Viscous Fluid Flow
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Module - 03
Steady Axisymmetric Flows
Lecture - 03
Steady Flow Between Rotating Cylinders

Hello everyone. So, today we will continue with the exact solutions of Navier Stoke equation in cylindrical coordinate. Today, we will consider Steady Axisymmetric ah approach Between Rotating Cylinders. Let us consider the flow in the annulus between two rotating cylinders. These are the two cylinders.

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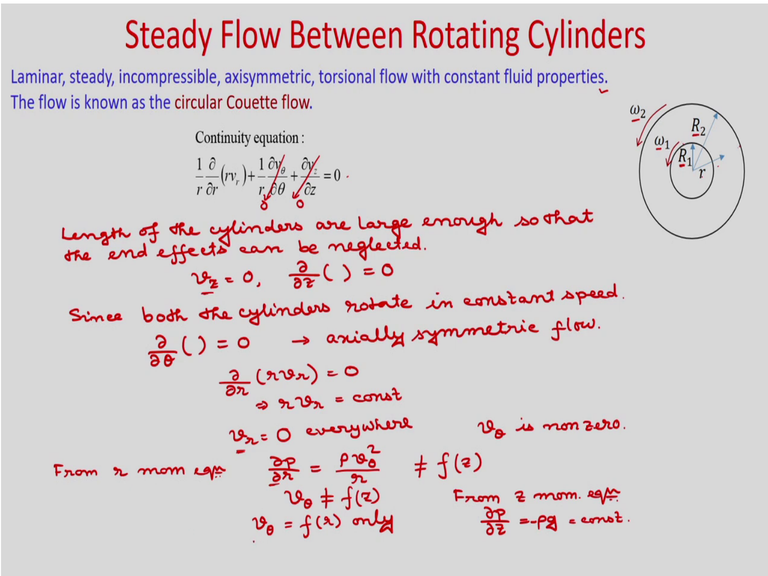

So, this is the inner cylinder of radius R 1 and the outer cylinder of radius R 2. Inner cylinder is rotating with the angular velocity omega 1 and the outer cylinder is rotating with the angular velocity omega 2 and the flow is taking place in the annulus of this two cylinders.

So, we are considering laminar steady incompressible axisymmetric torsional flow with constant fluid properties and this flow is known as Circular Couette flow. So, first we will start with the continuity equation and we will invoke the assumptions and we will see the velocity in the theta direction is non-zero and other velocities are 0.

So, we are considering these two cylinders in perpendicular of this plane is infinite; that means, in the z direction, the cylinder length is infinite so that we can neglect the end effects. That means, v z is equal to 0 and any gradient in that direction is 0. Let us consider that length of the cylinders are large enough so that the end effects can be neglected; that means, v z is equal to 0 and del of del z of any parameter is 0.

And since both the cylinders rotate in constant speed, so this is del of del theta of any quantity is 0; that means, it is axially symmetric flow ok. So, now, if you consider this continuity equation, so you can see that del of del z of v z will be 0 because z direction is infinite and it is axisymmetric flow. So, this will be 0.

So, we will get del of del r r v r is equal to 0; that means, r v r is equal to constant ok. Now, you can see that these cylinders are inoperable, so that means, the perpendicular velocity at the wall is 0; that means, in the radial direction the velocity will be 0 everywhere inside the flow field.

Because you can see here the perpendicular velocity is 0; that means, in the radial direction velocity is 0 and; obviously, at the outer cylinder surface the radial velocity is 0. That means, v r is equal to 0 everywhere. So, you can see that v z is 0 everywhere, v r is 0 everywhere; so, only v theta is non-zero ok.

If you consider the r-momentum equation and invoke this assumptions, then you will get from r-momentum equation, that del p by del r is rho v theta square by r. What does it mean? It means that centrifugal force on an element of fluid balances the force produced by the radial pressure gradient; that means, you can see that del p by del r obviously in the r direction.

So, we are calculating the pressure gradient and this is obviously not function of z. So, this is not function of z. So, that means, v theta is not function of z. Because we are calculating the pressure gradient in the radial direction. So, it should not depend on the z direction. So, that means, it is not function of z. So, v theta is not function of z.

