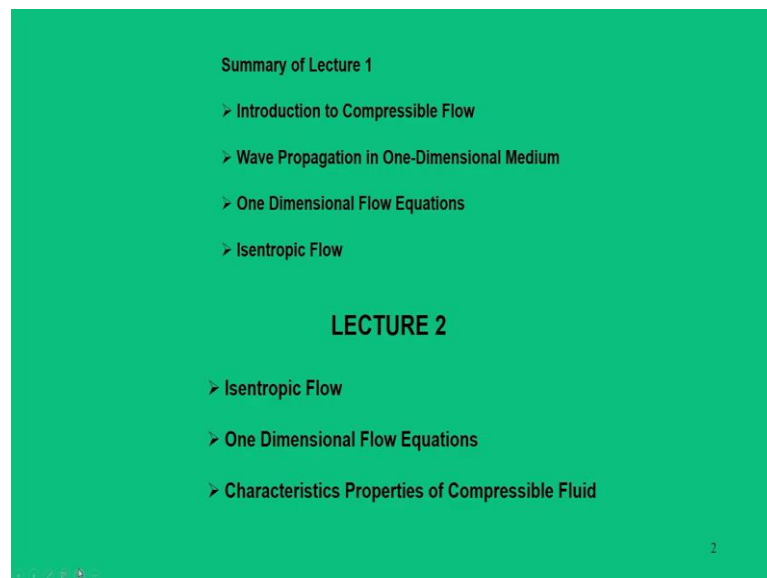


**Fundamentals of Compressible Flow**  
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**Module – 02**  
**Wave Propagation in Compressible Medium**  
**Lecture - 05**  
**Wave Propagation in Compressible Medium - II**

Welcome you again for this course; we are in the module 02 and this is the 2nd lecture in the module 02. The topic of this module 02 is Wave Propagation in Compressible Medium.

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Now, just to give the brief introduction what we have seen in the 1st lecture of this module that is introduction to compressible flow. We will say that we bring into parameter that is compressibility into account.

Then you discussed how wave propagates in a compressible medium. And when the smallest wave is generated in the medium, we call this as a sound waves. Now, for the sound waves we derived this one dimensional flow equations, then we talk something about isentropic flow.

Now, moving further we will continue it from here again and here also we will continue with isentropic flows. Also we will talk about this one dimensional flow equation, but in

a different context. Apart from this the last component of this lecture is the characteristics properties of a compressible fluid.

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**Isentropic Flow**

**Isentropic Flow Equations**

- It may be emphasized that the stagnation parameters can be calculated from actual flow conditions at a given point in the flow field.
- But, the actual flow does not have to be adiabatic/isentropic from one point to other.
- In other words, the choice of 'isentropic process' is only a definition to define 'stagnation/total conditions' at a point.

Now, when I said that in a isentropic flow field, we say that a system undergoes a change of state from 1 to 2 and when the process is isentropic. So, the essentially basic feature is that when such a change takes place 1 and 2 we can represent that change to happen in a small volume we call this as a control volume as shown is this figure.

So, the flow fields before this is condition 1, flow field after this control volume is condition 2 and this process undergoes in isentropic manner. But in other words in the last lectures we also define the concept of stagnation properties. That is again a situation where we hypothetically bring the state isentropically also to the condition which is called as stagnation.

So, when a state 1 becomes a stagnation state we define this properties to be  $p_{01}$  that is pressure,  $T_{01}$  temperatures,  $\rho_{01}$  that is density,  $e_{01}$  is total energy. And now when you bring this flow to stagnation situation we say that you are bringing this  $u_1$  to be 0. Now, similarly since condition 2 is also another arbitrary situation we can also bring this flow conditions to be also stagnation.

So, that situation also we will have  $u_2$  is also equal to 0,  $p_2$  will become  $p_{02}$ , temperature will be  $T_{02}$ , density will be  $\rho_{02}$  and then total energy will be  $e_{02}$ .

Now, these 2 conditions also needs to be same. So, what it means that any arbitrary flow fields can be brought to a stagnation states. And essentially stagnation states are nothing, but the another definition of a given flow field and we also assign them with certain properties.

