Viscous Fluid Flow

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Module - 03
Steady Axisymmetric Flows
Lecture - 01
Hagen - Poiseuille Flow

Hello everyone. So, today we will consider Navier-Stoke equations in cylindrical coordinate and we will try to find few exact solutions in pipe flow. So, today, we will solve two different problems, one is Hagen-Poiseuille Flow that means, fully developed pipe flow and next, we will consider the flow through annulus.

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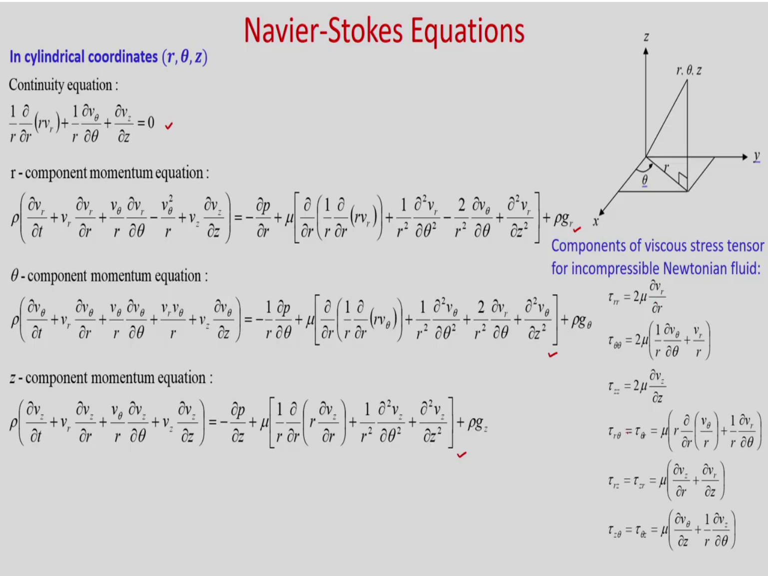

If you consider cylindrical coordinate, then you can write the continuity equation as this then r component of momentum equation, then theta component of momentum equation and this is the z component of momentum equation and components of viscous stress tensor for incompressible Newtonian fluid, you can write like this so, these are normal stresses first three and next three, you can see these are shear stresses ok.

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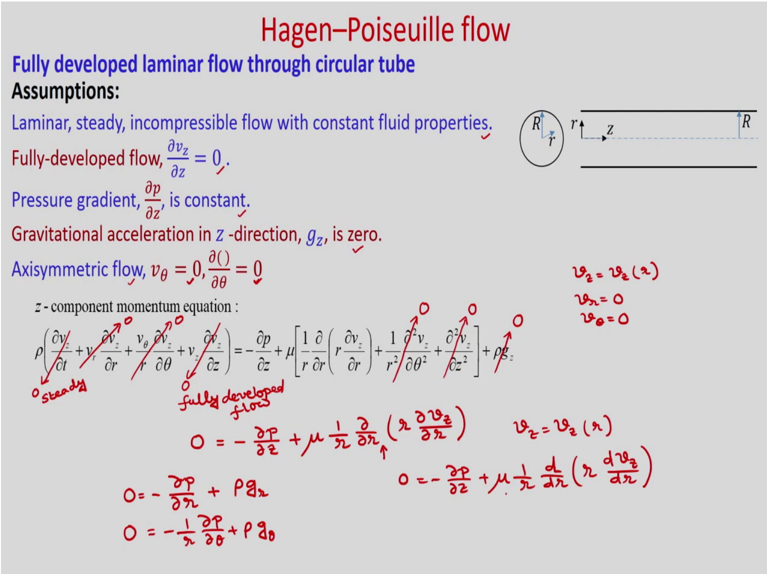

So, first, let us consider Hagen-Poiseuille flow. This is the fully developed flow inside pipe. The following assumptions we will take the laminar, steady, incompressible flow with constant fluid properties. It is a fully developed flow so, the velocity gradient in the axial direction is 0 del v z by del z is equal to 0. Pressure gradient, del p by del z is constant. Gravitational acceleration in z-direction is zero.

And more importantly, we will consider axisymmetric flow. What is axisymmetric flow? That means, in circumferential direction, the velocity is 0 and gradient in the direction of any quantity is 0. So, you can see here, v theta is equal to 0; del of del theta of any quantity is 0. So, this is known as axisymmetric flow.

So, first let us take the z component of momentum equation. So, you can see this is the z component of momentum equation. Now, invoke the assumptions ok. So, first is steady flow so obviously, this is 0 because it is a steady flow. From the continuity equation, you can see that we have v z which is function of r only and v r is equal to 0 and anyway it is axisymmetric flow so, v theta is equal to 0.

