Viscous Fluid Flow
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Module - 04
Transient One-dimensional Unidirectional Flows
Lecture - 01
Stokes First Problem

Hello everyone, till now we have solved steady one-dimensional problem and we have seen that velocity is function of one special coordinate. Today we will start to solve some unsteady flow problem.

So, in today's class, we will consider Stokes first problem. What is Stokes first problem? So, if there is a stationary plate in an infinite stationary fluid medium and suddenly or impulsively. If this plate starts moving, then we will get velocity profile which is function of one special coordinate and time. So, this problem is known as Stokes first problem.

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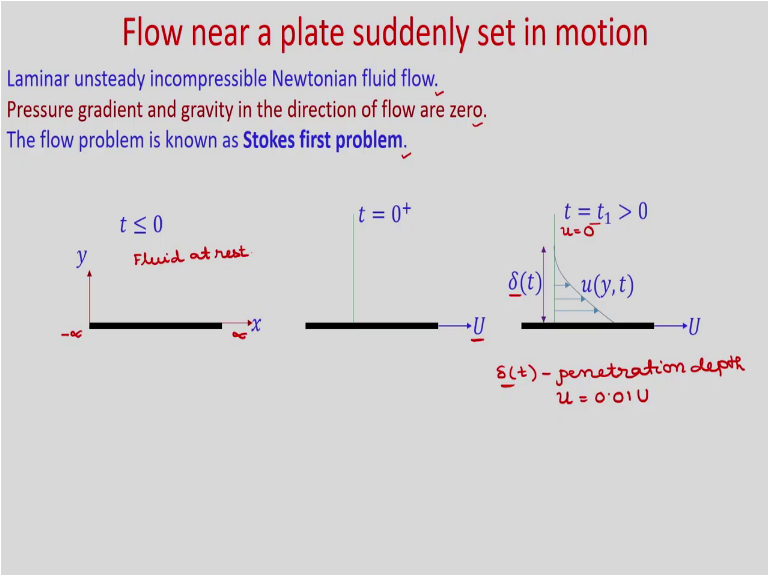

So, let us consider infinite plate. So, it is in x-direction, it goes to infinity to minus infinity; in z-direction also it is infinite and y is measured from the plate. So, this stationary plate is kept in a stationary fluid medium. Now, suddenly, if it starts moving with a constant velocity U in the x-direction parallel to the its own axis, then there will be some velocity profile generating inside the fluid domain.

So, it is just when t is equal to 0 plus means it starts impulsion, it starts moving impulsively with a constant velocity U. And if t is greater than 0, let us say at any time t 1, you will see that this disturbances will propagate inside and d 2, there is no slip condition at the wall the fluid will have some velocity in the x-direction.

And as you go in y direction, this velocity will decrease and y tends to infinity, you will get that fluid is at rest that means velocity is 0. So, as y tends to infinity, u still it will be 0. So, this velocity profile will be function of one special coordinate y and time t.

So, as time progresses, these disturbances will propagate more inside the fluid domain. And if up to which this velocity; up to which this velocity have this effect inside the fluid domain, so that distance let us say that delta t which is known as penetration depth. That means, up to this distance, the effect of this moving flat plate will have the effect inside the fluid domain ok. So, once this u becomes 0.01 of U, then we say that that is the penetration depth delta t at that particular time.

Now, let us consider laminar unsteady incompressible Newtonian fluid flow. Pressure gradient and gravity in the direction of flow are 0 and as we discussed that this flow problem is known as Stokes first problem. Now, to solve this problem, first let us start with the continuity equation, and then we will consider the momentum equations, and we will derive the governing equations for this Stokes first problem.

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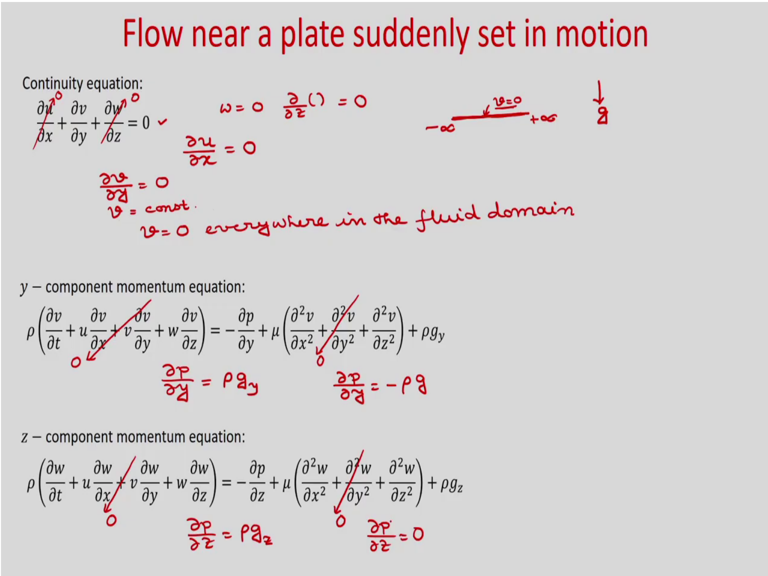

So, if you consider the continuity equation, so this is the continuity equation for incompressible fluid flow. Now, we have already told that in z-direction, it is infinite, so obviously, w will be 0, and any gradient in that direction will be 0.