So, obviously, v theta is function of r only ok. So, v theta is function of r only and from z momentum equation, we can show that del p by del z is equal to rho g sorry minus rho g. So, that means, it is the hydrostatic pressure gradient ok. So, and it is constant. So, from here, you can see that v theta is function of r only. Now, let us consider theta momentum equation and we will invoke all the assumptions and we will simplify the partial differential equation.

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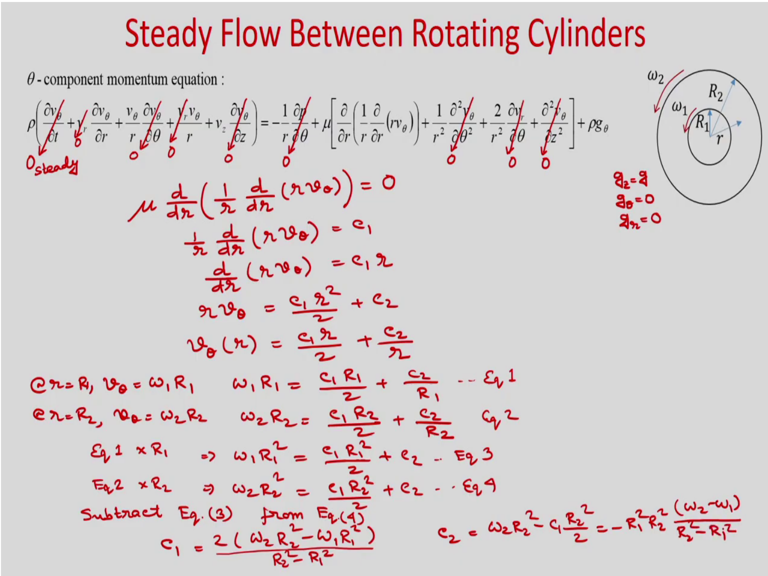

So, you can see this is the theta momentum equation and we are considering the gravitational acceleration in z direction only ok. So, g z is equal to g only ok. So, other gravitational accelerations are 0 and g r is equal to 0. So, if you consider that, then you can see this will be 0 because steady flow and v r is 0. Here axisymmetric flow, so this is 0; v r is 0. So, it is a long cylinder, so del of del z of v theta is 0. Then, it is axisymmetric, so this is 0. Axisymmetric flow, so this is 0; axisymmetric flow, this is 0 and this is a long cylinder, so this is 0.

So, you can see that we are left with mu d of d r 1 by r d of d r r v theta is equal to 0. So, we can see we have written the ordinary differential equation because we know that v theta is function of r only ok. So, now, you can see that from theta momentum equation which was partial differential equation, now we have converted to ordinary differential equation.

Now, you integrate twice and find the velocity distribution, circumferential velocity distribution. So, you can see you can write 1 by r d of d r r v theta is equal to c 1 and d of d r r v theta will be c 1 r and r v theta will be c 1 r square by 2 plus c 2 or v theta which is function of r only it will be c 1 r by 2 plus c 2 by r.

Now, let us apply the boundary conditions and find two integration constants c 1 and c 2. So, what are the boundary conditions? So, at r is equal to r 1, the tangential velocity is omega 1 into r 1 and also, in r is equal to r 2, tangential velocity is omega 2 r 2. So, at r is equal to R 1, v theta is omega 1 R 1.

So, we can write omega 1 R 1 is equal to c 1 R 1 by 2 plus c 2 by R 1 and at r is equal to R 2, v theta is equal to omega 2 R 2. So, if you put it here, you will get omega 2 R 2 is equal to c 1 R 2 by 2 plus c 2 by R 2 ok. So, now, let us find the constants c 1 and c 2 from these two equations.

So, let us multiply these equations. So, let us say this is equation 1 and this is equation 2. So, what we will do? We will just multiply equation 1 into R 1. So, what you will get? Omega 1 R 1 square is equal to c 1 R 1 square by 2 plus c 2 and if you write equation 2 into R 2, then you will get omega 2 R 2 square is equal to c 1 R 2 square by 2 plus c 2. Now, if you tell this equation 3; this is equation 4. Now, subtract equation 3 from equation 4 ok. So, what you are going to get? So, you can see you are going to get; so, c 2 will get cancelled.