So, you can see the convection term so, you can see v r is 0 so obviously, this term is 0 as v r 0, then v theta is 0 so, this is 0 and it is a fully developed flow so, this is 0 because it is a fully developed flow so, del v z by del z is 0 and a pressure gradient is constant.

And in the right-hand side, if you see the viscous term so obviously, the second term you can see this is 0 because it is axisymmetric flow and as it is fully developed flow, del v z by del z is equal to 0 everywhere so, del 2 v z by del z square will be also 0 as it is fully developed flow and we have considered g z as 0 so, this is 0.

So, now you can see this z component of momentum equation you can write as left-hand side, all terms are 0, in the right hand side, you have constant pressure gradient del p by del z plus mu and the first viscous term 1 by r del of del r r del v z by del r.

Now, if you considered r component and theta component of momentum equations and invoke this assumptions, you will finally, get del p left-hand side it will be 0 so, it will be minus del p by del r plus rho g r so, this you will get and from the theta momentum equation, you will get 1 by r del p by del theta plus rho g theta ok. So, you can see that if the components of this gravitational acceleration g and g theta is 0, then obviously, you will get del p by del r is equal to 0 and del p by del theta is equal to 0.

So, now, you can see that v z is function of r only ok. So, these partial differential we can write as a ordinary differential ok. So, we can write 0 is equal to minus del p by del z plus mu 1 by r d of dr r d v z by dr ok. In this equation, you can see that minus del p by del z, it is a favorable pressure gradient and as this is constant so, and this is ordinary differential equation so, you will be able to integrate with proper boundary conditions.

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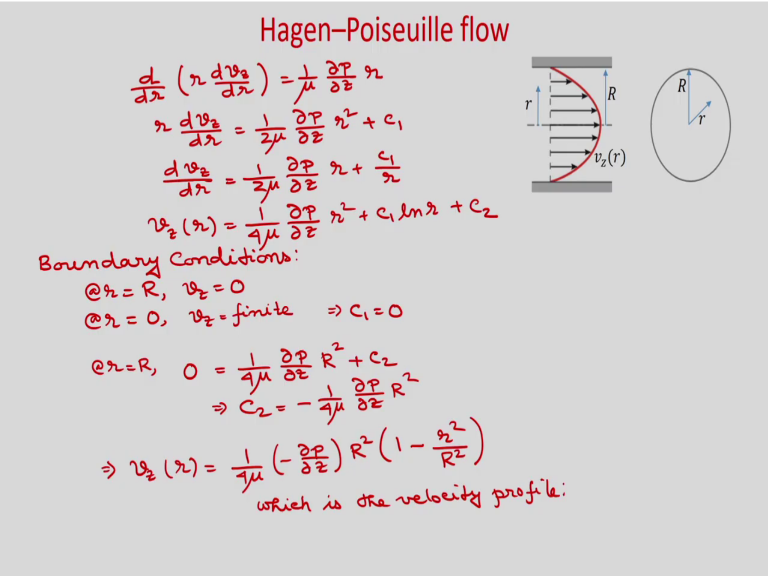

So, what are the boundary conditions? You can see that at r is equal to Rr, if capital R is the radius of the pipe, then obviously, there is a no slip condition and v z will be 0 and you can see as it is axisymmetric flow and due to symmetry.

Because it is a circular pipe, then obviously, you will get the maximum velocity at the center line and that means, that r is equal to 0. And hence, you can see that your v z is finite and the del v z by r will be 0 because the you are getting maximum velocity at the center line.

So, now, let us consider this ordinary differential equation and invoke the boundary conditions and find the velocity distribution inside fully developed pipe flow. So, our governing equation is now d of dr r d v z by dr is equal to 1 by mu del p by del z into r. So, if you integrate this equation, you will get r d v z by dr is equal to 1 by twice mu del p by del z r square plus c 1.

So, if you divide both side by r, you are going to get d v z by dr is equal to 1 by twice mu del p by del z r plus c 1 by r and again, if you integrate, then you will get the velocity profile v z which is function of r as 1 by 4 mu del p by del z r square plus c 1 lnr plus c 2. So, c 1 and c 2 are integration constants that you can find invoking two boundary conditions.

So, let us write the boundary conditions. So, at r is equal to R capital R, v z is equal to 0 and at r is equal to 0, v z is finite right because at the center line you have finite velocity. So, now, if you put it here, you can see that at r is equal to 0 so, if v z is finite, then obviously, c 1 must be 0 because c 1 ln 0 so, to have the left-hand side v z finite, c 1 must be 0, c 1 is equal to 0 and at r is equal to R, v z is equal to 0.

