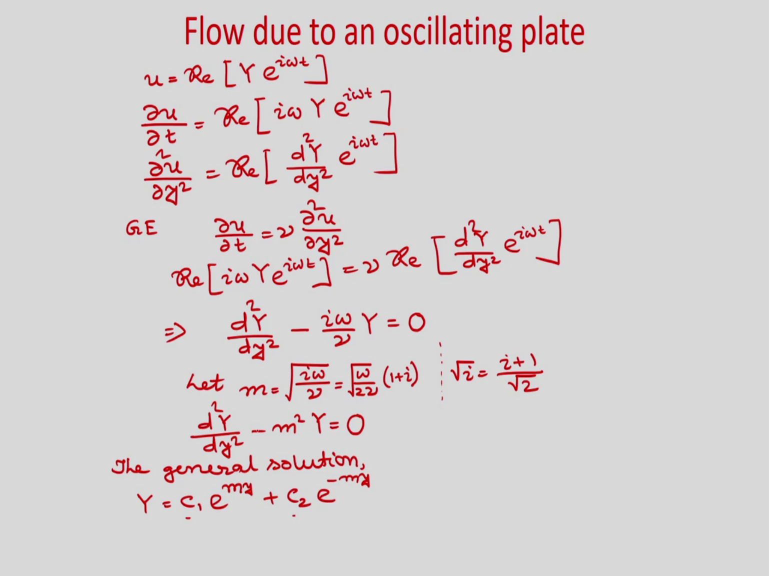

So, we have u is equal to real part of Y e to the power i omega t, ok. So, we can see we can write del u by del t, ok. So, real part of i omega Y e to the power i omega t, and delta 2 u by del y square if you find then it will be real part of this d 2 Y, because Y is function of Y only, so you can write in terms of ordinary derivative e to the power i omega t, ok.

So, if you put these in the governing equation. So, what is our governing equation? Del u by del t is equal to nu del 2 u by del y square and if you put it here then, you can see it will be i omega Y e to the power i omega t is equal to nu real part of d 2 Y by dy square e to the power i omega t, ok. If you simplify it, then you will get d 2 Y by dy square minus i omega by nu, where nu is the kinematic viscosity of the fluid, Y is equal to 0, ok.

So, let us now, let m is equal to root i omega by nu, ok. And root i is equal to you know i plus 1 divided by root 2, so this you can write it now root omega by twice nu 1 plus i, ok. So, you can write now d 2 Y by dy square minus m square Y is equal to 0. Now, the solution of this ordinary differential equation we can write in terms of exponential function as well as in terms of sine hyperbolic and cos hyperbolic.

So, in this case as you can see from the boundary condition that one boundary condition is at y tends to infinity is equal to 0, so it is better to write the solution in terms of exponential function, ok. So, we can write now the solution, where Y; the general solution, Y is equal to c 1 e to the power m y plus c 2 is e to the power minus m y and this integration constant c 1 and c 2, we will find from the boundary conditions.

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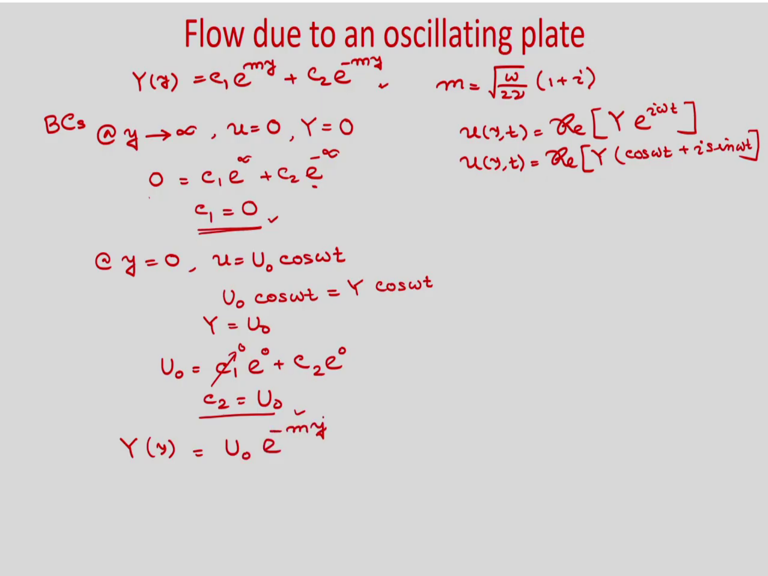

So, we have Y as c 1 e to the power m y plus c 2 e to the power minus m y, where m is equal to root omega by twice nu 1 plus i, ok. And now boundary conditions at y tends to infinity, u is equal to 0, ok. And we have seek the solution u y, t as real part of Y e to the power i omega t, ok. And e to the power i omega t also you can write as Y cos omega t plus i sin omega t, ok.