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**Isentropic Flow**

Isentropic Relations

$$\frac{p_0}{p} = \left(\frac{T_0}{T}\right)^{\frac{\gamma}{\gamma-1}} = \left(\frac{\rho_0}{\rho}\right)^{\gamma}$$

*2nd Law*

$$c_p \ln\left(\frac{T_2}{T_1}\right) - R \ln\left(\frac{p_2}{p_1}\right) = 0; \quad c_p \ln\left(\frac{T_2}{T_1}\right) + R \ln\left(\frac{\rho_2}{\rho_1}\right) = 0$$

*Isentropic*

*Stagnation*  
 $p_0, T_0, \rho_0$

*Calorically perfect gas*

$$\ln\left(\frac{T_2}{T_1}\right) = \frac{R}{c_p} \ln\left(\frac{p_2}{p_1}\right)$$

$$\left(\frac{u_2}{u_1}\right) = \left(\frac{T_2}{T_1}\right)^{\frac{1}{\gamma-1}}$$

$$\frac{h_2}{h_1} = \left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma-1}}$$

$$\frac{p_0}{p} = \left(\frac{T_0}{T}\right)^{\frac{\gamma}{\gamma-1}}$$

$$\frac{\rho_0}{\rho} = \left(\frac{T_0}{T}\right)^{\frac{1}{\gamma-1}}$$

$$c_p = \frac{\gamma R}{\gamma-1}$$

$$c_v = \frac{R}{\gamma-1}$$

Now, coming back to these things, when we talked about all this case we say that process undergoes a change of state from 1 to 2, that process is isentropic. But what you are trying to say is that when both the conditions become stagnation and they are essentially same we call this as  $p_0, T_0, \rho_0$ . So this  $p_0, T_0, \rho_0$ ; how you are going to evaluate?

So, for this reason second law of thermodynamics talks about a property called entropy and in this case the flow is isentropic so change in entropy from state 1 to 2 is 0. And the entropy equations has been given through these fundamental principles where there are 2 equations essentially, when you know the static properties like pressure, temperature and specific volume one can calculate the entropy change.

So, let us talk about the first equations when we are talking about gas we are talking about this calorically perfect gas. When you say calorically perfect gas you are using two important relation that is  $C_p = \frac{\gamma R}{\gamma-1}$  and  $C_v = \frac{R}{\gamma-1}$ . Now, these two things we are going to put here and we shall be solving these equations.

$$C_p \ln\left(\frac{T_2}{T_1}\right) - R \ln\left(\frac{p_2}{p_1}\right) = 0; \quad C_v \ln\left(\frac{T_2}{T_1}\right) + R \ln\left(\frac{\vartheta_2}{\vartheta_1}\right) = 0$$

When we are solving these equations we can write  $\ln\left(\frac{T_2}{T_1}\right) = \frac{R}{C_p} \ln\left(\frac{p_2}{p_1}\right)$ .

Now, when you put this  $\frac{R}{C_p}$  from this equation then this happens to be in this situation

like we can say  $\frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma-1}}$ . Now, it happens to be the case that; if your condition 2 is

static and condition 1 is stagnation. So, you can write this as  $\frac{p_0}{p} = \left(\frac{T_0}{T}\right)^{\frac{\gamma}{\gamma-1}}$ .

Now, from this similar expression also we can do same way, but here we have to use the

relations for  $C_v = \frac{R}{\gamma-1}$ . Now, from these equations we can rewrite  $\frac{\vartheta_2}{\vartheta_1} = \left(\frac{T_2}{T_1}\right)^{-\frac{1}{\gamma-1}}$ .

Now, here also we will say  $\vartheta = \frac{1}{\rho}$ . So this equation can be written in terms of density.

So, finally, we are going to say the condition 2 is static and condition 1 is stagnation. So,

we say  $\frac{\rho_0}{\rho} = \left(\frac{T_0}{T}\right)^{\frac{1}{\gamma-1}}$ . So, now, we will move further to derive another relations.