So, if you put that at r is equal to R, v z is equal to 0 so, left-hand side it is 0 1 by 4 mu del p by del z and R is the radius of the pipe so, it is capital R square plus c 2 and c 1 is 0 so, c 2 you will get minus 1 by 4 mu del p by del z R square.

And now, if you invoke these values of constant c 1 and c 2, then you can write v z which is function of r as c 1 is 0. So, if you put c 2 after rearranging, you can write 1 by 4 mu minus del p by del z will take R square outside, then if you can write 1 minus r square by R square ok. So, velocity profile is parabolic. So, you can see that this is the velocity profile which is the velocity profile ok.

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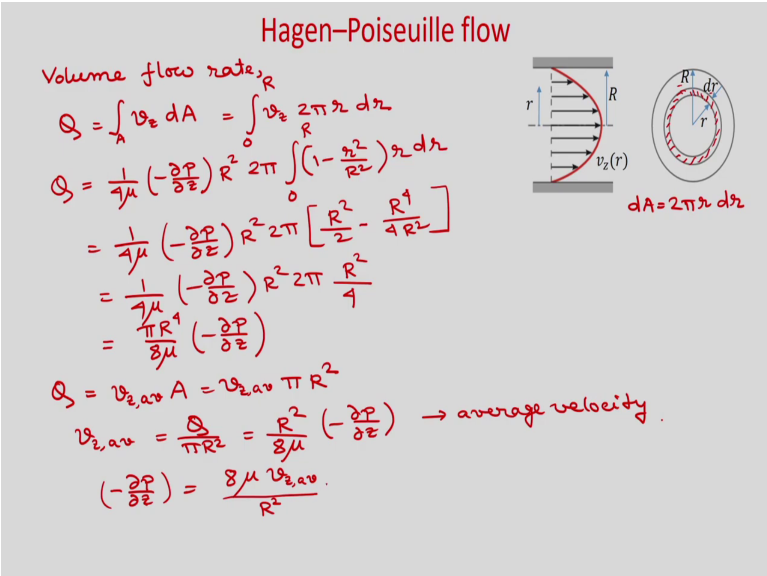

Now, let us find what is the volume flow rate inside this pipe ok. So, let us consider volume flow rate ok. So, Q is area integral v z dA ok. So, what is dA in this case? So, if you see that this is a circular pipe so, of radius capital R. So, if you consider one elemental strip at radius r of thickness dr, then this is the flow area, elemental flow area ok; this is the elemental flow area.

So, these v z into d, this you have to integrate over this area so obviously, you can see dA will be twice pi r into dr. So, if you put it here so, you will get integral. Now, we will write the limits from r is equal to 0 to capital R v z twice pi r dr ok. Now, let us put the velocity profile v z here and you can write Q 1 by 4 mu minus del p by del z into r square that we can take outside because these are constants 2 pi integral 0 to R 1 minus r square by R square into r dr ok.

So, you perform the integration 1 by 4 mu minus del p by del z R square twice pi so, we can see this will be R; R square by 2 so, capital R square by 2 it will be R cube. So, R to the power 4 by 4 so, it will be R to the power 4 by 4 R square after putting the limits. So, this you can write as 1 by 4 mu minus del p by del z r square twice pi and now, this will be R square by 4. So, we can see the Q will be 1 by 8 mu R to the power 4 minus del p by del z into pi.

Now, if we want to find the average velocity inside the pipe, then obviously, we can write average velocity as volume flow rate divided by the area. So, we know Q is equal to v z average into flow area which is your v z average into so, it will be pi R square right. So, now, v z average you can write as Q divided by pi R square. So, from here, you can see that it will be R square by 8 mu minus del p by del z. So, this is the average velocity.

So, now, you can see we can write the pressure gradient del p; minus del p by del z which is your favorable pressure gradient in terms of the average velocity 8 mu v z average by R square ok. So, you can see v z average is obviously, positive value, mu and R positive so, right-hand side is positive ok. So, minus del p by del z, if it is greater than 0, then obviously, you will get the favorable pressure gradient and in the positive z direction, flow will occur.