So, if you can see that as y tends to infinity u is equal to 0 so, obviously if u is equal to 0 Y must be 0, ok. So, from here the boundary condition Y will be 0. So, if you can see here, if you put it in the left hand side it is 0 c 1, e to the power infinity plus c to the power plus c 2 e to the power minus infinity, ok. So, you can see, obviously this term will be 0, but left hand side is 0, so to keep left hand side 0 c 1 must be 0, ok. So, c 1 will be 0.

Now, other boundary condition we have at y is equal to 0, u is equal to U naught cos omega t, right. So, if u is equal to U naught cos omega t. So, from here you can see what will be the real part. So, in terms of y, so you can see if you write U naught cos omega t, U left hand side and real part of this, real part of this is Y cos omega t. So, in terms of y you can see we have boundary condition at Y is equal to 0, Y is equal to U naught.

So, now Y is equal to U naught, if you put in this equation at Y is equal to 0. So, what you will get? Left hand side Y is equal to U naught is equal to c 1 e to the power 0 plus c 2 e to the power 0. And, obviously c 1 is 0. So, c 2 will be U naught. So, now, we have found c 1 as 0 c 2 is equal to U naught. So, now, you can write Y as c 2 is U naught e to the power minus m y.

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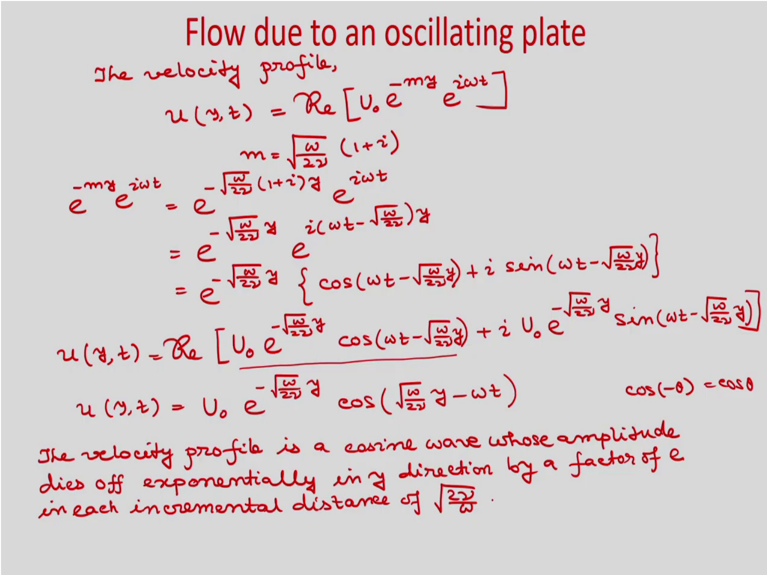

So, now, we know the Y, so we can write the velocity profile as, the velocity profile, u y, t is equal to real part of, ok. So, what is y? So, y is U naught e to the power minus m y and we have e to the power i omega t. And what is m? m if you remember we have written as omega by twice nu 1 plus i, ok.

So, let us write this term e to the power minus m y, e to the power i omega t as e to the power minus omega by twice nu 1 plus i y e to the power i omega t. So, this if you rearrange you will get e to the power minus omega by twice nu y and e to the power i omega t minus root omega by twice nu y, ok.

And this term if you write in terms of cos and sine, then you can write e to the power minus omega by twice nu y. So, you can write this term as cos omega t minus root omega by twice nu plus i sin omega t minus root omega by twice nu, ok.

So, if you put in the velocity profile. So, u y, t will be real part of now it will be U naught then this term e to the power minus omega by twice nu y cos omega t minus omega by twice nu and another term you will have i U naught e to the power minus root omega by twice nu y sin omega t minus root omega by twice nu. So, here there will be y omega t y, ok.