So, you will get c 1 is equal to 2 into omega 2 R 2 square minus omega 1 R 1 square divided by R 2 square minus R 1 square and c 2 you will get omega 2 R 2 square minus c 1 R 2 square by 2. So, you will get minus R 1 square R 2 square omega 2 minus omega 1 R 2 square minus R 1 square.

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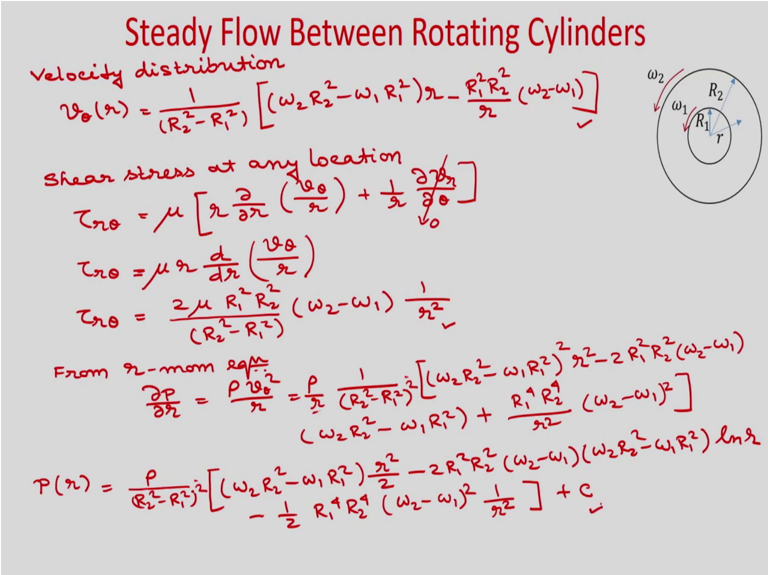

So, now, if we put this constants c 1, c 2 in the velocity distribution, you will get final velocity distribution for this particular case as v theta is equal to 1 by R 2 square minus R 1 square omega 2 R 2 square minus omega 1 R 1 square into r minus R 1 square R 2 square divided by r into omega 2 minus omega 1.

So, now, we are interested to find the shear stress ok, tangential shear stress. So, shear stress you can find. Shear stress at any location tau r theta will be mu into r del of del r v theta by r plus 1 by r del v r by del theta. So, this term is 0 because it is axisymmetric flow. So, tau r theta will be just mu into r d of d r v theta by r ok. So, this is v theta.

So, v theta by r, if you take the derivative with respect to r, then you will get tau r theta will be twice mu R 1 square R 2 square divided by R 2 square minus R 1 square omega 2 minus omega 1 into 1 by r square. So, now, let us find the pressured variation along the radial direction ok. So, we have already shown that from the r-momentum equation, we have written del p by del r is equal to rho v theta square by r 2.

So, we will get from r-momentum equation, del p by del r is equal to rho v theta square by r. So, v theta square by r if you put it here, you are going to get rho by r 1 by R 2 square minus R 1 square omega 2 R 2 square minus omega 1 R 1 square whole square r square minus 2 R 1 square R 2 square omega 2 minus omega 1 into omega 2 R 2 square minus omega 1 R 1 square plus R 1 to the power 4, R 2 to the power 4 divided by r square omega 2 minus omega 1 square ok. So, divided by r we have written here.

So, now, if you integrate it with respect to r, then you will get p as a function of r as rho by R 2 square minus R 1 square omega 2 R 2 square minus omega 1 R 1 square. So, here you will get r because here one r is there. So, if you integrate it, then you will get r square by 2 ok.

Then, here you can see. So, here in denominator, we will have r. So, you will get l n r. So, minus 2 R 1 square R 2 square omega 2 minus omega 1 omega 2 R 2 square minus omega 1 R 1 square. So, integration 1 by r d r means ln r and here, it will be r cube. So, R cube means R to the power minus 3.

So, you will get minus half R 1 to the power 4 R 2 to the power 4 omega 2 minus omega 1 square. So, it will be 1 by r square plus integration constant c. So, you can see, so for this particular case, where two cylinders are rotating with a constant angular velocity omega 1 and omega 2, this is the velocity distribution, this is the shear stress distribution and the this is the radial pressure variation. Let us now consider 4 special cases for this case the flow between two rotating cylinders. So, first case is that inner cylinder is stationery ok. So, that means, omega 1 is 0.