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**Isentropic Flow**

Relation Between Static and Stagnation Properties

$$\frac{T_0}{T} = 1 + \left(\frac{\gamma-1}{2}\right) M^2; \frac{p_0}{p} = \left[1 + \left(\frac{\gamma-1}{2}\right) M^2\right]^{\frac{\gamma}{\gamma-1}}; \frac{\rho_0}{\rho} = \left[1 + \left(\frac{\gamma-1}{2}\right) M^2\right]^{\frac{1}{\gamma-1}}$$

Energy eqn:  $h_0 = h + \frac{u^2}{2}$

$$\Rightarrow c_p T_0 - c_p T = \frac{u^2}{2}$$

$$\Rightarrow T_0 = T + \frac{u^2}{2c_p}$$

$c_p = \frac{\gamma R}{\gamma-1}$   $\Rightarrow \frac{T_0}{T} = 1 + \frac{u^2}{2 \left(\frac{\gamma R}{\gamma-1}\right) T}$

$M = \frac{u}{a}$   $\Rightarrow \frac{T_0}{T} = 1 + \frac{\gamma-1}{2} \left(\frac{u^2}{a^2}\right)$

$a = \sqrt{\gamma R T}$   $\Rightarrow \frac{T_0}{T} = 1 + \left(\frac{\gamma-1}{2}\right) M^2$

$\frac{h_0}{p} = \left(\frac{T_0}{T}\right)^{\frac{\gamma}{\gamma-1}}$

$\Rightarrow \frac{h_0}{p} = \left[1 + \left(\frac{\gamma-1}{2}\right) M^2\right]^{\frac{\gamma}{\gamma-1}}$

$\frac{p_0}{p} = \left(\frac{h_0}{p}\right)^{\frac{1}{\gamma}}$

$\frac{p_0}{p} = \left[1 + \left(\frac{\gamma-1}{2}\right) M^2\right]^{\frac{1}{\gamma-1}}$

$\frac{\rho_0}{\rho} = \left(\frac{h_0}{p}\right)^{\frac{1}{\gamma-1}}$

$\frac{\rho_0}{\rho} = \left[1 + \left(\frac{\gamma-1}{2}\right) M^2\right]^{\frac{1}{\gamma-1}}$

Stagnation

So, in the previous relations we talk about pressure, temperature and density. Now, we did not bring into the flow speed into account and we say that compressible flow is also a function of the flow speed. Now, flow speeds are essentially characterized through this non dimensional number that is Mach number.

Now, let us see that if any flow field has pressure, temperature and density given as  $p$ ,  $\rho$  and  $T$  and this flow has certain Mach number  $M$ . Then when it is going or it is moving through in a compressible medium what will happen to the corresponding stagnation properties;  $p_0$ ,  $T_0$  and  $\rho_0$ .

So, this is how we are going to calculate when a flow is isentropically brought to rest and to reach the stagnation state. So, to do that the first task to derive this relations we have to look into this particular expression that is temperature. So, the temperature is fundamentally related to this energy equation.

$$\frac{T_0}{T} = 1 + \left(\frac{\gamma-1}{2}\right) M^2$$

So, you recall these previous lectures where you talked the total enthalpy as

$h_0 = h + \frac{u^2}{2}$  for a one dimensional flow. And these equations can be written as

$$C_p T_0 - C_p T = \frac{u^2}{2}. \text{ So, bring } C_p \text{ to the other side so we can write } T_0 = T + \frac{u^2}{2C_p}$$

Again you recall  $C_p = \frac{\gamma R}{\gamma - 1}$ . When you put this equation here this equation turns out to

be and also we will also talk  $M = \frac{u}{a}$  where  $a$  is your speed of sound written by  $\sqrt{\gamma R T}$ .

So, when you use these equations here then this turns out to be

$$\frac{T_0}{T} = 1 + \frac{u^2}{2 \left( \frac{\gamma R}{\gamma - 1} \right) T}$$

So, this when you put these relations one can arrive at the fundamental lesson between

stagnation temperature and static temperatures as  $\frac{T_0}{T} = 1 + \frac{\gamma - 1}{2} \left( \frac{u^2}{a^2} \right)$  and  $\frac{u^2}{a^2}$  is nothing

but  $M^2$ . So, this means  $\frac{T_0}{T} = 1 + \left( \frac{\gamma - 1}{2} \right) M^2$ .