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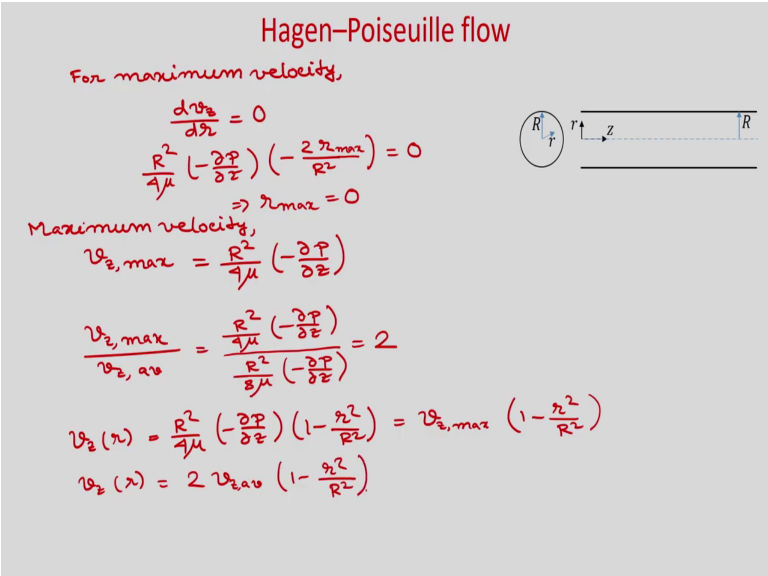

Where the maximum velocity will occur? So, obviously, to find the maximum velocity, you can write d v z by dr is equal to 0. So, for maximum velocity, the location where you can find that d v z by dr is equal to 0 ok.

So, now, you can see you will get R square by 4 mu minus del p by del z you will get minus 2 r by R square so, r max you can write where the maximum velocity will occur is equal to 0. So, from here, you can see that r max is equal to 0, so that means, at the center line, you will get the maximum velocity. So obviously, if you put r is equal to 0 in the velocity distribution, you will get the maximum velocity.

So, v z, max now, so, this is your maximum velocity v z, max will be just if you put R is equal to 0 in the velocity distribution expression, we are going to get R square by 4 mu minus del p by del z ok.

So, now let us see what is the ratio of this maximum velocity to average velocity ok. So, v z, max divided by v z, average so, you can see v z, max is R square by 4 mu minus del p by del z and v z, average already we have found that is R square by 8 mu minus del p by del z.

So, you can see this will be 2 that means, v z, maximum is 2 times the average velocity. So, if you remember for plane Poiseuille flow, we have found that maximum velocity is 1.5 times the average velocity. In case of fully developed pipe flow obviously, we are going to get maximum velocity is 2 times the average velocity ok.

So, now, if we write the velocity distribution in terms of average velocity or the maximum velocity, then we will get the expression as v z r is equal to R square by 4 mu minus del p by del z 1 minus r square by R square so, you can see this is your maximum velocity so, you can write v z, max 1 minus r square by R square and if you write in terms of the average velocity and obviously, we know that v z, max is equal to 2 times average velocity so, 2 v z, average 1 minus r square by R square.

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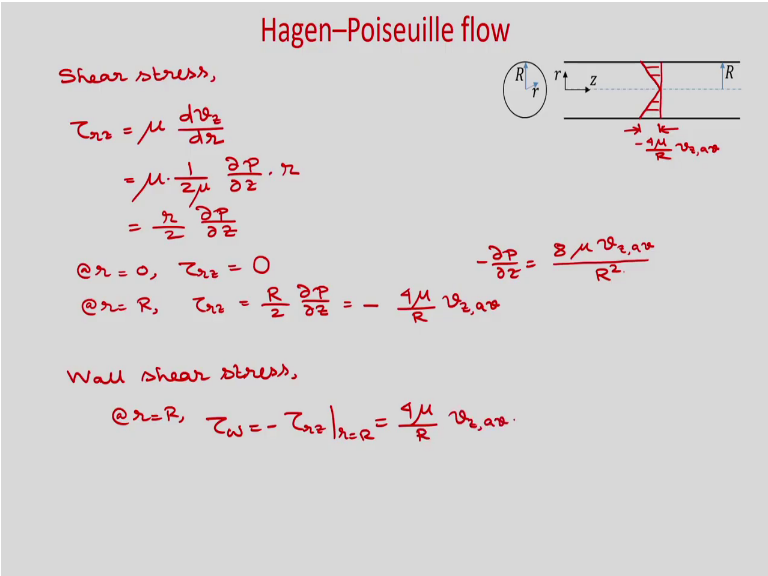

So, now let us find the shear stress distribution inside the flow domain so obviously, we will get only one component of the shear stress that is tau rz. So, shear stress if you find in this case, tau rz will be just mu d v z by dr ok. So, if you find this, you will get mu into 1 by twice mu del p by del z into r ok. So, you can see it will be just r by 2 del p by del z ok. So, you can see that shear stress varies linearly with the radial distance from the axis.