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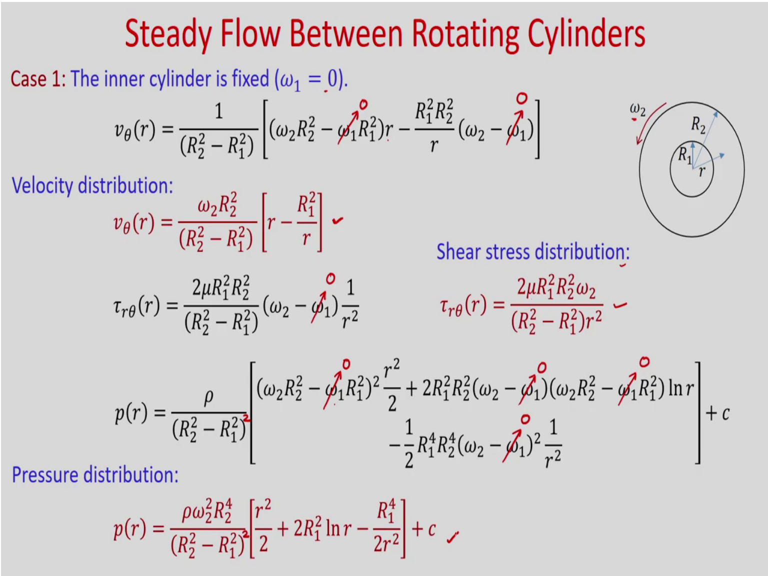

So, we can see the case 1, where inner cylinder is stationery and outer cylinder is rotating with velocity omega 2. So, you can put omega 1 is equal to 0 in this expression because this is the velocity distribution, we have derived. So, if you put omega 1 is 0, so this is 0 and this is 0. So, you can see if you take outside R 2 square omega 2 R 2 square, then you will get velocity distribution as omega 2 R 2 square divided by R 2 square minus R 1 square.

Here, you will get r, here you will get R 1 square by r. So, this is the velocity distribution and this is the shear stress distribution. So, if you put omega 1 as 0, then you will get the shear stress distribution tau r theta is equal to twice mu R 1 square R 2 square omega 2 divided by R 2 square minus R 1 square r square.

So, now the pressure distribution you can see here. So, pressure distribution will be rho by R 2 square minus R 1 square whole square and as omega 1 is 0. So, this term will become 0;

omega 1 is 0, omega 1 is 0 and omega 1 is 0. So, if you put it here and you will get finally, pressure distribution as rho omega 2 square R 2 to the power 4 R 2 square minus R 1 square whole square; r square by 2 plus twice R 1 square ln r minus R 1 to the power 4 divided by 2 r square plus constant c. Now, if you are interested to find the torque at the outer cylinder ok. So, this you can actually apply to find the viscosity using the viscometer ok.

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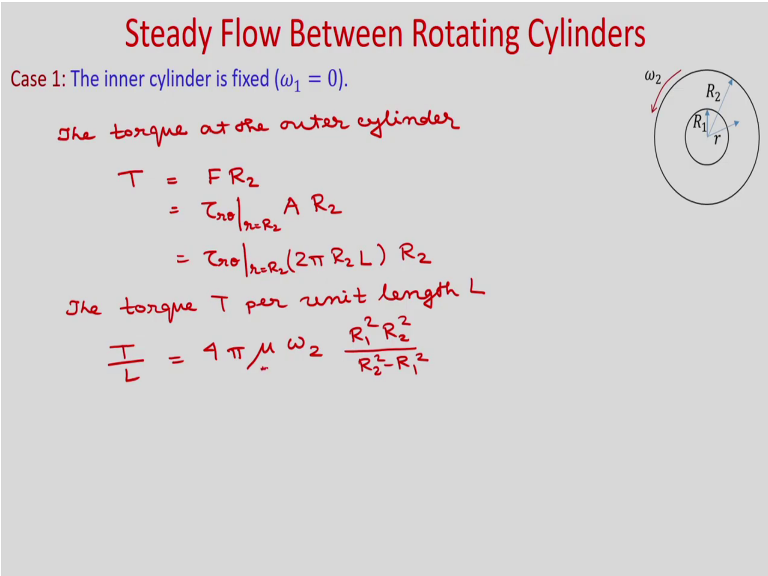

You can see that the torque at the outer cylinder, we will get T is equal to. So, it will be torque will be force into the radius outer cylinder radius is R 2. So, F into R 2 and F is the force. So, force is shear stress into area ok. So, tau r theta at r is equal to R 2 into area into R 2. So, tau r theta r is equal to R 2; what is the area?