Now, knowing these relations we can recall the similar relations between pressure and

temperature as  $\frac{p_0}{p} = \left( \frac{T_0}{T} \right)^{\frac{\gamma}{\gamma - 1}}$ . When you substitute these equations we will land up in

this expression  $\frac{p_0}{p} = \left[ 1 + \left( \frac{\gamma - 1}{2} \right) M^2 \right]^{\frac{\gamma}{\gamma - 1}}$ . Another relation of isentropic relation one can

bring out as  $\frac{\rho_0}{\rho} = \left( \frac{p_0}{p} \right)^{\frac{1}{\gamma}}$ .

So, you know relations  $\frac{p_0}{p}$  from these equations and when you substitute we will arrive

at this relation  $\frac{\rho_0}{\rho} = \left[ 1 + \left( \frac{\gamma - 1}{2} \right) M^2 \right]^{\frac{1}{\gamma - 1}}$ . So, this is how we derive the relations between

stagnation and static properties in terms of Mach number which is most important. And in fact, this is also called as isentropic relations.

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### Isentropic Flow

Relation Between Static and Stagnation Properties

- It may be emphasized that the stagnation parameters can be calculated from actual flow conditions at a given point in the flow field.
- The actual flow does not have to be isentropic from one point to other.
- The choice of 'isentropic process' is only a definition to define 'stagnation/total conditions' at a point.

$$\frac{p_0}{p} = \left( \frac{T_0}{T} \right)^{\frac{\gamma}{\gamma-1}} = \left( \frac{\rho_0}{\rho} \right)^{\gamma}$$

$$\frac{T_0}{T} = 1 + \left( \frac{\gamma-1}{2} \right) M^2; \quad \frac{p_0}{p} = \left[ 1 + \left( \frac{\gamma-1}{2} \right) M^2 \right]^{\frac{\gamma}{\gamma-1}}; \quad \frac{\rho_0}{\rho} = \left[ 1 + \left( \frac{\gamma-1}{2} \right) M^2 \right]^{\frac{1}{\gamma-1}}$$

$$a_0 = \sqrt{\gamma R T_0}; \quad a = \sqrt{\gamma R T}$$

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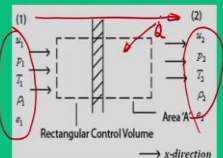
So, whatever I have told it is summarized here that is; what we did is we found the relation between static and stagnation properties for a given flow field.

So, here I need to emphasize that the actual flow field need not have to be isentropic, but we can assign the stagnation properties as a hypothetical state for a given arbitrary flow field. Now, we will move to the one dimensional flow equations in a different context.

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### One Dimensional Flow Equations

- Consider the flow through an arbitrary region of one dimensional flow. The flow properties can change as a function of  $x$  as the gas flows through the region.
- A rectangular control volume can be chosen that has equal area for left-hand and right-hand region of control volume.
- The flow properties of interest are pressure, temperature, density, velocity and total energy. These properties are uniform over left-hand and right-hand region of control volume.
- The fundamental equations under consideration are, continuity equation, momentum equation and energy equation along with perfect gas equation of state.



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So, in earlier situations we are talking about any arbitrary properties of a given flow field with respect to stagnation flow field and this is related through isentropic manner. That means; thermodynamic process has to be isentropic. Now we will see the another situations that when a thermodynamic states move from 1 to 2 through a control volume in an isotropic process, what will be the one dimensional flow equations.

So, essentially the properties which we are talking about the velocity, pressure, temperature, density and total energy and in fact, in the downstream side also we have same properties. So, here what we are assuming that we think about this flow field to happen in one dimensional manner; that is we have a constant area duct and entire flow field is one dimensional. And here in our subsequent analysis our work transfer is not important.

So, we will assume that the flow is changing its states without having any work transfer into the medium, that means; there is no work transfer. But still the control volume can do the energy interaction in terms of Q.

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### One Dimensional Flow Equations

Continuity, Momentum and Energy Equation

- Assuming a steady flow, all time derivatives of flow parameters can be assigned zero. Also, body forces are considered absent.