So, now, you can see that the real part of these terms inside the bracket, obviously this is the real part, ok. So, your velocity profile will be u y, t. So, we are considering only the real part, so real part will be U naught e to the power minus omega by twice nu y cos and if we take this term first omega by twice nu y minus omega t, because cos minus theta is equal to cos theta we know, ok. So, we can write this, ok.

So, now this is the velocity profile and you can see the velocity profile is a cosine wave, ok. Cosine wave with amplitude dies off exponentially in the y direction, ok. So, you can see the velocity profile is a cosine wave whose amplitude dies off exponentially in the y direction by a factor of e in each incremental distance of root twice nu by omega, ok.

So, now you can see that due to this oscillation of the plate, the disturbance will propagate in the y direction and in this particular case you can see that the plate is oscillating, ok. One time it is going in the positive x direction another time it is going negative x direction. So, due to that this penetration depth will be limited or constant for a particular frequency, ok.

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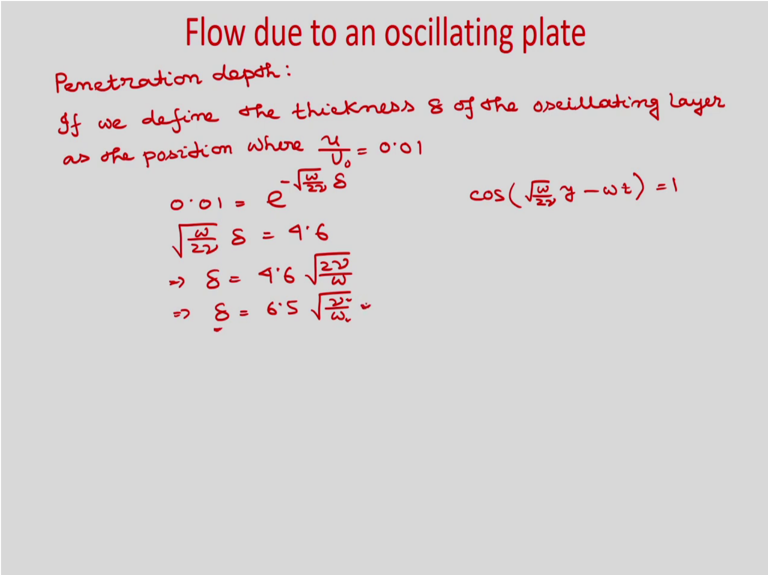

So, you can see that if you calculate the penetration depth, so if we define the thickness delta of the oscillating layer as the position, where u by U naught will be 0.01. So, if you show if it, so then 0.01 will be e to the power minus omega by twice nu delta, ok.

We have taken the this cos root omega by twice nu y minus omega t, so maximum value it will be 1, ok. So, for that reason we have written u by U naught as 0.01. So, obviously from here you can see that delta, so this will be this if you take omega by twice nu delta then it will become 4.6 and delta will be 4.6 into twice nu by omega and you will get delta as 6.5 nu by omega, ok.

So, you can see nu is the kinematic viscosity of the fluid. So, for a particular fluid if we have a constant frequency at a particular frequency if it is oscillating then delta will remain constant, ok. But with time it will vary, but it will not further increase with the increase in time, ok. So, maximum delta is limited, with this expression maximum penetration depth. Now, let us find what is the CST distribution, due to this oscillating plate in the fluid domain.

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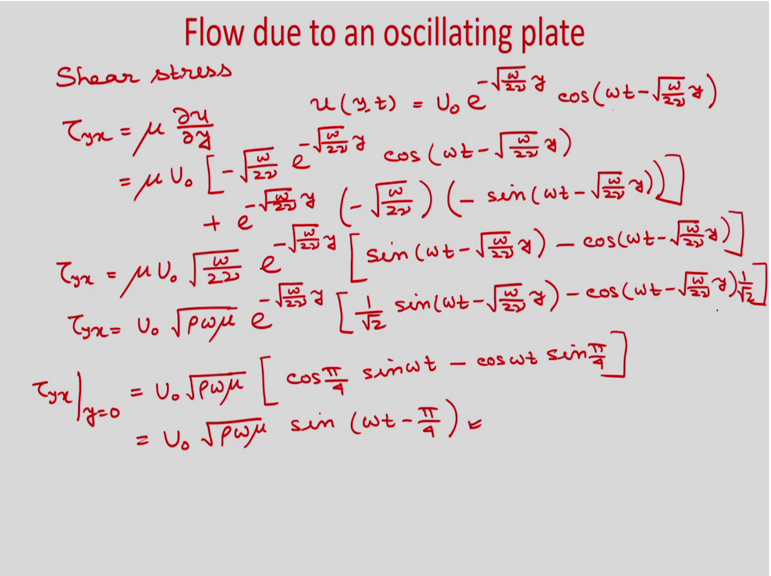