And let us consider that as x-direction is infinite. So, the plate is infinite, the plate is infinite in the x-direction. So, the x-component velocity profile is invariant in the direction parallel to the wall, that means, del u by del x will be 0 ok.

So, now, if you put that del w by del z as 0, del u by del x as 0, then del v by del y will be 0 ok, that means, v will be constant ok. Now, you can see that on the plate obviously, due to no slip condition v is 0, normal to this plate the velocity v is 0. So, from this condition, you can see that v will be 0 everywhere in the fluid domain ok.

So, now let us consider y-component of momentum equation. So, if we consider y-component of momentum equation as v is 0, obviously, all these terms in the left hand side will be 0, the viscous term will be 0. And you can write that del p by del y will be just rho g y.

So, this is hydrostatic pressure you can see. And z-component momentum equation if you see, and similarly w is 0, so all these term will become 0, this is 0 and del p by del z will be rho g z ok. And if you consider g in negative y direction as positive and other direction it is 0, then, obviously, del p by del y you can write as minus rho g, and del p by del z will be 0 ok.

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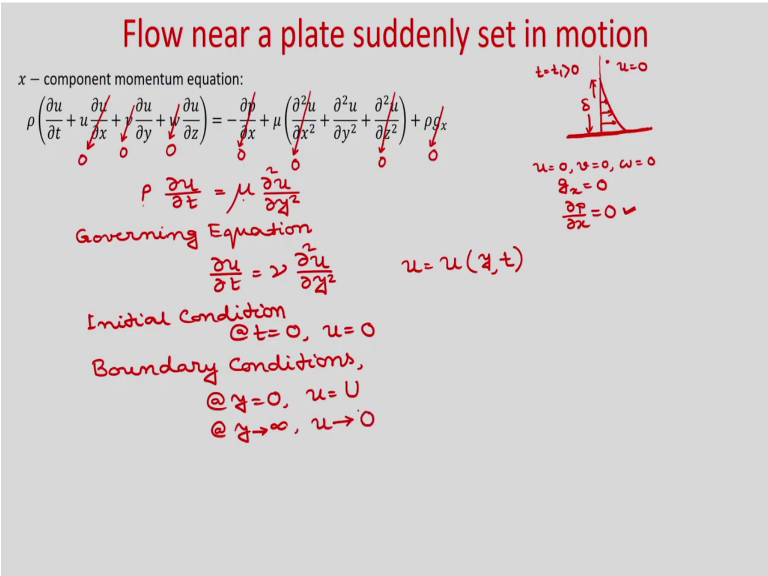

Now, let us consider a point outside the effect of this velocity means outside the penetration depth. So, if you consider that this is your plate let us say at a particular time t is equal to t 1 which is greater than 0, we have velocity profile like this.

The effect it has gone up to this distance which is your penetration depth. So, outside this, if you take one point and apply the x-component momentum equation, obviously, at this point still the fluid is stationary right. So, u will be 0. So, you can see there u is equal to 0, v is equal to 0, w is equal to 0 right. And already we have considered in that direction, gravitational acceleration is 0

So, if you consider this point outside this penetration depth, obviously, from this x-component momentum equation, you can see del p by del x will be 0 ok. So, as it is unconfined flow ok, the pressure gradient in the axial direction will be 0. So, now, let us derive the governing equation for this Stokes first problem. So, obviously, it is unsteady problem because you know the effect of fluid motion gradually increases inside the fluid domain with time.

So, if you consider this x-momentum equation, then you can see that we have already shown that del u by del x is 0, because the plate is infinite in the x-direction. So, the x-component velocity profile in is invariant in that direction parallel to the wall, v is 0, w is 0, del p by del x is 0 ok. As del u by del x is 0 everywhere, so del 2 u by del x square will be 0; and this is also 0, gravitational acceleration is 0.

So, you will get rho del u by del t is equal to mu del 2 u by del y square, that means, you can write the governing equation for this Stokes first problem will be del u by del t is equal to nu; nu is kinematic viscosity which is the ratio of dynamic viscosity divided by density of the fluid; del 2 u by del y square. So, you can see u is function of one space coordinate and time t.