$-\rho_1 u_1 A + \rho_2 u_2 A = 0$

$\rho_1 u_1 = \rho_2 u_2$  Continuity.

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$\rho_1 (-u_1 A) u_1 + \rho_2 (u_2 A) u_2 = -(-p_1 A + p_2 A)$

$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2$  Momentum.

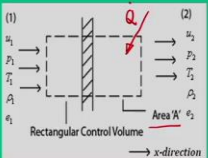
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$\frac{\dot{Q}}{A} - (-p_1 u_1 A + p_2 u_2 A) = -\rho_1 \left( u_{m1} + \frac{u_1^2}{2} \right) u_1 A + \rho_2 \left( u_{m2} + \frac{u_2^2}{2} \right) u_2 A$

$\frac{\dot{Q}}{A} + p_1 u_1 + \rho_1 \left( u_{m1} + \frac{u_1^2}{2} \right) u_1 = p_2 u_2 + \rho_2 \left( u_{m2} + \frac{u_2^2}{2} \right) u_2$

$\rightarrow \frac{\dot{Q}}{\rho_1 u_1 A} + \left( u_{m1} + \frac{p_1}{\rho_1} \right) + \frac{u_1^2}{2} = \left( u_{m2} + \frac{p_2}{\rho_2} \right) + \frac{u_2^2}{2}$

$\frac{h_1}{2} + \frac{u_1^2}{2} + q = \frac{h_2}{2} + \frac{u_2^2}{2}$  ✗



Energy.  $\frac{\dot{Q}}{\rho_1 u_1 A} \rightarrow \text{heat added per unit mass}$

So, with these assumptions we will now move to the one dimensional flow equations. So, here we say that Q is the heat added into the systems. So, obviously we will be talking about three important equations that is what we discussed in the last module; continuity equations, momentum equations and energy equations. And all these equations are addressed through the control volume approach.



So, through integral form of those equations one can write this continuity equations as  $\rho_1 u_1 = \rho_2 u_2$ . And similarly we can also derive this momentum equations that is the second part of these equations. I think this analysis is simple, we have to talk about the nature of the forces here and simplifying the equations. So, the momentum equations says that  $p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2$ . And moving further which is the energy equation and it is mostly important.

So, here we are saying that only the system undergoes the change of state only through heat transfer. So, that is what this heat transport term that is heat added into the system is to taken into account. And in fact, all for all the equations we are saying that area is a constant area so that is A. And another parameter of interest is the total energy.

Total energy consist of in this case here there is internal energy u and the kinetic energy  $\frac{u^2}{2}$ . So, here we represent this internal energy as  $u_{\text{int},1}$  that is for the state 1 and similarly

for state 2 and flow kinetic energy will have  $\frac{u_1^2}{2}$  and  $\frac{u_2^2}{2}$ . And we are neglecting the changes in the potential energy so that term is absent here.

$$\dot{Q} - (-p_1 u_1 A + p_2 u_2 A) = -\rho_1 \left( u_{\text{int},1} + \frac{u_1^2}{2} \right) u_1 A + \rho_2 \left( u_{\text{int},2} + \frac{u_2^2}{2} \right) u_2 A$$

And in fact, entire equations we will have A common. And finally, when you simplify that equations we turn out to reach at this stage where we define this each term of the equation in terms of energy per unit mass.

$$\frac{\dot{Q}}{\rho_1 u_1 A} + \left( u_{\text{int},1} + \frac{p_1}{\rho_1} \right) + \frac{u_1^2}{2} = \left( u_{\text{int},2} + \frac{p_2}{\rho_2} \right) + \frac{u_2^2}{2}$$

So, when you say energy per unit mass the first energy that is  $Q \frac{\dot{Q}}{\rho_1 u_1 A}$  is nothing, but

heat added per unit mass and this is represented in terms of q and internal energy and this flow energy or flow work is happens to be the enthalpy and kinetic energy term remains as it is. So, this is the one of the fundamental equations that is utilized in all subsequent analysis.