So, we will find tau yx as, so shear stress, so tau yx we will find as mu del u by del y, ok. And u, already we have written u as U naught e to the power minus omega by twice nu y cos omega t minus root omega by twice nu y, ok. So, if you find the derivative with respect to y then you will get mu U naught, ok.

So, the first term if you consider then it will be minus omega by twice nu e to the power minus omega by twice nu y cos omega t minus omega by twice nu y then plus U naught e to the power sorry, U naught we will not write e to the power minus omega by twice nu y and from here you can see from this term you will get minus root omega by twice nu, ok. And you will get derivative with respect to y of this cos it will become again minus sin omega t minus root omega by twice nu y, ok.

So, tau yx you can write as mu U naught root omega by twice nu e to the power minus omega by twice nu y. So, minus, minus plus, so this term I am writing first. So, it will be sin omega t minus omega by twice nu y and this will be minus cos omega t minus omega by twice nu y, ok.

So, now you can see this nu, nu is mu by rho. So, what we can write tau yx as, so, if you take mu inside, so you will get U naught rho omega mu e to the power minus omega by twice nu y and we have 1 by root 2. So, we will write here 1 by root 2 sin omega t minus omega by twice nu y minus cos omega t minus omega by twice nu y and 1 by root 2, ok.

So, now let us find what will be the shear stress at y is equal to 0. So, tau yx at y is equal to 0, so you will get now y is equal to 0. So, it will be just U naught root rho omega mu. And now this 1 by root we can write cos pi by 4 and at y is equal to 0, so it will be sin omega t and minus cos omega t and we can write sin pi by 4, So, you can write as U naught root rho omega mu sin omega t minus pi by 4, ok. So, this will be the as shear stress at y is equal to 0 and this is the shear stress distribution.

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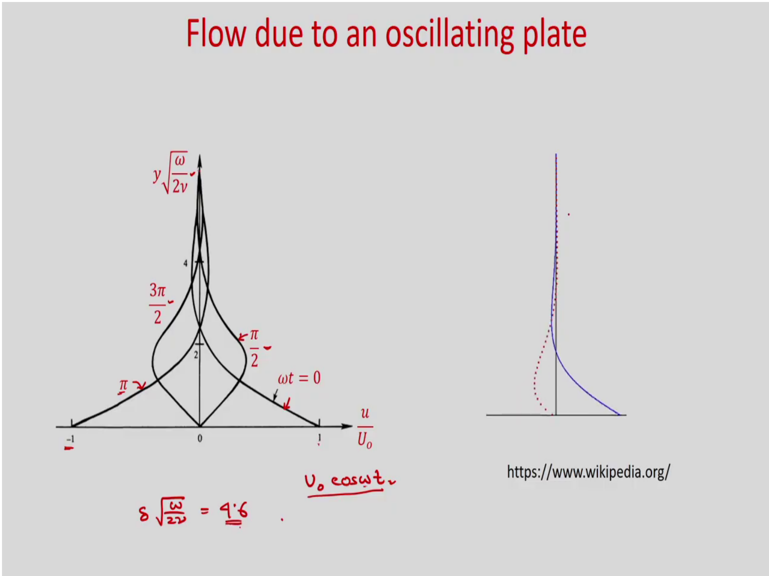

So, now, if you want to plot the velocity profile at different time instances, so we can see here say x axis is u by U naught, ok. So, maximum velocity will be U naught. So, obviously u by U naught will be one and in y axis we have put y root omega by twice nu, ok.

So, if omega t is equal to 0, ok, so you will get this as a velocity profile, so obviously you can see that if omega t is equal to 0, so u is equal to U naught cos omega t, right, sorry. At the wall we have U naught cos omega t, right. This is the wall velocity. So, if omega t is equal to 0. So, cos omega t will be 1, so u by U naught will become 1, ok. So, u by U naught is 1 and the velocity now is penetrating as y is increasing and we have already shown the penetration depth how it varies.