So, now what are the boundary conditions and initial conditions? So, from this governing equation, you know that you have time derivative is first ordered, so obviously, one initial condition is required; and spatial derivative is second ordered, then obviously, two boundary conditions are required.

So, you know that at y is equal to 0, for t greater than equal to 0, it is having the velocity u. So, u will be capital U. And if you go y tends to infinity still the fluid is at rest that means u tends to 0. And initial condition, you can see that at t is equal to 0, you will have this velocity suddenly it starts moving with a constant velocity u. So, initial condition at t is equal to 0, you can see the plate is stationary. So, u is 0.

Now, let us write the initial condition and boundary condition for this governing equations. So, initial condition, at t is equal to 0, it is stationary u is equal to 0. And boundary conditions, so at y is equal to 0 u is equal to capital U; and at y tends to infinity far away from the moving plate, u tends to 0.

So, now, you can see for this problem, we do not have any definite reference y scale and reference time scale. So, for this problem, you can see that we do not have any reference y scale and reference t scale, but we can derive it from the scale analysis. You know the governing equation. And if we do the scale of u as capital U, then we will be able to correlate with the reference y and reference t.

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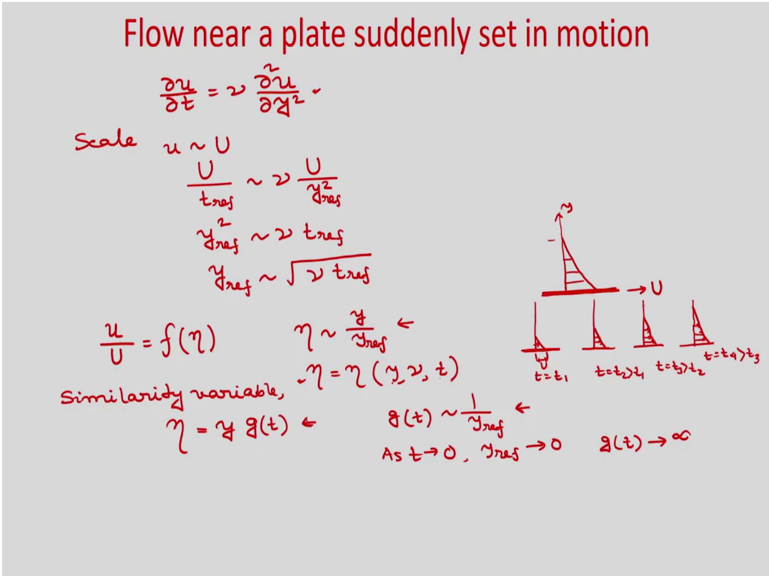

So, you can see we have the governing equation del u by del t is equal to nu del 2 u by del y square. So, if you take the scale of u as capital U, then you can see we can write U by some reference time is equal of the order of nu U divided by y reference square.

So, obviously, you can see that y reference square, you can write order of nu t reference or y reference you can write order of root nu t reference. So, from here, you can see that obviously the if you take the reference y scale as the penetration depth, then obviously, it will be directly proportional to the roots of kinematic viscosity into time.

So, you can see that you can get a scale of y as root nu t reference. So, now, we have the partial differential equation, in somehow we need to convert it into ordinary differential equation to get the velocity profile. One way is to solve this problem using Laplace transform, but we will not use it, we will use similarity variable approach.

So, when we use this similarity variable approach, obviously, we need to use some similarity variable as a function of two independent variables. So, here you can see that we have two independent variables y and t, and that we can use. And from the scale you can see that, obviously, y reference will be order of root of nu t reference.

And if you use that similarity variable, then we can show that the velocity profile u by capital U will be function of that similarity variable. Let us assume that similarity velocity are self similar, so that we can have the or we can use the similarity variable approach. And if similarity transformation exist, then we can get the ordinary differential equation from the partial differential equation.

So, what we will do now? We will use u by capital U as function of some similarity variable eta and this eta will be just y by y reference ok. So, obviously, this eta y reference you can see that t it is root nu t reference. So, eta obviously is function of y nu t ok. So, now, we will use this similarity variable as a function of y, and nu, t. But we do not know how it is a function of t. So, we will use this similarity variable eta as y into some function of time g t. And this g may be it will contain obviously, this kinematic viscosity.

And from here, you can see from this relation and this relation g t will be 1 by y reference. So, what does it mean? You can see if you have this plate moving with the constant velocity U as you go in the y-direction, so there will be the effect of this motion of this plate up to certain region and that is your y reference. So, you can see as t tends to 0; as t tends to 0, obviously, y reference will be tends to 0 because penetration depth will be tending to 0. So, g t will be tending to infinity from this relation ok.

