Viscous Fluid Flow
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Module - 03
Steady Axisymmetric Flows
Lecture - 02
Thin Film Flow and Annular Flow

Hello everyone. So, in last class we have solved the velocity distribution and the volume flow rate for Hagen Poiseuille flow. You have learnt how to derive the ordinary differential equation from the Navier Stoke equations using the assumptions. So, today we will also solve two problems Thin Film Flow and Annular Flow in cylindrical coordinate.

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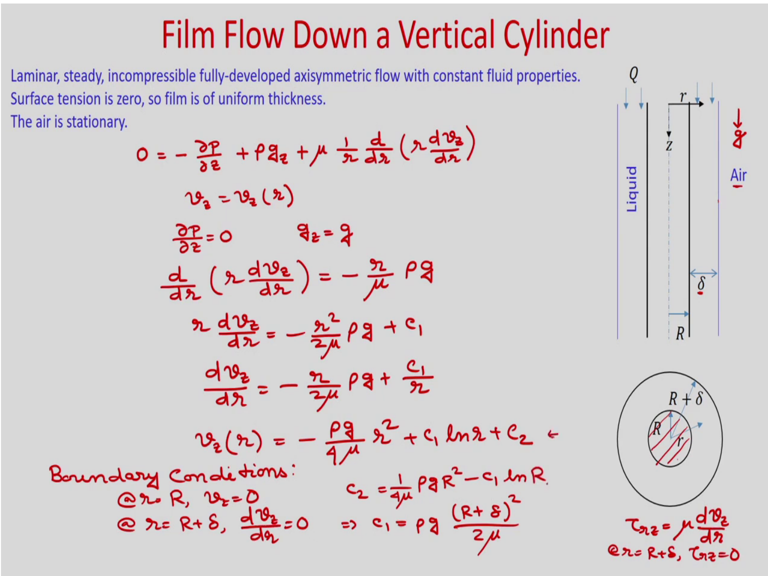

So, first let us consider film flow down a vertical cylinder. So, you can see a Newtonian liquid is flowing down in the outer surface of a infinitely long cylinder of radius r, this is the cylinder with radius r. And outside you can see it is a thin liquid film of delta ok. And outside of this film we have air which is stationary.

So, the coordinate is taken r in this direction and z in the downward direction. And, in this case we are assuming laminar steady incompressible fully developed axisymmetric flow with constant fluid properties. So, as it is axisymmetric flow so, you know that the circumferential direction velocity is 0 and the gradient in that direction of any parameter is 0.

We are assuming surface tension is 0 so the film has the uniform thickness delta. And the air is stationary such that the shear stress is 0 at the interface. So, you can see this is the cylinder actually and outside you have thin film of thickness delta. So, you can see that in this direction gravity will be there and let us say this is g acting in the positive z direction. So, we will start with the ordinary differential equation which we derived in the last class.

So, you know that we have derived in last class 0 is equal to minus del p by del z plus rho g z and you have plus mu 1 by r d of d r r d v z by d r. So, d v is the axial velocity and you know that v z is function of r only ok. So, in this case; obviously, you can see that it is a gravity driven flow, because del p by del z will be 0.

So, in this case del p by del z is 0 and due to gravity actually this liquid will go down along the z direction. And let us say in this direction we have the gravity g so; obviously, g z is equal to g. So, if you rearrange it you will get d of d r r d v z by dr is equal to minus r by mu rho g.

So, this equation now you can integrate twice and you will get the velocity distribution, and the integration constant you can find invoking the two boundary conditions. So, what are the boundary conditions? Obviously, at the cylinder surface at r is equal to r velocity is 0 v z is equal to 0. And at the interface at r is equal to r plus delta the shear stress is 0.

So, if you integrate twice you will get r d v z by d r is equal to minus r square by twice mu rho g plus c 1 d v z by dr is equal to minus r by twice mu we are dividing r in both side c 1 by r. And, if you integrate then you will get v z which is function of r only as rho g by twice mu r square r square by 2. So, it will be rho g by 4 mu plus c 1 l n r plus c 2 ok. And what are the boundary conditions?

Boundary conditions at r is equal to capital R v z is equal to 0 and at r is equal to R plus delta which is your interface d v z by d r is equal to 0, why? Because tau r z is equal to mu d v z by d r right. And tau r z is equal to 0 at r is equal to R plus delta tau r z is equal to 0. So, hence d v z by dr is equal to 0.

So, if you put it so, you will get c 1 is equal to rho g R plus delta square by 2 mu. And, if you invoke r is equal to R v z is equal to 0 from here you will get c 2 is equal to 1 by 4 mu rho g R square minus c 1 l n R.