So, you can see penetration depth whatever expression we have written, so delta, omega by choice nu is 4.6. So, you can see this will be around 4.6, because y root omega y twice nu it will be 4.6, so up to this it will have the effect of these disturbances, ok. Now, if it is omega t is equal to pi by 2.

So, if omega t is equal to pi by 2, obviously it will be 0 velocity of the wall will be 0 at that time instances and you will get this as a velocity profile and the penetration will be up to this point 4.6. Similarly, this is for omega t 3 pi by 2. And when omega t is equal to pi, so omega t is equal to pi; that means, this will become minus U naught and u by U naught will become minus 1. So, at that time instances the velocity profile will look like this, ok.

And you can see this is the temporal evolution of the velocity profile the blue line whatever you are seeing, so you can see this is the velocity profile oscillating with time. So, at the wall this plate is oscillating with U naught cos omega t and u by U naught it is varying from 1 to minus 1 and you can see how the velocity is oscillating and the penetration it is happening in the y direction and at y is equal to 4 point, y into omega by twice nu at 4.6. So, up to that point the disturbance effect will be there.

Now, if you consider that plate is stationary, but the fluid is oscillating with U naught cos omega t, ok. So, if you have a some oscillating pressure gradient and due to that the velocity is oscillating as U naught cos omega t, but the plate is stationary. So, in that case, obviously the similar analysis you will do only the boundary conditions will change and you can get the velocity profile as oscillating fluid medium with stationary plate, ok.

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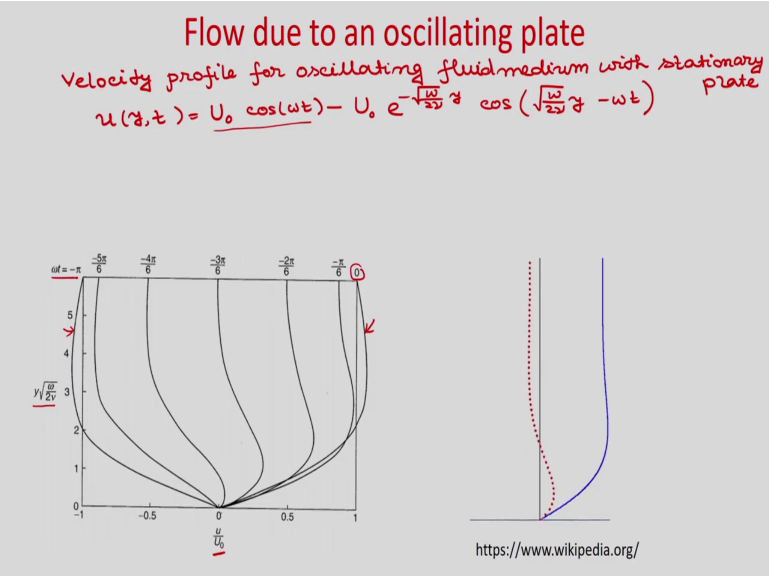

So, in this case, the velocity profile will be u y, t is equal to U naught cos omega t minus U naught e to the power minus omega by twice nu y and cos root omega by twice nu y minus omega t, ok.

So, this is the velocity profile and you can see u by U naught in the x direction in y direction y root omega by twice nu and, obviously at different omega t value you can see the instantaneous velocity distribution, obviously plate velocity will be 0 at y is equal to 0 and y tends to infinity.

You will have this U naught cos omega t velocity and you can see at different omega t value how the velocity will look like. So, this is the omega t is equal to 0. So, this is the velocity profile and omega t minus pi this is the velocity profile and you can see the temporal evolution of this velocity profile blue color solid line. It is giving the velocity profile with time, and obviously the plate velocity will be 0.

Now, let us consider Stokes-Couette flow. So, in case of Couette flow, we have seen that the bottom plate is stationary and upper plate is moving with a constant velocity u. Now, in this particular case we will consider that upper plate is oscillating with the velocity U naught cos omega t, ok. So, this problem is known as Stokes-Couette flow.