So, now, you can see that why we can use this similarity transformation for this solution of this governing equations. If you see that at a different time, if you plot the velocity profile, so let us say t is equal to t 1, t is equal to t 2 which is greater than t 1, t is equal to t 3 which is greater than t 2, and t is equal to t 4 greater than t 3. So, if you plot the velocity profile, so slowly the effect of this moving plate will go inside the fluid domain, and the velocity profile may look like this where this is your capital U. Again it will penetrate more, again will it will penetrate more and so on ok.

And if you see that, if you can use some similarity variable such that if you bring down this velocity profile to a scale, it will fall in the same graph and that is why you are using this similarity variable eta is equal to y g t. And later we will see that if this similarity solution exist, then we can convert this partial differential equation to ordinary differential equation as well as we will see that velocity profile will collapse into a same curve or single curve.

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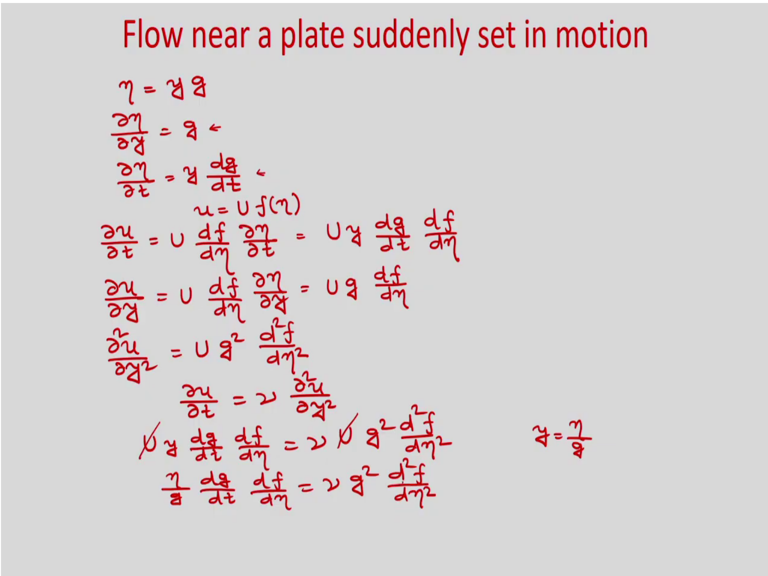

So, now let us write the governing equation in terms of derivative of the with respect to the similarity variable eta. So, we have assumed eta is equal to y into g, where g is function of t.

So, del eta by del y we can write as g and del eta by del t, obviously, g is function of t. So, you can write g we can write y into d g by d t ok.

And we can write del u by del t now and we know u is U into some function of eta right. So, u by U is function of one variable that is eta and eta is the similarity variable which is function of two independent variables y and t. So, del u by del t now we can write U into d f by d eta into del eta by del t. And del eta by del t we know this one, so you can write U y d g by d t d f by d eta.

Similarly, you can write del u by del y as U d f by d eta into del eta by del y ok. So, del u eta by del y is g. So, you can write equal to U into g d f by d eta. Now, if you write del 2 u by del y square, then obviously it will be U into g square d 2 f by d eta square. So, now, we have the governing equation del u by del t is equal to nu del 2 u by del y square.

So, if you put all these derivative del u by del t and del 2 u by del y square here what you will you are going to get? So, in the left hand side, it will be U into y d g by d t d f by d eta; and right hand side, nu U g square d 2 f by d eta square. So, both side U is there, so you can cancel and you can see, y you can write eta by g right, y is eta by g. So, here you can write eta by g d g by d t d f by d eta is equal to nu g square d 2 f by d eta square.

So, you can see in this equation that d g by d t g, and here g square, if you take in the left hand side, then it will become function of t only. And eta d f by d eta if you take in the right hand side, then it will become d 2 f by d eta square divided by eta into d f by d eta and that will be function of eta only. So, we are separating the variables. So, if you use the if you separate the variables where left hand side is function of t and right hand side is function of eta only, then it will be equal to some constant ok.

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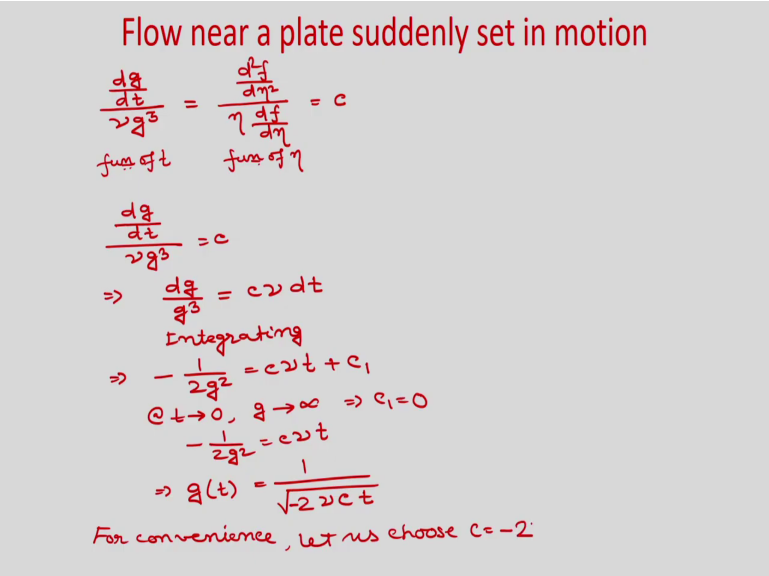