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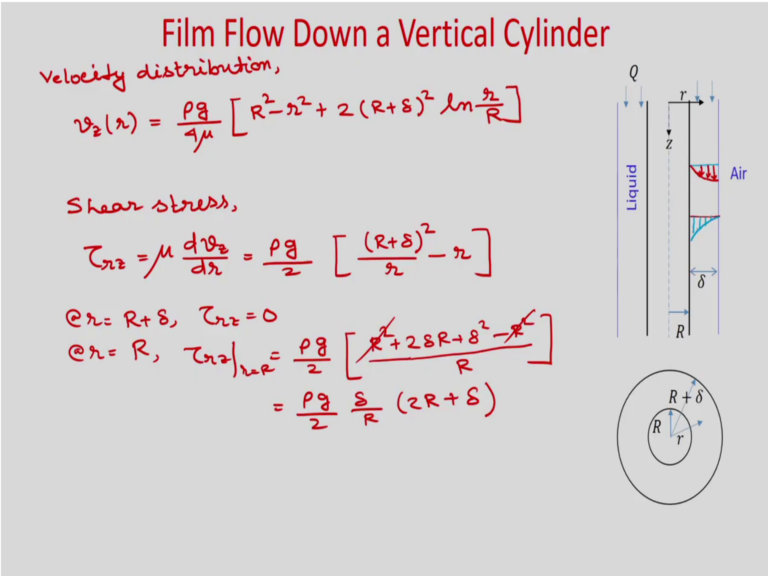

Now you put this values in this velocity distribution, and you will get final expression for velocity distribution as v z, which is function of r is equal to rho g by 4 mu R square minus r square plus 2 R plus delta square l n r by R ok.

So, if you plot the velocity profile it will look like this so; obviously, v will be 0 at the solid wall so, velocity will look like. Now, if we want to find the shear stress, then you can find tau r z inside the film thickness. So, shear stress tau r z you will get mu into d v z by d r and this if you find you will get rho g z by 2 R plus delta square divided by r minus r ok.

So; obviously, at r is equal to R plus delta you know that tau r z is equal to 0 and at r is equal to R tau r z will be rho g. So, if you put in this expression r is equal to R. So, you will get R square plus 2 delta R plus delta square R minus R square. So, this will get cancel so, you will get rho g by 2 delta by R twice R plus delta ok. So, if you want to draw the shear stress profile inside the fluid domain, then you will get the shear stress is 0 here right. So, it will vary like this.

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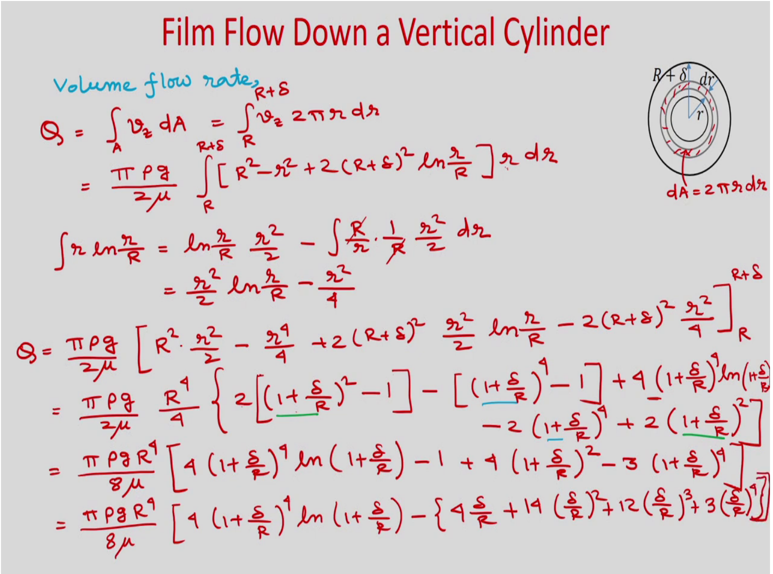

Now, we are interested to find the volume flow rate, and if volume flow rate is known then you can find the film thickness delta. So, first let us find what is the volume flow rate ok? So, if you want to calculate the volume flow rate, then you can write Q is equal to area integral v z d A.

So, in this particular case you can see that at a radius r, if you take a small elemental thickness d r, then this is the flow area ok. And the elemental area d A will be twice pi r into d r ok. So, if you put the velocity distribution v z here, and if you integrate it and the constant if you take so, it will be v z twice pi r d r and you have to integrate from R to R plus delta R to R plus delta.

So, here now you can see that it will be the constants you can take outside the integral, pi rho g by twice mu R to R plus delta R square minus r square plus twice R plus delta square l n r by R into r d r. So, now you multiply this r with these terms and integrate it so, you will get. So, first let us integrate this term integration by parts r into l n r by R ok.