So, at r is equal to 0, obviously, tau rz is equal to 0 because you will; you are getting maximum velocity there and at r is equal to R, tau r z will be R by 2 del p by del z and if you see del p by del z so, del p by del z we have written in terms of the average velocity. So, we can see we have written minus del p by del z is 8 mu v z, average by R square. So, minus del p by del z is equal to 8 mu v z, average by R square ok. So, if you write tau rz at r is equal to R, we are going to get minus 4 mu by R v z, average.

So, now, let us plot the shear stress inside the flow domain. So, you can see that inside the flow domain, it will vary linearly from the center line to the wall of the pipe. So, we can see that this will vary linearly like this ok. So, this is shear stress profile and this is obviously, you can see inside the flow domain, it will be negative so, it will minus 4 mu by R v z, average.

So, what will be the wall shear stress? So, wall shear stress obviously, it will be negative of this tau rz. So, at r is equal to R, you are going to get the wall shear stress. So, wall shear stress you are going to get as at r is equal to R so, tau wall is equal to minus tau rz at r is equal to R. So, this will be 4 mu by R v z, average.

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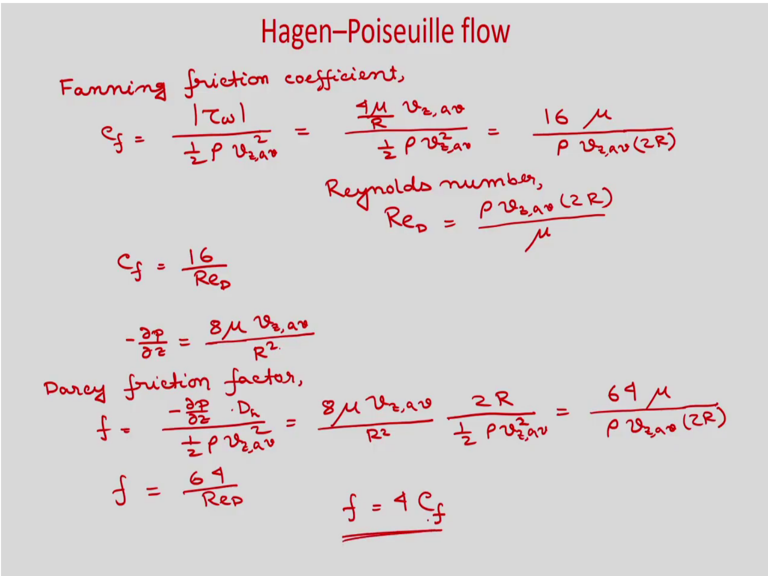

Now, let us write this shear stress in non-dimensional form and that is known as fanning friction factor. So, fanning friction factor, if you find, fanning friction coefficient so, this will be just non-dimensional representation of the shear stress divided by half rho v z, average square ok. So, tau w already we have found so, this is 4 mu by R v z, average divided by half rho v z, average square ok.

So, now if you rearrange, you are going to get as 16 mu by rho v z, average into 2R ok. So, now, we will define the Reynolds number ah; Reynolds number as Re based on the diameter as rho v z, average twice R by mu. So, you can write the fanning friction coefficient c f as 16 by Re D. Now, let us write the Darcy friction factor which is the representation of non-dimensional pressure gradient. So, we know the pressure gradient minus del p by del z we have written as 8 mu v z, average divided by R square.

So, now if we want to write the Darcy friction factor so, f is just minus del p by del z into hydraulic diameter divided by half rho v z, average square ok. So, from here, you can see it will be 8 mu v z, average divided by R square D h, hydraulic diameter in this case it is 2R divided by half into rho v z, average square ok.

So, now, if you rearrange so, you can write as 64 mu divided by rho v z, average into 2R ok. So, again we can write Darcy friction factor as 64 by Re D and from this two expressions, c f is equal to 16 by Re D and f is equal to 64 by Re D so, you can write Darcy friction factor is 4 times fanning friction coefficient ok.

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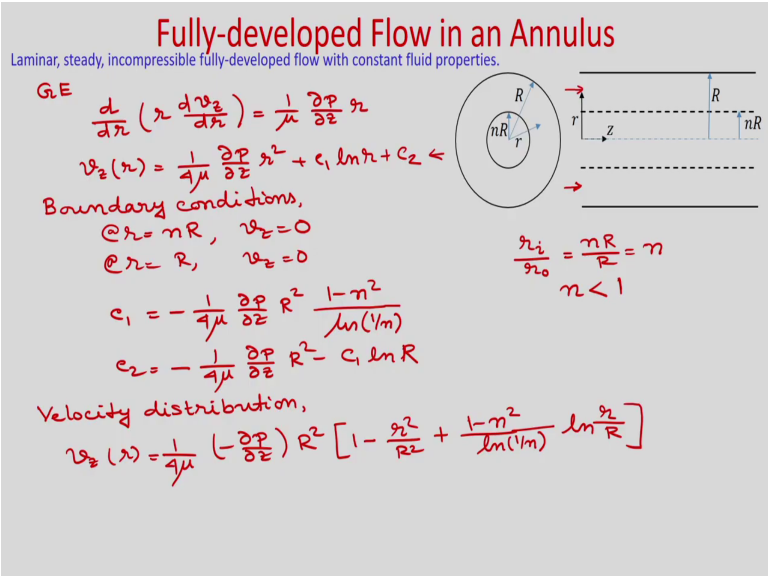