So, now, we are taking the d g by d t in the left hand side divided by eta into g cube is equal to d 2 f by d eta square divided by eta into d f by d eta ok. So, you can see left hand side is function of t only right, because g is function of t, and nu is constant and right hand side, it is function of eta only ok. So, we have separated the variables, so that now you can write that equal to some constant c ok. So, you can choose any c value, because if you change the value of c, obviously, g will change and accordingly f will change.

So, now if you consider the first term d g by d t divided by nu g cube is equal to c, then from here we will get the function g ok. So, we can write d g by g cube is equal to c nu d t. So, if you integrate it, you will get minus 1 by 2 g square is equal to c nu t plus integration constant c 1. Now, we have already discussed that as t tends to 0, obviously, penetration depth will be tending to 0 that means y reference will be tending to 0. And hence g t will be tending to infinity.

So, hence if you put it here, you will get the integration constant as 0. So, at t tends to 0, g tends to infinity, so that will give c 1 is equal to 0. So, you will get minus 1 by 2 g square is equal to c nu t and here you will get g is equal to 1 by root 2 nu c t with a minus sign. Now, you can see that g should be real number right. So, c value has to be a negative, then we will get g as real. And for convenience now let us take c is equal to 2 ok, so that here it will become 4.

So, you can choose any value of c because we have already told that c is a constant. If you change the value of c, accordingly your g will change, and obviously, your velocity profile f will change. So, for convenience, now we are taking the value of c as minus 2 ok, and negative we are taking to make the g as real. So, for convenience, let us choose c is equal to minus 2 ok.

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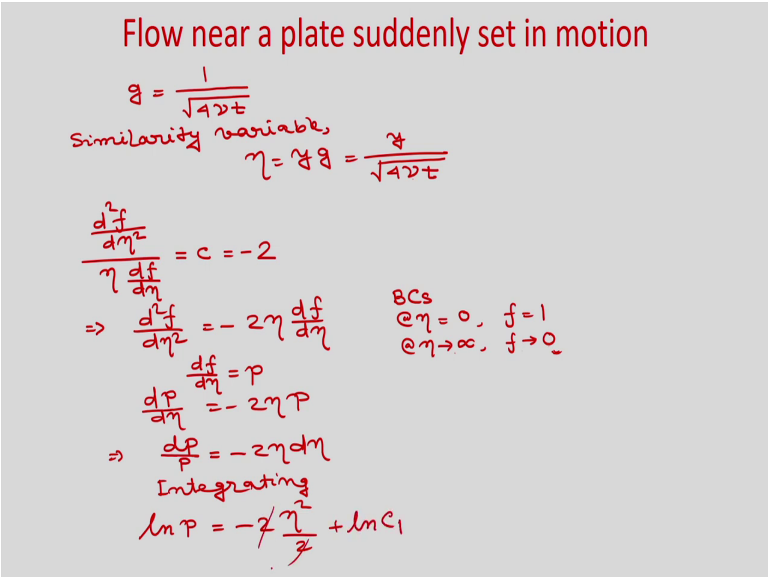

So, if you take c is equal to minus 2, so obviously, g will become 1 by root 4 nu t ok. So, similarity variable will become eta which is your y into g, so it will be y by root 4 nu t ok. So, you can see eta is function of y nu and t and t is nu is anyway constant, so it is function of y and t.

Now, let us take the other part. So, if you take d 2 f by d eta square divided by eta into d f by d eta is equal to c, and c we have already chosen that is as minus 2. So, from here you can see that you can write d 2 f by d eta square is equal to minus 2 eta d f by d eta. So, you can see that we started with partial differential equation, now you have converted it to ordinary differential equation, because this is the second order ordinary differential equation.

Now, what are the boundary conditions? So, now, you can see that at eta tends to 0 ok; eta tends to 0, so u tends to U right. So, it will become f is equal to 1. And at eta tends to infinity, so eta is equal to 0 let us write and eta tends to infinity, obviously, you will become 0 and f will be tending to 0.