So, this is integration by parts you will get in the third term. So, this you can see you can write l n r by R, then integral r d r r square by 2 minus integral. So, it will be d r by d l n r by R d of d r of l n r by R. So, you will get R by r into 1 by R, then again integral r dr means r square by 2 d r ok.

So, these R R will get cancel here you will get 1 R. So, you can write r square by 2 l n r by R minus so, it will be again r square by 2. So, it will be r square by 4 ok. Now, we can write this integral so, pi rho g by twice mu. So, you can see first term it will be R square into R. So, it will be R square into r square by 2 minus it will be r cube. So, r to the power 4 by 4 and plus 2 R plus delta square into integral r l n r by R.

So, this is the term so, you can write r square by 2 l n r by R minus 2 R plus delta square r square by 4 ok, and limits R to R plus delta ok. Now, if you put the limits you can see that it will be R plus delta whole square minus r square. So, everywhere if you take pi rho g by twice mu, you take R to the power 4 by 4 outside the bracket ok.

So, then you can write you can see it will be just so, 2 into 1 plus delta by R square here, then minus 1 ok, because R square will be there minus r square, then minus so, R to the power 4 by 4 we have taken outside. So, it will be 1 plus delta by R to the power 4 minus 1, then this term ok. So, it will be so, 4 we have taken outside. So, it will be plus 4 1 plus delta by R to the power 4 ok.

And we will get l n 1 plus delta by R, then minus 2 1 plus delta by R to the power 4 and one term will become 0 because it will be l n 1. So, l n 1 is 0 so, this term we are not writing and you will get plus 2 into 1 plus delta by R whole square ok. So, if you put R then; obviously, R to the power 4 you are taking outside. So, it will be 1 plus delta by R whole square. So, if you rearrange it so, you will get pi rho g R to the power 4 divided by 8 mu. So, you can see here 2 into 1 plus delta by R whole square right.

So, first let us take this term 4 into 1 plus delta by R to the power 4 l n 1 plus delta by R ok. Then, here we have minus 2 and plus 1 so, it will be minus 1, then we have 4 into 1 plus delta by R square. So, this is 2 into 1 plus delta by R whole square this term and this term. And now we have left with this term and this term.

So, you can see it will be minus 3 1 plus delta by R to the power 4. So, after simplification you are going to get pi rho g R to the power 4 by 8 mu 4 1 plus delta by R to the power 4 l n 1 plus delta by R ok. And if you rearrange it then you can write as 4 delta by R plus 14 delta by R square plus 12 delta by R cube plus 3 delta by R to the power 4.

So, you can see this is the expression for the volume flow rate of this film. Now, if Q is known then you will be able to find the film thickness delta from this expression. And if delta is known; obviously, the volume flow rate you will be able to find. Now, let us consider one limiting case ok, where film thickness is very very small compared to the radius of the cylinder.

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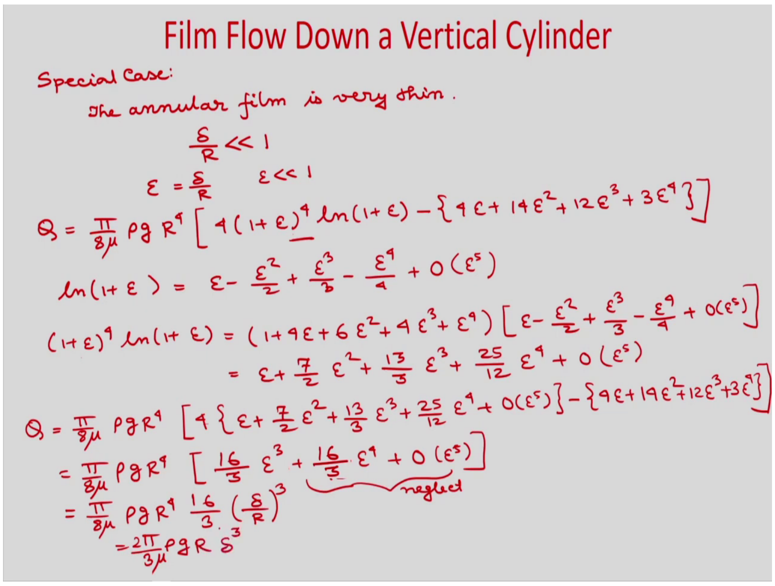

So, special case where the annular film is very thin ok, and it can be approximated as a thin planar film ok. So, delta by R is much much smaller than 1 ok. So, now, let us define epsilon which is delta by R so; obviously, epsilon will be much much less than 1.

So, now if you express the volume flow rate in terms of epsilon, then we can write pi by 8 mu rho g R to the power 4, 4 into now delta by R we are putting epsilon, l n 1 plus epsilon minus 4 epsilon plus 14 epsilon square plus 12 epsilon cube plus 3 epsilon to the power 4 ok.