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### One Dimensional Flow Equations

Governing Equations – Summary

- The fundamental equations for steady one-dimensional constant area flow are the algebraic equations that relates flow properties at two different locations.
- The momentum equation neglects body forces and viscous stresses while energy equation does not include work done due to viscous stresses, shaft work, heat transfer due to thermal conduction or diffusion and changes in potential energy.

Continuity:  $\rho_1 u_1 = \rho_2 u_2$  ✓

Momentum:  $p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2$  ✓

Energy:  $h_1 + \frac{u_1^2}{2} + q = h_2 + \frac{u_2^2}{2}$  ✓

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So, if the summary of these equations we will have continuity, we will have momentum equations; we will have energy equations. So, these equations are the thermodynamic change of state from 1 to 2.

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### Characteristics Properties of Compressible Fluid

Characteristics Mach number ✓

- Consider a point 'A' in an arbitrary flow field at which the fluid element has a finite value of velocity, static pressure, static temperature and Mach number.
- An imaginary state can be reached by the fluid element when its Mach number becomes unity in an adiabatic manner (either by slowing down or speed up).
- When the fluid element arrives at this imaginary state from its initial state, a new temperature can be specified at which one can define the speed of sound and characteristics Mach number.

$$a^* = \sqrt{\gamma R T^*}$$

$$M^* = \frac{u}{a^*}$$

$$M = \frac{u}{a}$$

$$M^* = \frac{u}{a^*}$$

Stream line.

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Now moving further we will now, talk about another segment that is characteristics properties of the compressible fluids. In all previous situations we discussed about the thermodynamic property, we discussed about fluid mechanics properties, we discussed

about the stagnation properties of a given flow. Here we will now bring into another concept which is called as characteristics properties.

So, what does this mean? So, in all previous cases when we talk about these properties we give the thermodynamic definition for the change of state. Like in the stagnation states we say that fluid was brought isentropically to rest. So, in this case what we are trying to say is that in a given arbitrary state of the fluid; now this flow when it changes its state to a situations where. So, let me redraw it, when a medium consisting of large number of streamlines, we take certain fluid elements which has some velocity  $V$ ; obviously, it has some Mach number  $M$ . So, this is any arbitrarily flow.

Now you define a state which we call as a hypothetical imaginary state. So, you define a state hypothetical or imaginary state in which we will say that fluid is brought to a Mach number  $M$  which is equal to 1. So, in a given flow field when which is arbitrary in nature it has a velocity  $v$  or  $u$  and the Mach number  $M$ ; obviously, it has associated temperature also.

So, in this case when you are bringing it to 1, the thermodynamic definition that we give is that the process has to be adiabatic. Now, when you say Mach number 1 the fluid has to either slowdown or we increase the speed; that means, fluid element has to be slowed down so that Mach number becomes 1 or you increase its speed so that its Mach number becomes 1.

Now, when such a state is defined we say its a  $*$  conditions that is known as this characteristics condition. Now, when you say characteristics conditions when you either you slow down or increase the speed; obviously, temperature is going to change. So, in star conditions we define another new temperature which is called as  $T^*$ . So, when I say  $T^*$  also we can define the speed of sound at  $T^*$  that is  $a^*$ . And that is what we define the speed of sound  $a^* = \sqrt{\gamma R T^*}$ .

So, if you say the Mach number of this fluid is  $M$  which is  $\frac{u}{a}$  in general flow field and when you are defining this  $a^*$  again with reference to this imaginary state. So, we can define a another Mach number which is  $M^*$  which is known as characteristics Mach number and this is defined as the ratio of  $\frac{u}{a^*}$ .

So, this is how the concept of characteristics Mach number is defined. Likewise whichever is related to the \* conditions we can define any situations as a characteristic states. But very basic bottom line is in this case the characteristics state is raised when the fluid is brought to sonic state adiabatically, this is the most important definition to it.