Now, let us consider fully developed flow inside annulus. So, we will consider the assumptions, it is a laminar, steady, incompressible, fully developed flow with constant fluid properties.

So, you can see, this is the inner pipe so that is your inner and this is the outer radius capital R and flow is taking place inside this annulus; inside this annulus. So, this is the axial direction z and this the radial direction r and this is the center line ok. So, flow is taking place inside the annulus.

So, we are considering fully developed and axisymmetric flow so, obviously, we can write the differential equation which we have already written for as an Poiseuille flow only in this case, boundary conditions are different. So, we will write the governing equation d by dr r d v z by dr is equal to 1 by mu del p by del z into r. So, that we have already derived in the beginning in today's class. So, now, if you integrate twice so, we will get v z which is function of r as 1 by 4 mu del p by del z r square plus c 1 lnr plus c 2.

So, in this case, now boundary conditions at r is equal to nR, the velocity is 0 as well as at r is equal to capital R so that is there also velocity is 0 so, obviously, boundary conditions you can write at r is equal to nR so, it is v z is equal to 0 and r is equal to capital R, v z is equal to 0. So, you can see the ratio of these radius is n. So, if inner radius divided by outer radius so, obviously, it is nR by R is equal to n and obviously, you can see that inner radius always it will be smaller than r o so, n is less than 1 ok; n is less than 1.

So, now, if you put this boundary condition and find the constants, then you will get c 1 as minus 1 by 4 mu del p by del z R square 1 minus n square divided by ln 1 by n and c 2 will be minus 1 by 4 mu del p by del z R square minus c 1 ln R.

So, if we put these values in this expression, then we are going to get the velocity distribution as v z which is function of r as 1 by 4 mu minus del p by del z into R square 1 minus r square by R square plus 1 minus n square divided by ln 1 by n into ln r by R ok. So, this is the velocity distribution inside annulus.

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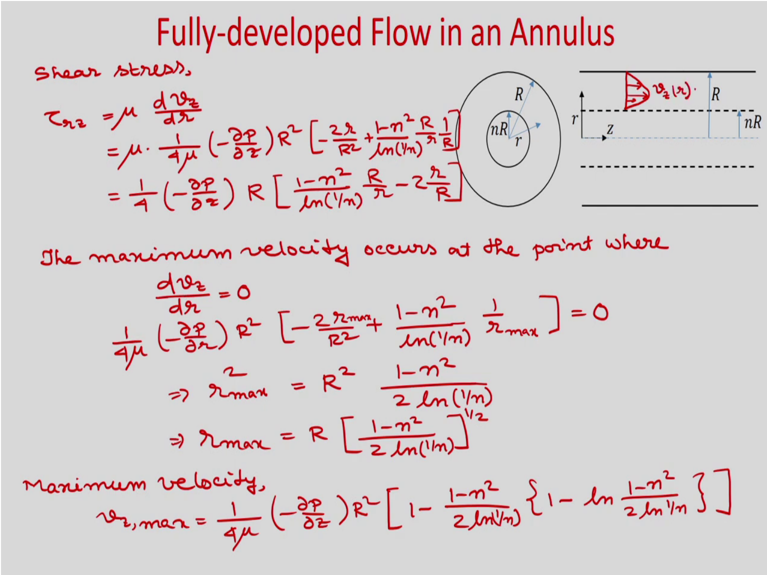

Now, let us find the shear stress first, then we will find where the maximum velocity will occur ok. So, if you find the shear stress so, tau rz will be just mu d v z by dr and if you find this so, you will get mu into 1 by 4 mu minus del p by del z R square into minus 2 r by R square plus 1 minus n square divided by ln 1 by n R by r into 1 by R ok.

So, if you rearrange, you are going to get 1 by 4 minus del p by del z R into 1 minus n square divided by ln 1 by n R by r minus 2 r by R ah. The location in which the maximum velocity occurs so that you can find just putting d v z by dr is equal to 0. So, the maximum velocity occurs at the point where d v z by dr is equal to 0. So, from here, you can see d v z by dr is equal to 0 if you put so, it will be 1 by 4 mu minus del p by del z R square so, minus 2 r by R square plus 1 by minus n square ln 1 by n into 1 by r so, let us write r max and r max here so, this would be 0.