Now, we can see it will be twice pi R 2 and if the length of the cylinder is L, then twice pi R 2 into L. So, this is the area into R 2 ok. So, if you put the expression of tau r theta here, then you can write the torque T per unit length L. So, that will be T by L is equal to 4 pi mu omega 2 R 1 square R 2 square divided by R 2 square minus R 1 square.

So, you can see if mu is unknown. So, you if you can measure the torque per unit length, then from the given parameters, you will be able to find the viscosity of the fluid. Now, let us consider that the gap between the two cylinders are very small ok, where inner cylinder is stationary and outer cylinder is moving with angular velocity omega 2. So, now, in that particular case, actually this circular couette flow will become plane couette flow and the velocity will be linearly varying.

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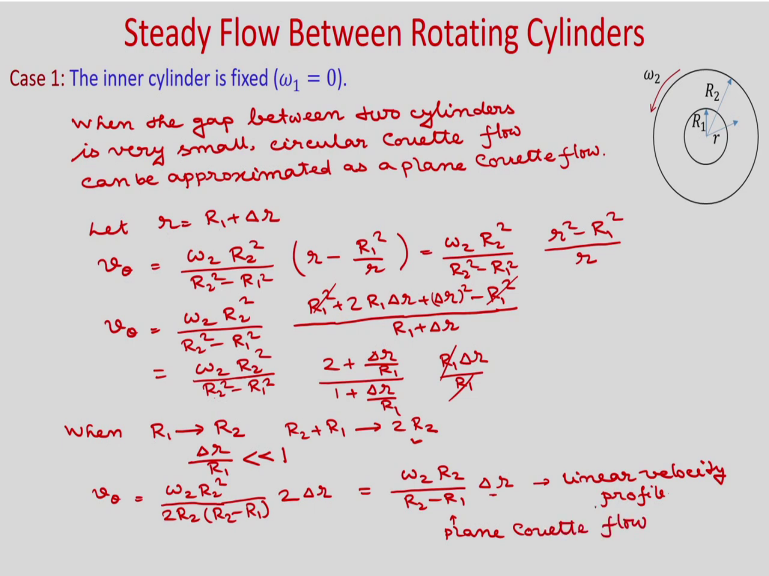

So, in this particular case now when the gap between two cylinders is very small, circular couette flow can be approximated as a plane couette flow ok. So, let r is equal to R 1 plus delta r and v theta we have already derived for this particular case is omega 2 R 2 square divided by R 2 square minus R 1 square into r minus R 1 square by r.

So, we can write omega 2 R 2 square divided by R 2 square minus R 1 square, r square minus R 1 square divided by r. So, now, you put the value of r as R 1 plus delta r. So, v theta will become omega 2 R 2 square divided by R 2 square minus R 1 square. So, r square is R 1 square plus twice R 1 delta r plus delta r square minus R 1 square divided by r; so, it will be R 1 plus delta r.

So, this R 1 square, this R 1 square will get cancelled and you can write omega 2 R 2 square divided by R 2 square minus R 1 square here, you just take R 1 delta r outside. So, you will get 2 plus delta r by R 1 and outside we can write R 1 into delta r. And here, you just take outside R 1. So, you will get one plus delta r by R 1 ok.

So, now, in this particular case, the gap is very small; that means, R 1 tends to R 2 and R 2 plus R 1 will tend to twice R 2 and delta r by R 1 is much much smaller than 1 ok. So, you can see that it will be very smaller than 1. So, you can just write 2 and this denominator, you can write 1 and here you can write R 2 plus R 1 into R 2 minus R 1. So, R 2 plus R 1, we will write as twice R 2. So, v theta will become omega 2 R 2 square. So, this we are writing R 2 plus R 1 into R 2 minus R 1. So, R 2 plus R 1 will become 2 R 2 as R 1 tends to R 2.