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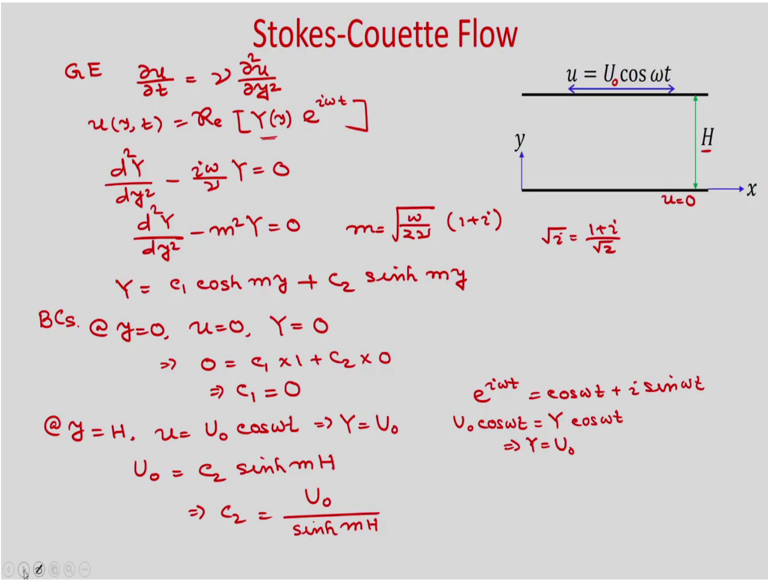

So, you can see the bottom plate is stationary u is equal to 0, x is measured in this direction, y is from the y is measured from the bottom plate and the distance between two plates is H and this upper plate is oscillating, with u cos omega t or let us say U naught cos omega t, ok.

So, you can see the governing equation will remain same and the solution procedure will remain same, only the boundary condition will be different, ok. So, you can see that governing equation, in this case will remain same as earlier case. So, del u by del is t is equal to nu del 2 u by del y square and your velocity profile will assume as real part of this Y which is function of y e to the power i omega t, ok.

And we will get the differential equation of y d 2 Y by dy square minus i omega by nu Y is equal to 0 which we can write as d 2 Y by dy square minus m square Y is equal to 0, where m is equal to we will write as omega by twice nu 1 plus i ok. Because root i is 1 plus i divided by root 2, ok. So, now, solution will be y is equal to c 1.

So, now in this case, you can see that we have a finite distance in y because y is equal to 0 to H, so obviously we will write the solution in the hyperbolic function, ok. So, we will write c 1 cos hyperbolic m y plus c 2 sin hyperbolic m y.

And now boundary conditions you can see at y is equal to 0 u is equal to 0, ok. So, if u is equal to 0, so from this expression, obviously Y will be 0, ok. So, you can see it will be 0 c 1 cos hyperbolic 0 is 1 plus c 2 sin hyperbolic 0 is 0. So, from here you can see c 1 will be 0.

So, now, another boundary condition at y is equal to H. So, the plate is oscillating, right u is equal to U naught cos omega t, ok. So, you can see that from this expression if e to the power i omega t, you can write cos omega t plus i sin omega t, so obviously in the left hand side you will get U naught cos omega t.

And in the right hand side real part of this will be Y into cos omega t, right, so Y will be U naught. So, obviously in terms of Y, it will be Y is equal to U naught. So, if you put it here. So, you can see left hand side will be U naught and c 1 is 0, c 2, and sin hyperbolic mH, ok. So, c 2 will be just U naught by sin hyperbolic m H.

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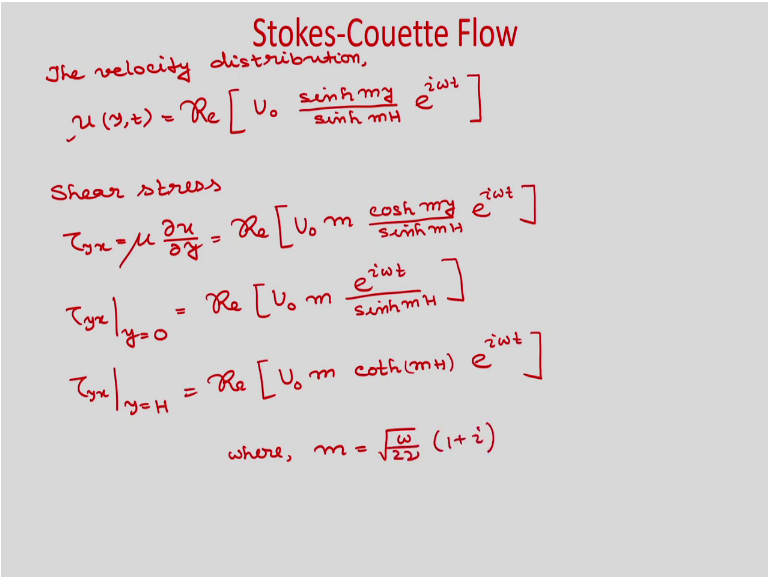