So, you can write this r max square is equal to R square 1 minus n square divided by 2 ln 1 by n. Then, you can see that r max will be at this location R 1 minus n squared divided by 2 ln 1 by n to the power half. So, maximum velocity you will get v z, max is equal to 1 by 4 mu minus del p by del z R square. So, if you put this r max in the velocity distribution, you will get 1 minus 1 minus n square divided by 2 ln 1 by n 1 minus ln 1 minus n square divided by 2 ln 1 by n ok. So, after putting the value of r max, you will get this as maximum velocity ok.

So, if we want to draw the velocity profile so, you can see, this will look this. So, maximum velocity will not occur at the mid position here so, it will be somewhere like this so, this is the maximum velocity. So, this is v z which is function of r.

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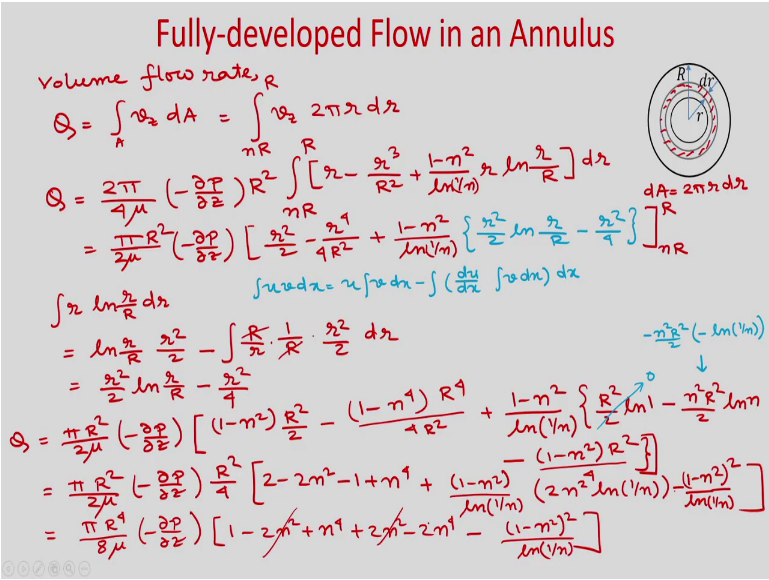

Now, let us find the volume flow rate and then, we will calculate the average velocity inside the annulus to find the volume flow rate so, we will use Q is equal to integral area integral v z dA. So, in this case, elemental area is same, elemental area we will consider as dA is equal to twice pi r dr. So, you can see in this figure obviously, the elemental area dA so, this is the elemental flow area so, dA will be twice pi r into dr ok. So, volume flow rate we can write Q is equal to area integral v z dA and if you write in terms of dr so, you can write twice pi r dr and we have to integrate from inner radius nR to outer radius capital R ok.

So, now you put the expression of v z here and which are constant that you can take outside the integral and perform the integration. Then, you can find Q is equal to if you put the value of v z and all the constant terms if you can take outside, then it will be twice pi divided by 4 mu minus del p by del z R square integral nR to R and if you multiply this R in the velocity distribution, then you will get r minus r cube by R square plus 1 minus n square divided by ln 1 by n r ln r by R dr.

So, you can write it as pi by twice mu R square minus del p by del z so, it will be r square by 2 minus r to the power 4 by 4 R square and now, it will be 1 minus n square divided by ln 1 by n and this so, this integration now, integration by parts you can use ok. So, if we use integral r ln r by R dr.

So, you can use this integration by parts you know that it integral u v dx is equal to u integral v dx minus integral du by dx integral v dx dx ok. So, if we use it, then you can find this integration so, it will be so, ln r by R is u so, it will be r square by 2 minus integral so, du by dx so, that you will get R by r into 1 by R and integral r dx so, it will be r square by 2 dr.

So, you can write this r square by 2 ln r by R minus so, you can see these R, R will get cancel, you will get r by 2 and integral r dr is r square by 2 so, it will be r square by 4 ok. So, now, if you put these value here so, you will get r squared by 2 ln r by R minus r square by 4 ok. So, now, you put the limit from nR to R ok.

So, now, you can write Q is equal to pi R square by twice mu minus del p by del z. So, now, if you put this limit so, you will get 1 minus n square R square by 2 minus 1 minus n to the power 4 R to the power 4 by 4 R square and here, you will get 1 minus n square divided by ln

1 by n. So, if you put here so, it will be R square by 2 ln 1 so, this will be 0 minus n square R square by 2 ln n right and again, you will get minus 1 minus n square R square by 4.