And you can see you have one initial condition and that initial condition as t tends to 0, u tends to 0, so that means, eta tends to infinity f tends to 0. So, you can see that one initial condition and one boundary condition actually together, you are representing as at eta tends to infinity, f tends to 0.

So, now let us integrate twice this ordinary differential equation and find the value of f which will give the velocity distribution. So, let us take d f by d eta as p. So, we can write d p by d eta is equal to minus 2 eta p. So, it will be d p by p is equal to minus 2 eta d eta. So, integrating, you will get ln p is equal to minus 2 eta square by 2, and integration constant let us say ln c 1. So, this 2, 2 will get cancelled.

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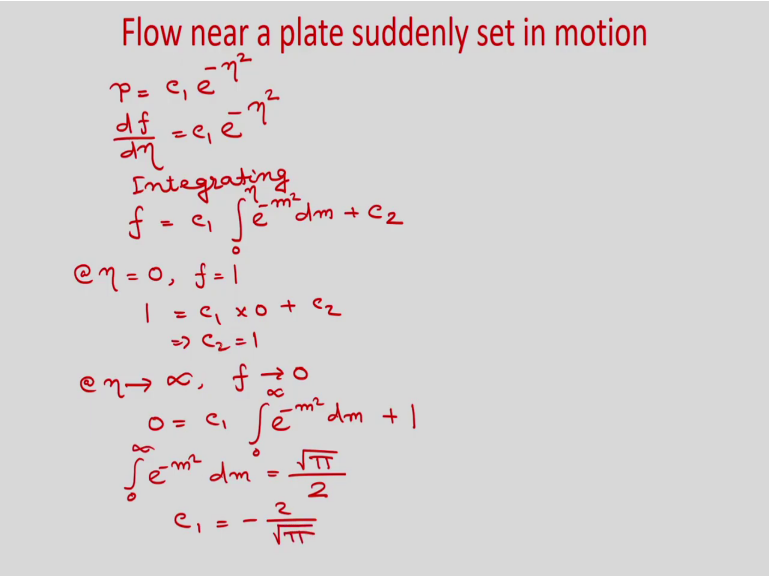

So, if you write it, so you will get p is equal to c 1 into e to the power minus eta square and we know p is d f by d eta, so it will be c 1 e to the power minus eta square. So, again if you integrate, so you will get f is equal to c 1 integral. Now, we are integrating eta from 0 to eta e to the power minus m square dm. So, this we have used dummy variable, because we are putting the limit eta plus integration constant c 2.

So, now apply the boundary conditions, so at eta is equal to 0, f is equal to 1. So, you can see from here f is 1, so it will be c 1 into 0 plus c 2, so that means, c 2 will be 1. And at eta tends to infinity f tends to 0, so it will be 0 c 1 0 to infinity e to the power minus m square d m plus 1 ok. So, this infinite integral whatever it is there, so this will have the value as root pi by 2 ok. So, now, if you put it here, so you are going to get c 1 is equal to minus 2 by root pi ok.

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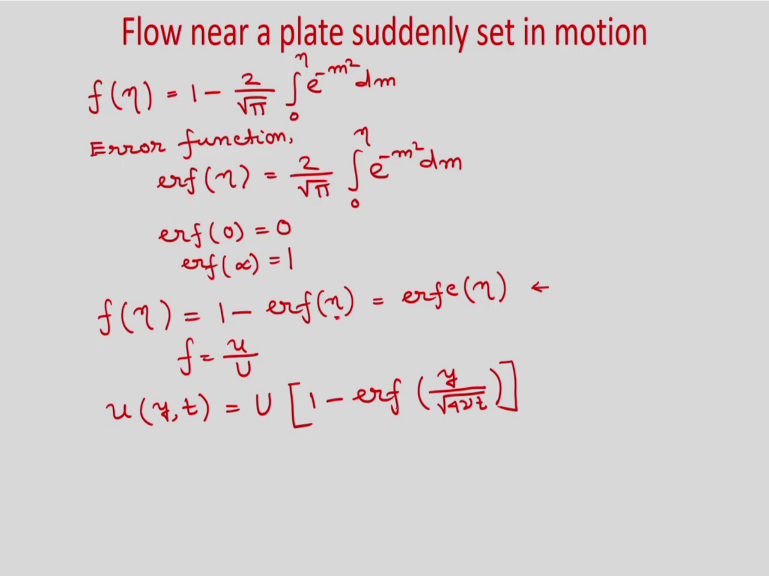

So, now, you know c 1 and c 2 value you put it here. So, you will get f which is function of eta is equal to 1 minus 2 by root pi 0 integral 0 to eta e to the power minus m square dm.