So, now let us expand this l n 1 plus epsilon ok. So, using Taylor series so, l n 1 plus epsilon ok, if you expand into Taylor series you are going to get epsilon minus epsilon square by 2 plus epsilon cube by 3 minus epsilon to the power 4 by 4 plus order of epsilon to the power 5.

So, we can write now 1 plus epsilon to the power 4 this term we can write now l n 1 plus epsilon ok. So, this term if you write then you will get 1 plus 4 epsilon plus 6 epsilon square plus 4 epsilon cube plus epsilon to the power 4 and these terms, epsilon minus epsilon square by 2 plus epsilon cube by 3 minus epsilon to the power 4 by 4 plus order of epsilon to the power 5.

So, now if you multiply then you will get epsilon plus 7 by 2 epsilon square plus 13 by 3 epsilon cube plus 25 by 12 epsilon to the power 4 plus order of epsilon to the power 5. So, now, if you see the volume flow rate Q so, this term you can put it here. So, pi by 8 mu rho g R to the power 4.

So, you have to multiply with 4 epsilon plus 7 by 2 epsilon square plus 13 by 3 epsilon cube plus 25 by 12 epsilon to the power 4 plus order of epsilon to the power 5. And this term minus 4 epsilon plus 14 epsilon square plus 12 epsilon cube plus 3 epsilon to the power 4 so, this is the term.

Now, if you rearrange it you will get pi by 8 mu rho g R to the power 4 16 by 3 epsilon cube 16 by 3 epsilon to the power 4 plus order of epsilon to the power 5. So, you can see these will get cancelled, because you have 4 epsilon here minus 4 epsilon here. So, if you do the rearrangement you will get this term.

Now, in this case we have assumed that epsilon is much much smaller than 1. So, you can neglect the higher order terms keeping only the first term ok, then you will get the volume flow rate, when you have a very thin film ok.

And that actually we will now we can write so, the first term you keep ok. And other terms you can neglect, neglect these terms only the first term if you keep, then you will get pi by 8 mu rho g R to the power 4 16 by 3 and epsilon is nothing, but delta by R to the power 3. So, now you can see it will be 2 pi by 3 rho g R delta cube ok.

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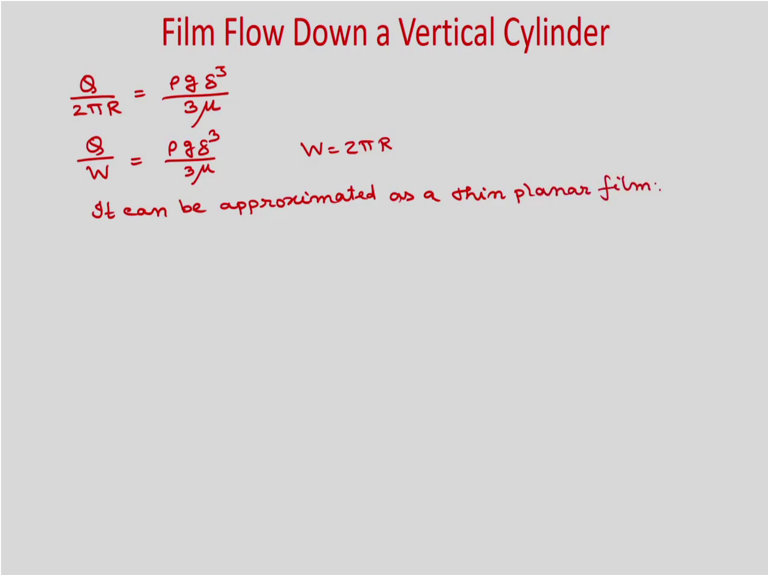

Now, if you rearrange it Q by twice pi R, then you can write rho g delta cube by 3 mu ok. So, you can see in this case if you consider twice pi R which is actually the width W, then the volume flow rate per unit width you are getting the expression which actually we got it in Cartesian coordinate ok.

So, if film thickness is very very small compared to the R, you are going to get the volume flow rate same as what you got the thin film thickness in a vertical flat plate with the thin film. So, that results you are getting so; that means, it will be Q by W rho g delta cube by 3 mu, where W is equal to twice pi R. So, you can see that it can be approximated as a thin planar film.

Now, let us consider the second problem ok. So, this is annular flow with the outer cylinder moving with a constant velocity V. So, in cylindrical coordinate you have 2 coaxial cylinders and the flow is taking place in the annulus.