Then, R 2 minus R 1 and this will become 2 and this is delta r. So, you can see this will become omega 2 R 2 divided by R 2 minus R 1 into delta r. So, we can see this is your linear velocity profile. So, what is delta r? Delta r is actually if you take y from the inner cylinder, then delta r will represents equivalent to y and R 2 minus R 1 is the gap; that means, for plane couette flow it is h.

So, and omega 2 R 2 is the velocity, tangential velocity. So, that is nothing but the velocity u. So, you can see this is u by h into y delta r ok. So, you can see this is the linear velocity profile and velocity profile for the plane couette flow ok. Now, let us consider the second special case, where both the cylinders rotate with same angular velocity omega; that means, omega 2 is equal to omega 1 is equal to omega.

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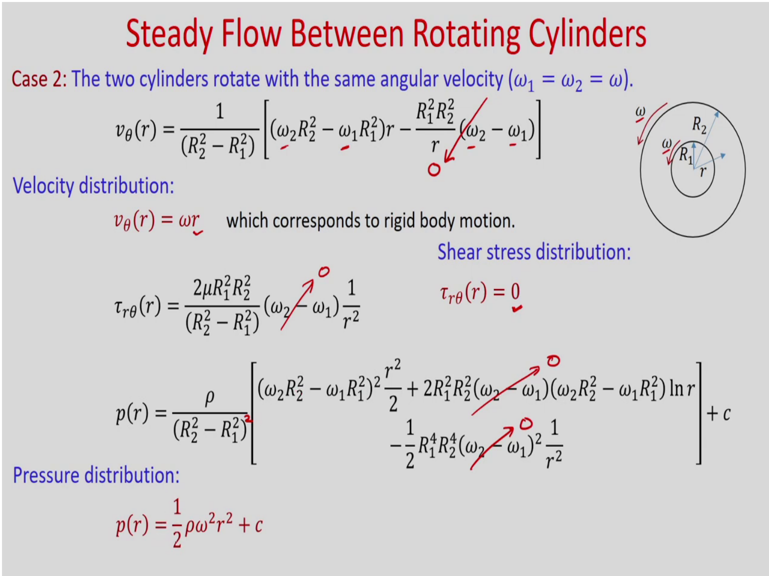

So, you can see the inner cylinder is rotating with angular velocity omega and outer cylinder is rotating with constant angular velocity omega. So, in this case, omega 1 is equal to omega 2 is equal to omega. So, you can see this omega 2, you can write omega, here you can write omega, this will become 0 because omega minus omega will become 0. So, this term will become 0.

So, from here, you can see if you write omega then R 2 square minus R 1 square and this will get cancel. So, you will get v theta as only omega r which corresponds to rigid body motion and here, tau r theta, so omega 2 minus omega 1 will become 0. So, tau r theta is 0.

So, tangential shear stress is 0 and pressure distribution, so obviously, you can see that this you can write omega 2 minus omega 1 will become 0, this will become 0. So, and here, omega 1 is equal to omega 2. So, if you take omega outside, it will become R 2 minus R 1 whole square and this will be squared. So, you can see this omega square if you take outside, then it will become R 2 square minus R 1 square whole square.

So, this will get cancelled. So, you will get half rho omega square r square plus c. Now, the third special case we will consider that inner cylinder is removed ok. So, inner cylinder is removed, so that means, R 1 tends to 0. So, in the velocity distribution whatever we have derived, if you put R 1 tends to 0, then what we will get? So, you can see this inner cylinder is removed. So, outer cylinder is moving with a constant velocity omega.