So, now, we have found these two integration constants c 1 and c 2, and let us put in the velocity distribution. So, you will get the velocity distribution as u y, t is equal to real part of, ok, so it will be just U naught sin hyperbolic m y divided by sin hyperbolic mH, right, e to the power i omega t. So, this is the velocity distribution. So, a real part of this you have to take to get the u velocity, and if you want to find the shear stress, so tau yx will be just mu del u by del y. So, real part of this, now you can see U naught, so del u by del y.

So, the sin hyperbolic m y, so you will get m cos hyperbolic m y by sin hyperbolic mH, e to the power i omega t. And at y is equal to 0, if you want to find the shear stress, so you will get, obviously you can see that cos hyperbolic 0 will be 1. So, you will get real part of U naught m e to the power i omega t divided by sin hyperbolic mH and tau yx at y is equal to H on the upper plate. So, it will be the real part of U naught m, sorry here it will be e to the power i;, we have written earlier.

So, now, you can see here, so at y is equal to H cos hyperbolic mH and divided by sin hyperbolic mH. So, you will get cot hyperbolic mH, e to the power i omega t, where m is equal to root omega by twice nu 1 plus. So, we have to take the real part of this to find the shear stress.

So, next we will consider flow between parallel plates with a oscillating pressure gradient, ok. And what will be the velocity profile? Obviously you will get a tangent velocity profile. So, in this particular case the two plates are stationary, but we have a imposed pressure gradient which is oscillating, ok.

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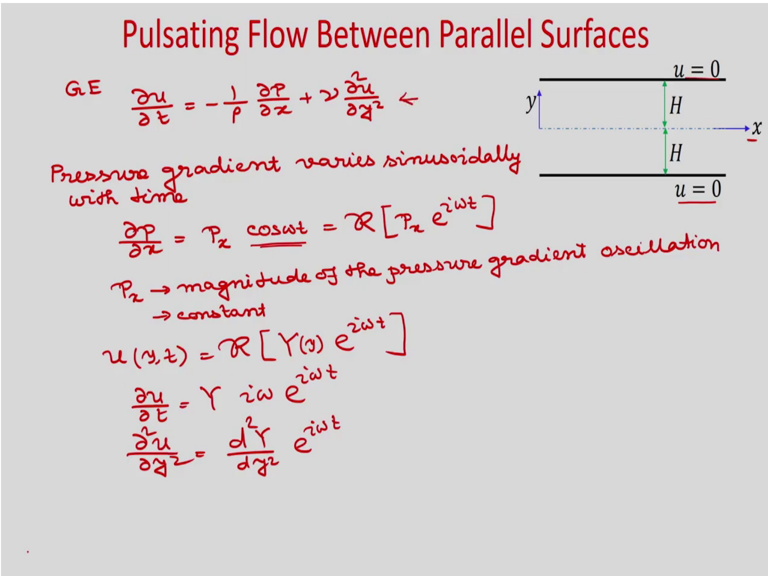

So, for this the governing equation will be del u by delta t, is equal to minus 1 by rho del P by del x because there is a imposed pressure gradient plus nu del 2 u by del y square. So, nu is the kinematic viscosity of the fluid. So, you can see bottom plate is stationary, upper plate is stationary, and we have a imposed pressure gradient del P by del x which is oscillating. We are taking y from the center line and this is the axial direction x and the distance between two parallel plates is 2 H, ok.

So, in this particular case now, we will consider that pressure gradient varies sinusoidally with time, ok. So, del P by del x we will consider as let us say some let us say P x cos omega t and that you can write as real part of P x e to the power i omega t, where P x is the magnitude of the pressure gradient oscillation, and this is constant, ok. P x is constant only it is varying with time with cos omega t.