So, let us take outside, you can see you will get here R square and this anyway it will be 0 and here, R square and here R square is there so, you can take R square by 4 outside ok. So, R square by 4 outside minus del p by del z R square by 4. So, we can see here you will get 2 minus 2 n square, here you will get minus 1 plus n to the power 4 so, you can see this term we can write as minus n square R square by 2 minus ln 1 by n ok.

So, you can see if you multiply with these, then ln 1 by n and this ln 1 by n will get cancelled and obviously, R square by 4 we have taken outside so, you can write minus 2 n square. So, it will be minus minus plus 2 n square. So, you can write 1 minus n square divided by ln 1 by n so, it will be 2 n square ln 1 by n ok. So, this is one term and another you will get minus 1 minus n square ln 1 by n and this will become 1 minus n square whole square ok.

So, now, you can rearrange it. So, you will get pi by R to the power 4 by 8 mu minus del p by del z. So, you can see 2 minus 1 so, you can write 1 minus 2 n square plus n to the power 4 and if you multiply this so, this will get cancel so, you will get plus 2 n square minus 2 n to the power 4 minus 1 minus n square whole square by ln 1 by n. So, you can see these 2 n square 2 n square will get cancelled and you will get 1 minus n to the power 4.

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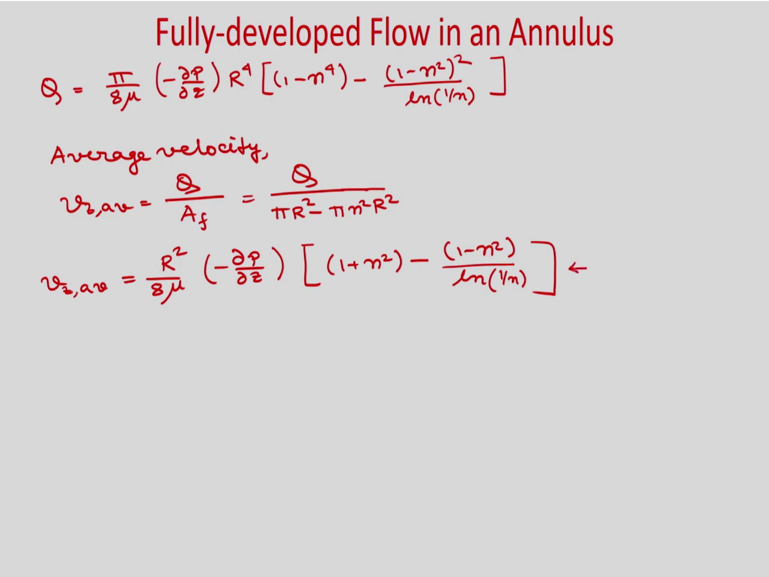

So, finally, you can write the volume flow rate Q as pi by 8 mu minus del p by del z R to the power 4 1 minus n to the power 4 minus 1 minus n square divided by ln 1 by n. So, now if you want to find the average velocity so, we can write the average velocity as the volume flow rate divided by the annulus flow area and what is the annulus flow area? So, you can write average velocity v z, average is equal to Q divided by flow area and in this case, flow area is pi R square, this is the outer area and inner area is pi n square R square ok.

So, if you divide, you will get the average velocity v z, average as 1 by 8 mu R square minus del p by del z 1 plus n square minus 1 minus n square divided by ln 1 by n ok. So, this is the average velocity expression.

So, in today's class, we considered the Navier-Stoke equations in cylindrical coordinate and invoking the proper boundary conditions, we could write the ordinary differential equation from the partial differential equation. So, first we consider the Hagen-Poiseuille flow which is fully developed pipe flow. In this case, we considered that it is a axisymmetric flow. So, that in circumferential direction velocity is 0 and any gradient is 0 in that direction.

We considered fully developed flow that means, del v z by del z is equal to 0 where z is the axial direction and from here, just invoking the boundary conditions and solving the ordinary differential equation, we express the velocity distribution which is your parabolic in nature. Then, we calculated the shear stress distribution inside the fluid domain which linearly varies from the center line to the wall.

Then, we wrote the non-dimensional expression for the shear stress as well as non-dimensional expression for the pressure gradient and we wrote the fanning friction coefficient and Darcy friction factor.

Next, we considered the flow inside the annulus. So, in this case, we found the velocity distribution, then we calculated the shear stress distribution and found the location where the maximum velocity will occur. Next, we calculated the volume flow rate and from there, we calculated the average velocity.

Thank you.