So, you can see this is the velocity distribution and now you have to know the expression of error function, because in the right hand side whatever it is there that is known as error function eta means 2 by root pi integral 0 to eta e to the power minus m square dm ok. And error function 0, you can see if you put here, it will be 0 and error function infinity if you put, so obviously, it will become root pi by 2, so it will be 1 ok.

So, if you put in this expression, so your velocity distribution will become f eta is equal to 1 minus error function eta and 1 minus error function eta, so it is error function c eta. So, you can see that f eta we know f is equal to u by U, so you can write u which is function of y and t, we can write as U into 1 minus error function y by root 4 nu t. So, if you see this expression, obviously, you can see that if you write the u by U which is your non-dimensional velocity, it is function of eta only ok. And in terms of y and t, if you write the velocity, then obviously, this will be function of y and t ok.

So, now, let us see that how it penetrates inside with time ok. So, the distance up to which the effect of moving plate penetrates inside that is known as penetration depth. And we have told that distance we will measure when u will become almost 0, that means, we will tell that 1 percent of the moving plate velocity, that means, u is equal to 0.01 U.

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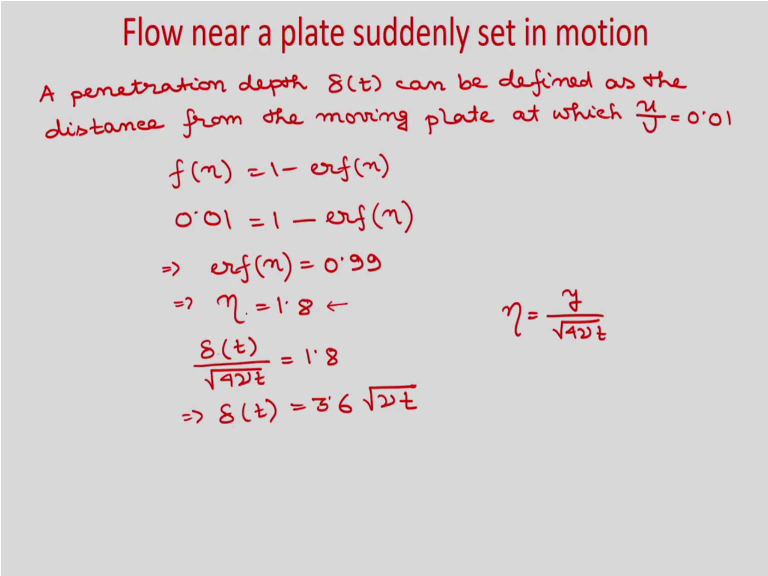

So, if you write that, so a penetration depth delta can be defined as the distance from the moving plate at which u by U is 0.01 ok. So, you know f eta is equal to 1 minus error function eta. So, f eta is u by U that is 0.01 is equal to 1 minus error function eta. So, you can see this will become eta error function eta is equal to 0.99. So, if you see the value of error function ok, 0.99 at which this eta will become 1.8. So, we can see if you see in terms of this similarity variable eta, the penetration depth is always constant, and it is 1.8 ok.

And in terms of delta if you write it is y by root 4 nu t right we have already shown that eta is equal to y by root 4 nu t. So, this is penetration depth at that particular time t, then it will become 1.8. So, the penetration depth delta t will be 3.6 into root nu t. So, you can see that penetration depth is proportional to the root of t and as t increases, obviously, penetration depth increases. But if you see in terms of similarity variable, then obviously, it will be always constant as 1.8.

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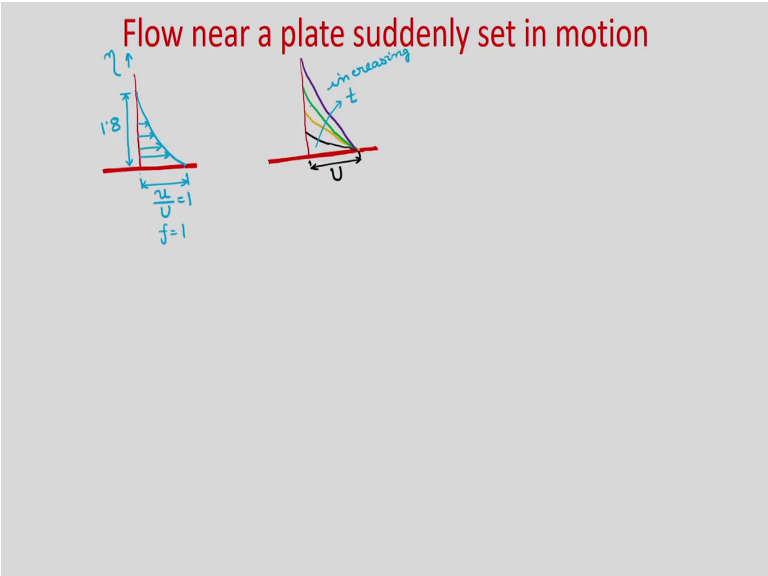

So, now let us plot the velocity profile. So, you can see, so this is the plate and if you want to plot the velocity profile in terms of eta, then it will be look like this. So, this is the velocity profile ok. So, this is obviously, u by U is equal to 1, and this is. So, this side is eta ok, so this will be 1.8 ok. So, in it is actually f, so f is 1 f is equal to 1. So, if you plot eta versus f, then obviously, this is 0 to 1; and it is 0 to 1.8 ok. So, you can see all the velocity profiles collapse into one profile ok.