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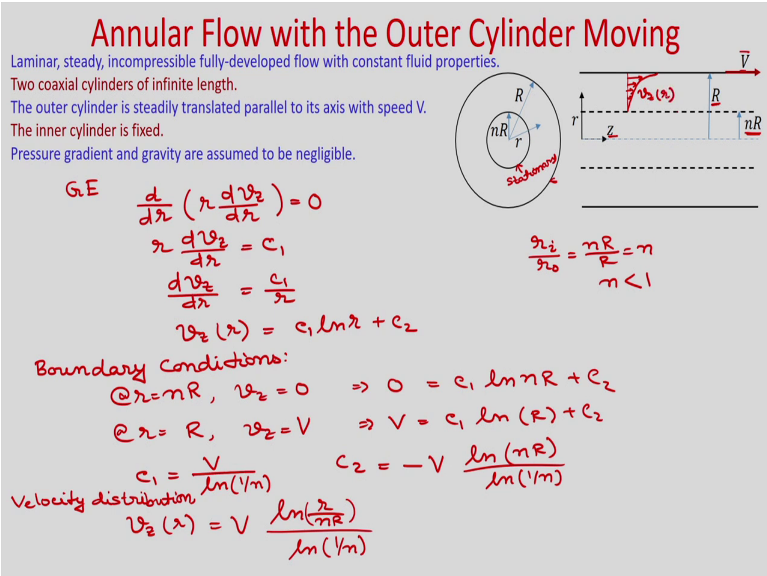

So, in this case you can see this is the inner cylinder of radius n R and outer cylinder is of radius R. And this outer cylinder is moving with a constant velocity V in the positive z direction ok, and inner cylinder is stationary. So, these are coaxial cylinders r is measured from the central line. So, you can see this is this cylinder is stationary, and this cylinder is moving with a constant velocity V.

And we are assuming laminar steady incompressible fully developed flow with constant fluid properties. And 2 coaxial cylinders of infinite length the outer cylinder is steadily translated parallel to its axis with speed V, inner cylinder is fixed and pressure gradient and gravity are assumed to be negligible.

So, you can see that it is a purely shear driven flow ok. So, because pressure gradient is 0 and; obviously, gravity is 0 then it is a purely shear driven flow. So, now first we will derive the velocity distribution, then the volume flow rate and then with limiting case will see that what happens if the inner cylinder is not there and, if the inner cylinder radius is almost close to the outer cylinder radius.

So, first let us start with the governing equation so; obviously, pressure gradient is 0 gravity is 0. So, you can write the equation as d by d r r d v z by d r is equal to 0, if you integrate twice you will get the velocity profile and d v z by d r is equal to c 1 by r and v z which is function of r only you will get c 1 l n r plus c 2 and what are the boundary conditions, boundary conditions are at r is equal to n R ok.

So, you can see in this case inner cylinder radius divided by outer cylinder radius is n R by R that means, n so, n obviously is less than 1 ok. So, at r is equal to n R v z is equal to 0 so; that means, you will get 0 is equal to c 1 l n n R plus c 2 and at r is equal to R. That means, outer cylinder it is moving with a constant velocity V. So, it will get V is equal to c 1 l n R plus c 2.

So, if you solve for this constants you get c 1 is equal to V by l n 1 by n and c 2 as minus V l n n R divided by l n 1 by n. So, if you put this constants in the velocity distribution, you will get v z is equal to V l n r by n R divided by l n 1 by n ok.

So, this is the velocity distribution ok. So, how the velocity profile will look like. So, velocity profile will look like. So, you can see it is a purely shear driven flow velocity is 0 here, and you have velocity V at the outer cylinder ok. So, this is v z which is function of r.

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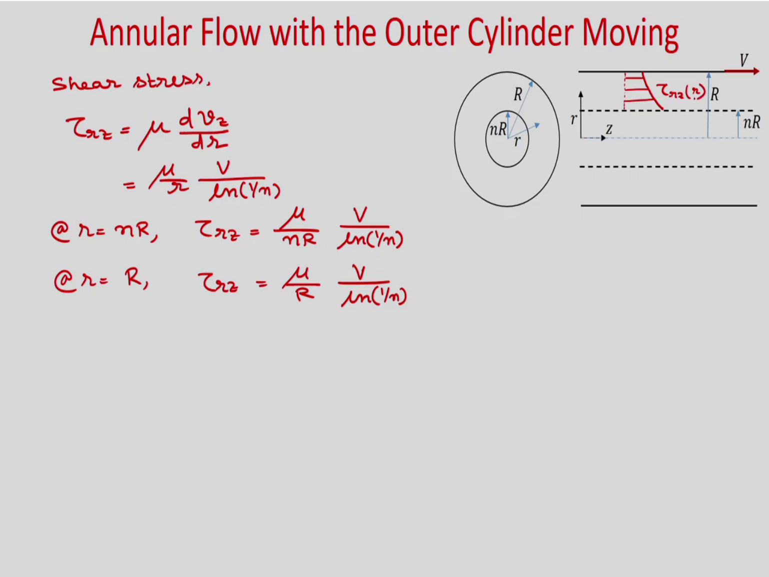

Now, let us calculate the shear stress distribution. So, shear stress tau r z you can write as mu d v z by d r ok. So, from here you can write mu by r V divided by l n 1 by n ok. So, at r is equal to n R tau r z will be just mu by n R V by l n 1 by n and at r is equal to R tau r z will be mu by R V l n 1 by n ok.