Similar way we will seek the solution u y, t as real part of Y which is function of y only and e to the power i omega t, ok. And now, if you find del u by delta t, so it will be Y i omega e to the power i omega t del 2 u by del y square, so it will be d 2 Y by dy square e to the power i omega t.

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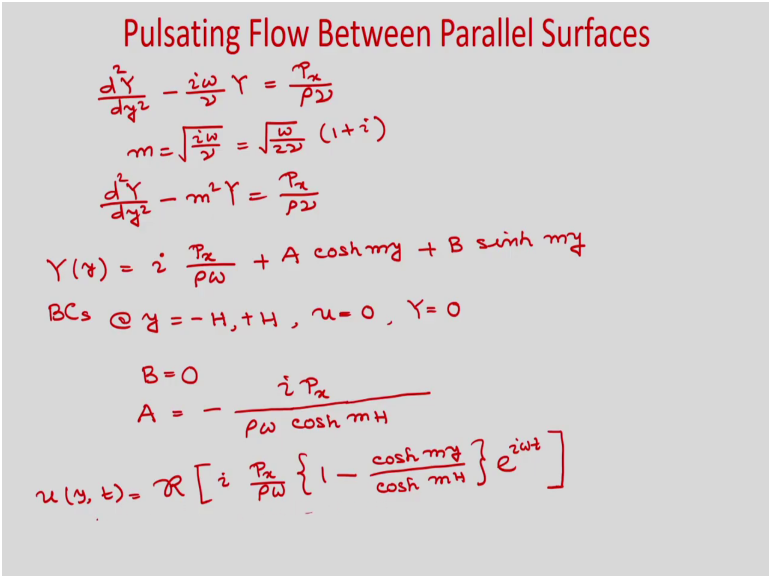

Now, if you put it in this governing equation you are going to get d 2 Y by dy square minus i omega by nu y is equal to P x by rho nu, ok. Now, we will put m is equal to root i omega by nu as root omega by twice nu 1 plus i. So, we will get d 2 Y by dy square minus m square Y is equal to P x by rho nu, ok. So, what will be the solution of Y? Y will be just i P x by rho omega plus A cos hyperbolic m y plus B sin hyperbolic m y, ok.

And what are the boundary conditions? At y is equal to minus H and plus H, ok. You can see velocity is 0, ok. So that means, if velocity is 0, so Y will be 0. So, from here you can find the constant, so B will be 0 and A will be minus i P x, P x is the pressure gradient ok, constant pressure gradient; divided by rho omega cos hyperbolic mH, ok. So, this will be the integration constant.

And velocity profile u, you will get as real part of this you have to consider i P x by rho omega 1 minus cos hyperbolic m y divided by cos hyperbolic mH e to the power i omega t, ok. So, the real part of this we have to consider to get the velocity profile u. So, in today's class we considered different flow situation where the plate is oscillating with time.

The first problem we considered where one plate is there in infinite fluid medium and this plate is suddenly set into motion with a velocity U naught cos omega t. And due to the movement of this plate the disturbance will propagate along the y and as y tends to infinity, obviously the disturbance will not reach and velocity will be 0.

So, as you can see as the plate is oscillating with a frequency omega, it is expected that the fluid velocity will also oscillate with the frequency omega with a phase lag. We seek the solution, as a real part of y into e to the power i omega t. And we put in the governing equation, then we found the boundary conditions and we wrote the final expression of velocity profile in terms of y and t.

So, you can see that for this particular case, U naught cos omega t will have the maximum velocity U naught, and these wall velocity obviously will vary from U naught to minus U naught. And we have plotted the velocity profile with in x direction u by U naught which varies between 1 to minus 1, and in y direction it is y into omega by twice nu. So, in this case, obviously the penetration depth you can see maximum penetration depth will be limited by the frequency of the oscillating plate omega.

And later we considered the Stokes-Couette flow where the flow inside two parallel plates where bottom plate is stationary and upper plate is oscillating with a velocity U naught cos omega t. So, in this particular case also, we found the velocity distribution in similar way as well as the shear stress distribution.

And at last we considered flow inside two parallel plates with a oscillating pressure gradient. And this pressure gradient is oscillating with cos omega t and the plates are stationary. So, in similar solution procedure we followed and we found the velocity distribution.

Thank you.