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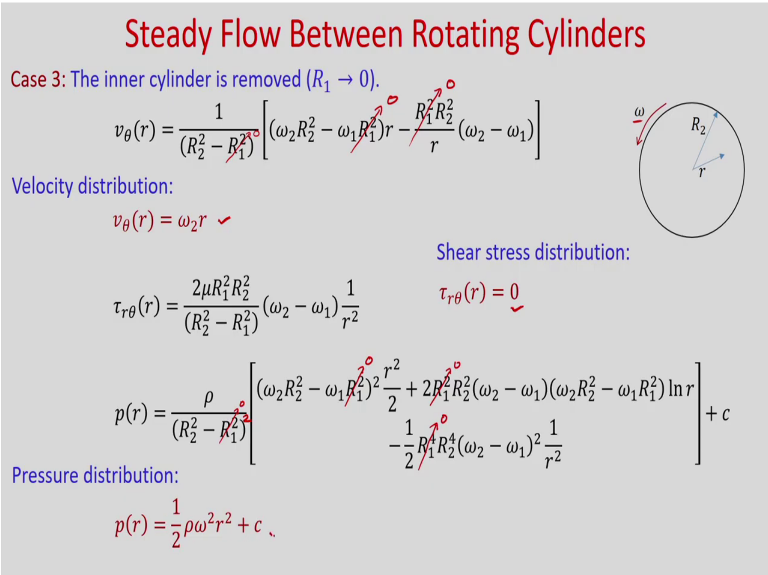

So, we can see as R 1 tends to 0, this is the velocity profile. So, this will become 0 ok and this will become 0. So, you will get and this is R 1 tends to 0; so, this is 0. So, we can see this will become this R 2 square R 2 square will get cancelled. So, you will get omega 2 into r and the shear stress distribution, similarly if you put R 1 tends to 0, then tangential shear stress will become 0 and the p r, so obviously, in this case also R 1 tends to 0.

So, this is 0, 0, this is 0; that means, this whole term will become 0; this is 0 means whole term will become 0. So, you will get half rho omega square r square plus c. So, now the fourth special case we will consider, where the outer cylinder is removed ok. So, that means, the inner cylinder is rotating in a infinite medium ok of fluid ok.

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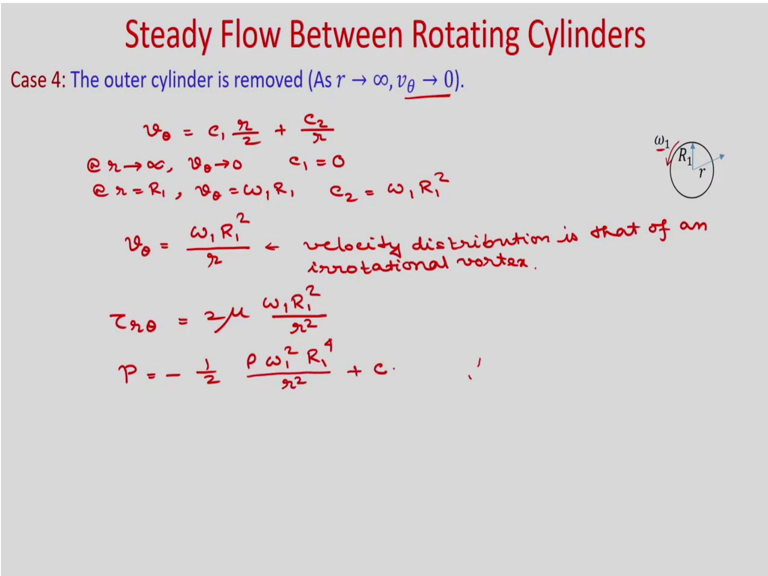

So, in that case, you can see that this inner cylinder is rotating with a constant velocity omega 1. So, as r tends to infinity, obviously far away from the cylinder, the v theta will tends to 0 ok. So, in this particular case, we will start from the velocity distribution which we actually derived after integrating the ordinary differential equation. So, that was c 1 r by 2 plus c 2 by r. So, you can see as r tends to infinity v theta tends to 0. So, if you put it here. So, you can see that c 1 must be 0 ok.

And as r is equal to R 1 v theta is equal to omega 1 into R 1. So, from here you will get c 2 is equal to omega 1 R 1 square. So, v theta will become omega 1 R 1 square divided by r ok. So, you can see velocity distribution is that of an irrotational vortex ok; velocity distribution is that of irrotational vortex ok. So, now, if you calculate the shear stress tau r theta, it will become twice mu omega 1 R 1 square by r square and the pressure distribution, p will become minus half rho omega 1 square R 1 to the power 4 divided by r square plus constant c.