So, if you draw the shear stress distribution; obviously, you can see that shear stress will be more in the on the inner cylinder. So, you will get tau r z ok.

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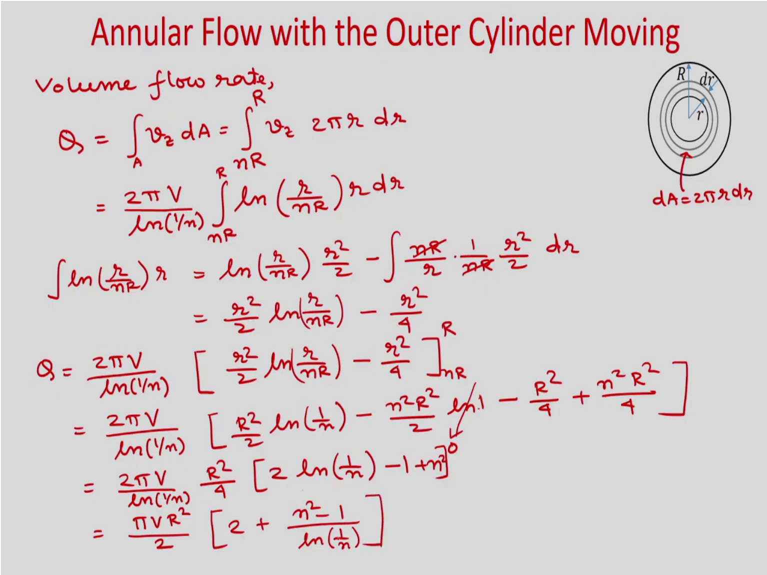

Now, we are interested to find the volume flow rate. So, volume flow rate you can find Q as area integral v z d A and obviously, in this case you can see here d A will be twice pi r d r. So, it will be integral so, n R to R v z twice pi r d r.

So, now you put the expression of v z here and take the constant outside the integral, then you can write twice pi V divided by l n 1 by n integral n R to R l n r by n R into r d r. So, again if you use the integration by parts, then you can find l n r by n R into r as so, it will be l n r by n R r square by 2 minus. Now, you can see here you will get d of d r of l n r by n R.

So, it will be n R by r into 1 by n R integral and you will get r square by 2 d r. So, it will be r square by 2 l n r by n R minus so, you will get integral r by 2 d r and that will be again r square by 4.

So, if you put it here then you are going to get twice pi V divided by l n 1 by n. So, it will be r square by 2 l n r by n R minus r square by 4 n R to R. So, if you put the limits you will get you see R square by 2 l n 1 by n minus n square R square by 2 l n 1 and l n 1 this will be 0 and minus R square by 4 and plus n square R square by 4 ok.

So, if you rearrange it if you take outside R square by 4, then it will be twice pi V divided by l n 1 by n R square by 4, if you take outside. So, you will get 2 l n 1 by n minus 1 plus n square. So, finally, you can write pi V R square divided by 2 into if you this denominator, whatever it is there if you take inside the bracket, then it will be 2 plus n square minus 1 divided by l n 1 by n. Now, we will discuss 2 special cases where n tends to 0 and n tends to 1.

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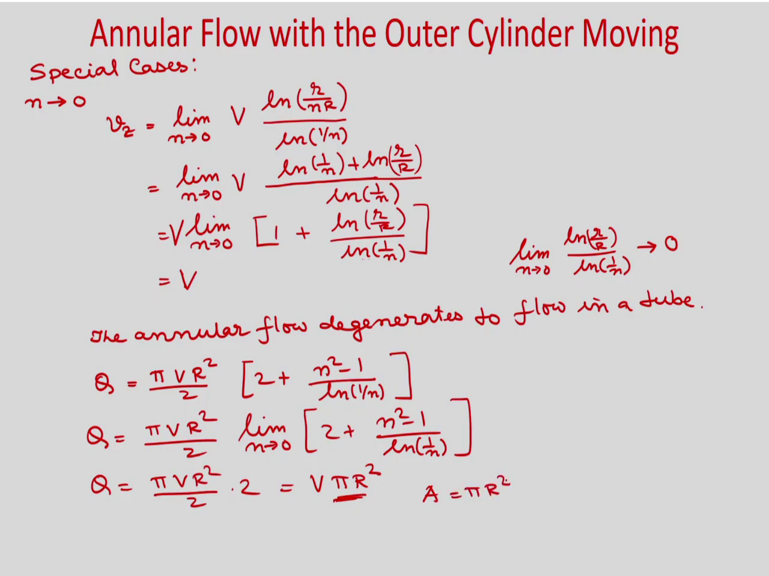

So, special cases first we will consider n tends to 0 ok. So, what will happen? So, if n tends to 0 then you can write v z is equal to limit n tends to 0 so, velocity you know V l n r by n R divided by l n 1 by n ok. So, this you can write limit n tends to 0 V so, this term you can write as l n 1 by n plus l n r by R divided by l n 1 by n.