And if you want to plot at different time, then, so we can see that as time increases, so let us say this is u, U is equal to u, so as time increases penetration depth will increase. So, it will look like this, then another time if you consider, it will be like this. As time increases, it will penetrate more, so you will get like this ok. So, you can see that this is t increasing t ok. So, as time increases, your penetration depth is increasing ok.

So, now, you want to find the shear stress distribution inside the fluid domain. So, what you will find that here we have only one nonzero velocity component u, so shear stress will be just mu del u by del y.

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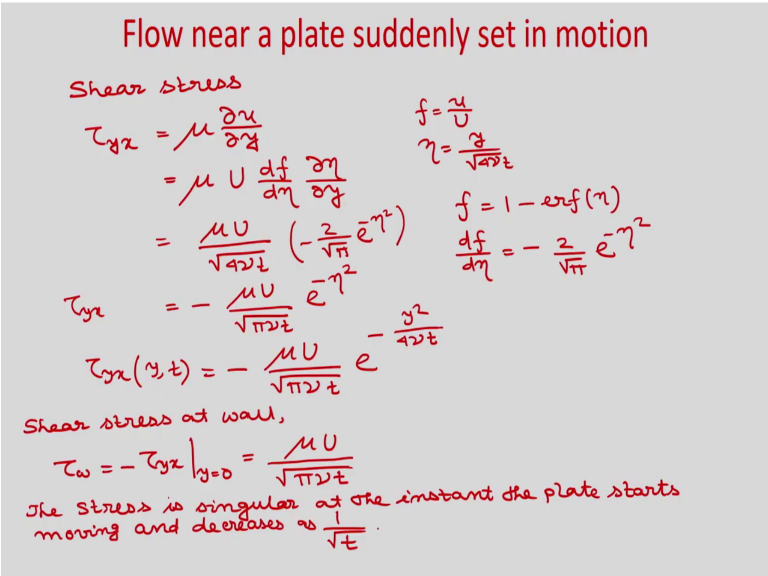

So, shear stress distribution inside the fluid domain will be just tau y x as mu del u by del y ok. So, obviously, you will get mu U, so f is u by U ok, and eta is y by root 4 nu t. So, del u by del y you can write U d f by d eta into del eta by del y. And del eta by del y from here you can write mu u by root 4 nu t.

And what about d f by d eta? So, we know the velocity profile f as function of eta as 1 minus error function eta. And if you take the derivative with respect to eta, then you will get d f by d eta as minus 2 by root pi e to the power minus eta square. So, you will get minus 2 by root pi e to the power minus eta square.

So, hence you will get the shear stress distribution as minus mu u by, so you can see this 2 and this 2 will get cancelled. So, you will write pi nu t e to the power minus nu square. So, we can see this tau y x, obviously, this nu square is nothing but y square by 4 nu t. So, we can write tau y x which is function of y and t as minus mu U by root pi nu t e to the power minus y square by 4 nu t ok. So, this is the shear stress distribution inside the fluid domain.

Now, if you want to find the shear stress acting at the wall, then tau w will be the negative of tau y x at y is equal to 0. So, shear stress at wall at wall, so tau w will become minus tau y x at y is equal to 0 ok. So, you can see from here you will get mu u by root pi nu t ok. So, you can see as t is equal to 0, it will become singular. So, the stress is singular at the instant the plate starts moving and decreases as 1 by root t.

So, in today's class, we started solving unsteady flow problem. So, we considered Stokes first problem. So, in this problem initially one stationary plate infinite plate was kept inside a infinite fluid domain. And suddenly this infinite plate starts moving with a constant velocity u. So, we wanted to find what is the velocity distribution and shear stress distribution inside the fluid domain.

As it starts impulsively as t is equal to 0 plus, so obviously, it is unsteady problem. And we used the similarity transformation technique to convert the partial differential equation to ordinary differential equation. So, we using similarity variable approach, we found the velocity distribution, and then we have calculated the penetration depth. And penetration depth we have seen that it is proportional to root t. And finally, we have calculated the shear stress distribution inside the fluid domain and at the wall.

Thank you.