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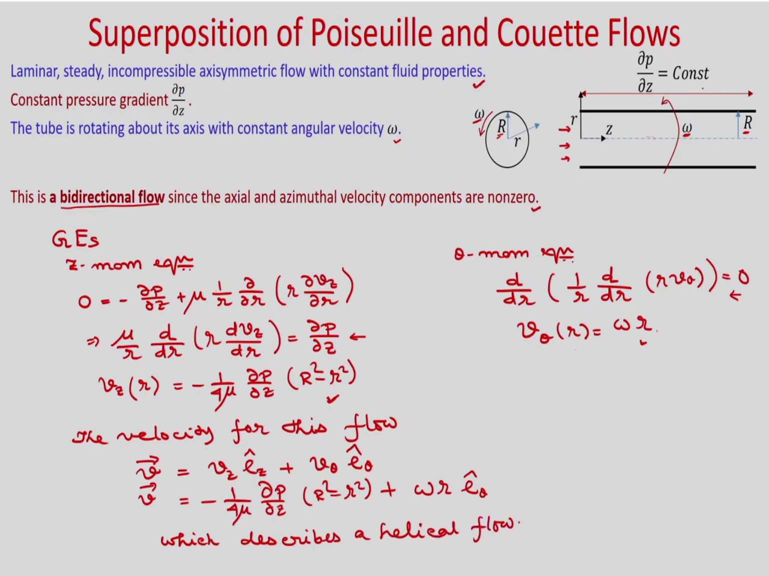

Now, let us consider steady incompressible flow over a circular tube which is actually rotating with a constant velocity omega. So, you can see this is the circular tube of radius R and it is rotating with constant angular velocity omega; z is the axial direction, r is the radial direction measured from the central line and in the z direction, there is a constant pressure gradient del p by del z.

So, if you see from the side view, so this solid cylinder is rotating with a constant velocity omega and R is the radius and obviously, there is a velocity in the axial direction. So, you can see that your axial velocity v z and azimuthal velocity v theta are non-zero.

So, let us see what are the assumptions we have taken. Laminar, steady, incompressible axisymmetric flow with constant fluid properties. We have a constant pressure gradient del p by del z. The tube is rotating about its axis with a constant angular velocity omega and this is a bidirectional flow since the axial and azimuthal velocity components are non-zero.

So, we can see that this is a super position of two different kinds of flow; one is Plane Poiseuille flow and another is Circular Couette flow. So, you know the governing equations for these two cases. So, you can see that from governing equations or from z-momentum equation, you will get 0 is equal to minus del p by del z plus mu into 1 by r del of del r r del v z by del r.

That means, you get mu by r d of d r r d v z by d r is equal to del p by del z and v z r will be minus 1 by 4 mu. So, if you solve this equation, you will get del p by del z into R square minus r square. So, this is the velocity profile. You can see inside this a tube. So, this is fully developed flow. We have assumed and this is the velocity profile v z you will get ok. And if you consider theta momentum equation, then you will get d of d r 1 by r d of d r r v theta is equal to 0. So, v theta will be omega into r ok.

So, the flow is taking place inside this tube and the velocity profile is this for plane poiseuille flow and as it is rotating, so here also you will get the circular couette flow and v theta is omega into r. And you can see the governing equations ok. So, these governing equations are linear ok.

So, you can actually superimpose the velocity profile. So, your final velocity profile for this particular problem, you can write the velocity for this flow v will be v z e z which is the unit vector in z direction plus v theta e theta. So, this will become minus 1 by 4 mu del p by del z R square minus r square plus omega into r e theta. So, you can see which describes a helical flow.

So, in today's class, we considered the flow in annulus between two rotating cylinders. So, in this particular case, we derived in general, where the inner cylinder is rotating with a constant velocity omega 1 and outer cylinder is rotating with a constant velocity omega 2 and we derived the velocity profile, shear stress distribution and the radial pressure distribution.

Then, we considered 4 special cases and first case we considered that the inner cylinder is stationary and second case we considered, where inner cylinder is removed; then, the third case we considered that both the cylinders are rotating with a same velocity omega. And the lastly, we considered that the outer cylinder is removed.

So, for this special cases, we calculated the velocity distribution, shear stress distribution and the radial pressure distribution. Then, we considered flow inside a pipe, where the pipe is rotating with a constant velocity omega.

Thank you.