So, you can see you can write limit n tends to 0, V you can take it outside because it is constant. So, it will be 1 plus l n r by R l n 1 by n ok. So, if n tends to 0 so you can see this will become l n infinity right. And l n infinity is infinity so, 1 by infinity will become 0. So, you can write this equal to V ok, because as n tends to 0 this second term will become 0 ok. So, limit n tends to 0 l n r by R divided by l n 1 by n this tends to 0 ok.

So, what is this flow, what is this flow? You can see that the fluid inside will have the same velocity which is having the magnitude V so; that means, it is a plug flow right. So, the annular flow degenerates to flow in a tube ok. So, this is a solid body translation you can see. Now, whatever the volume flow rate so, volume flow rate we have already calculated as Q is equal to pi V R square by 2 2 plus n square minus 1 divided by l n 1 by n ok.

So, you can see so Q when limit n tends to 0 n tends to 0 2 plus n square minus 1 by l n 1 by n. So, l n 1 by n will become infinity so, this second term will become 0. So, Q will be just pi V r square by 2 into 2. So, what is that that will be V into pi R square and pi R square is nothing, but the flow area right flow area is pi R square.

So, in this case for the plug flow, you can see that the flow area is; obviously, pi R square and uniform velocity you have inside that is V. So, volume flow rate will be just V into A; that means, V into pi R square and that we have already shown. Now, let us consider the other limiting case where n tends to 1 so; that means, the inner cylinder radius will come closer to the outer cylinder radius ok.

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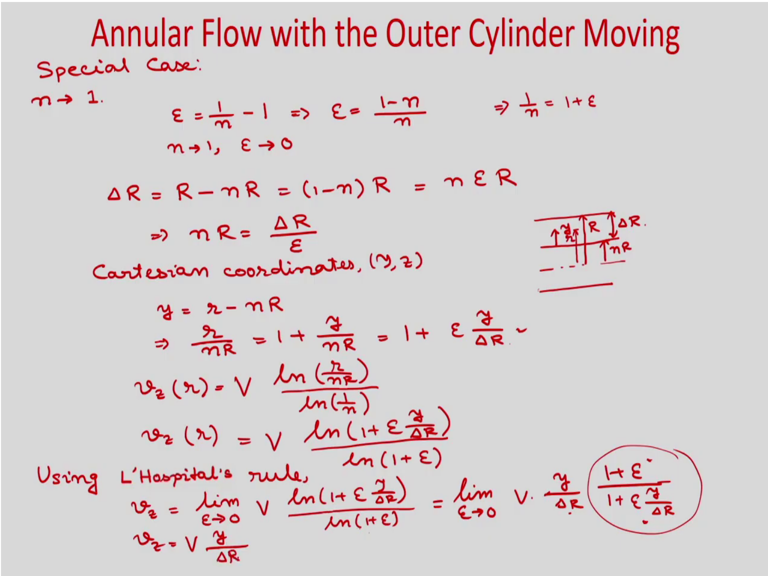

So, another special case where n tends to 1 ok. So, now, we will show that these flow actually will become the plane Couette flow ok, which we have already done in Cartesian coordinate ok. So, you can see that let us define epsilon as 1 by n minus 1. And you can see that n is less than 1 and; obviously, if n tends to 1, then as n tends to 1 epsilon will be tends to 0 ok.

So, now, let us define the delta R the difference between 2 radius is outer radius minus inner radius ok. So, it will be 1 minus n into R and from here, you can see epsilon is 1 minus n divided by n. So, 1 minus n you can write as n into epsilon into R. So, from here you can see n R we can write as delta R by epsilon ok. Now, you introduce the Cartesian coordinates ok.

So, with the origin on the surface of the inner cylinder so, y we are measuring. So, y we are measuring from here and this is R and this is n R and this is delta R ok. So, we can write so, n at any radial location R we can write y is equal to r minus n R right r minus inner radius. So, you can write r by n R is equal to 1 plus y by n R and y by n R is you can write so, n R is equal to delta R by epsilon. So, you can write 1 plus epsilon y by delta R ok.

So, now you substitute this r by n R this expression in the velocity distribution. So, what is your velocity distribution? So, v z we have the V into l n r by n R divided by l n 1 by n. So, if you put in this expression then v z you can write V into l n. So, r by n R now we are substituting 1 plus epsilon y by delta R divided by l n. Now, 1 by n so, 1 by n you can see from here 1 by n we can write 1 plus epsilon so, 1 by n now you want to write 1 plus epsilon.

So, you can see now let us find when epsilon tends to 0 ok. So, a plus epsilon tends to 0, then we will use using L' Hospital's rule v z we can write as limit epsilon tends to 0 V l n 1 plus epsilon y plus delta R divided by l n 1 plus epsilon so, these if you write using L' Hospital's rule. So, you have to take the derivative of this denominator and numerator with respect to epsilon.

So, you will get so, limit epsilon tends to 0 V y by delta R 1 plus epsilon divided by 1 plus epsilon y by delta R ok. Because, you can see so this will become 1 by 1 plus epsilon y by delta R. So, this you are getting and if you take this derivative, then y by delta R and in the denominator 1 by 1 plus epsilon and it will come in the numerator.

So, now if epsilon tends to 0 so, it will become this term will become 1 ok. So, you will get v z is equal to V into y by delta R. So, now, you can see that it is just the expression of velocity for plane Couette flow, where delta R is the distance between 2 flat plates ok. So, that is we considered earlier case as H and it is varying linearly with y ok

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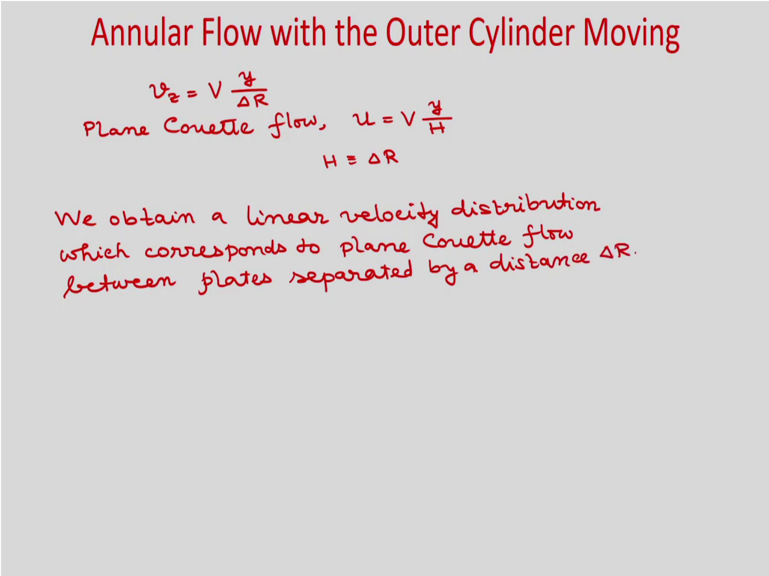

So, v z is equal to V into y by delta R. So, for plane Couette flow we calculated u as V into y by H if you remember ok. So, H is the distance between 2 plates. So, H is equivalent to delta R ok. So, we are getting the linear velocity profile ok. So, we obtain a linear velocity distribution which corresponds to plane Couette flow between plates separated by a distance delta R. So, in today's class first we considered the liquid flowing outside a cylinder surface in the downward direction.

So, assuming surface tension 0 we considered uniform film thickness delta. So, for this particular case we can see that as outer ambient air is stationary. And they are; obviously, you have the ambient pressure as atmospheric pressure so; obviously, there will be no pressure gradient in the z direction.

So, del p by del z is equal to 0 and we have only gravity driven flow and for that we express the velocity distribution in terms of R. Then, we calculated the volume flow rate and we considered a special case where delta by R is very very small; that means, your thin you have a film thickness which is very thin. So, in this particular case this actually degenerates to a thin film flow in a Cartesian coordinate.

Next we considered flow inside 2 coaxial cylinders where outer cylinder is moving with a constant velocity V and inner cylinder is stationary. So, in this particular case we calculated the velocity distribution, as well as the volume flow rate and two special cases we considered 1 is n tends to 0; that means, inner cylinder is removed. So; obviously, this degenerates to a plug flow where you have a uniform velocity V and we have also shown that the volume flow rate will become just V into the area which is pi R square.

Then the special case we considered where n tends to 1; that means, this inner cylinder is coming closer to the outer cylinder. So, we defined the delta R which is the thickness between these two cylinders that is as outer radius minus the inner radius. And, we defined one small thickness epsilon which is much much smaller than 1 ok.

So, in this particular case now we have seen that when epsilon tends to 0 this flow actually degenerates to plane Couette flow in Cartesian coordinate and the velocity profile becomes linear.

Thank you.