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**Wave Propagation in Compressible Medium**

**Maximum speed** ( $u_{max}$ )

- A gas can attain its maximum speed when it is hypothetically expanded to zero pressure. The static temperature (K) is also zero corresponding to this state.
- Maximum speed of a gas represents the speed corresponding to complete transformation of kinetic energy associated with random motion of the gas molecules into directed kinetic energy.

$$u_{max} = \sqrt{\frac{2\gamma RT_0}{\gamma - 1}}$$

$$u \rightarrow u_{max} = \sqrt{\frac{2\gamma R T_0}{\gamma - 1}}$$

$$h_0 = h + \frac{u^2}{2} \quad c_p = \frac{\gamma R}{\gamma - 1}$$

$$T_0 = T + \frac{u^2}{2 c_p}$$

$$\Rightarrow T_0 = T + \left( \frac{\gamma - 1}{2\gamma R} \right) u^2$$

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Now, when given flow field is defined as its velocity  $u$ , what is the maximum speed it can attain? So, this is we call as a maximum speed that is  $u_{max}$ . So, a gas can attain a maximum state when it is hypothetically expanded to zero pressure. So, when it is at a zero pressure, but the static temperature is also 0 that is represented in terms of Kelvin corresponding to the state.

So, what it physically means that it is a maximum speed of the gas represent the state corresponding to complete transformation of kinetic energy associated with the fluid. That is fluid has its own kinetic energy and it gets transferred completely to the gas molecules. Its associated motion of the gas is into the directed kinetic energy.

So; that means, the complete transformation of the kinetic energy is associated with the random motion of these gas molecules. So, when it is becomes 0 there is effectively no random motion. So, that means; fluid has attained its maximum velocity. So, again we can define this for a given state we can define that; what is the maximum speed it can attain?

So, again we can revisit the energy equation of this form  $h_0 = h + \frac{u^2}{2}$  or  $T_0 = T + \frac{u^2}{2C_p}$ ;

we say  $C_p = \frac{\gamma R}{\gamma - 1}$ . So, this turns out to be  $T_0 = T + \left( \frac{\gamma - 1}{2\gamma R} \right) u^2$ .

Now here we say  $u$  has to be  $u_{\max}$ . So, when you say  $u$  goes to  $u_{\max}$ , this can happen when

$T$  becomes 0; that is absolute temperature. So,  $u_{\max} = \sqrt{\frac{2\gamma R T_0}{\gamma - 1}}$ . So, this is how we get

this maximum velocity for a given arbitrary velocity of  $u$ . Again we will talk about a characteristics speed as I mentioned that characteristic speed is nothing but the speed of the gas when it is at the sonic state.

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**Characteristics Properties of Compressible Fluid**

Characteristics speed

➤ It is the speed of the gas at sonic state.

$$a^* = \sqrt{\frac{2\gamma R T_0}{\gamma + 1}}$$

Energy eqn

$$T_0 = T + \left( \frac{\gamma - 1}{2\gamma R} \right) u^2$$

$$T_0 = T^* + \left( \frac{\gamma - 1}{2\gamma R} \right) u^{*2}$$

Solve for  $u^*$

$$u^{*2} = \left( \frac{2\gamma R T_0}{\gamma - 1} \right) - \left( \frac{2\gamma R T^*}{\gamma - 1} \right)$$

$$u^* = \sqrt{\left( \frac{2\gamma}{\gamma + 1} \right) R T_0}$$

$$u^* = \sqrt{\frac{2\gamma R T_0}{\gamma + 1}} = \sqrt{\frac{2a_0}{\gamma + 1}}$$

$$a^* = \sqrt{\frac{2a_0}{\gamma + 1}}$$

$u \rightarrow u^*$

$$u^* = a^*$$

$$M = 1$$

$$\gamma R T^* = a^{*2} = u^{*2}$$

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Again also we can find out same these expressions using the energy equation. So, we can know that how that energy equation is very vital in all this analysis. So, everywhere you are using this energy equations here; we will use this energy equation for stagnation

temperature  $T_0 = T + \left( \frac{\gamma - 1}{2\gamma R} \right) u^2$ .

So, this derivation was made in one of the previous slides which is this. So, here what we are assigning is that  $u$  becomes  $u^*$ ; or  $u^* = a^*$  speed of the sound at sonic state, so that  $M$  becomes 1. So, Mach number becomes 1.

So, for that case we can rewrite this equation as  $T_0 = T^* + \left(\frac{\gamma-1}{2\gamma R}\right) u^{*2}$  that is arbitrary state becomes a star conditions, so you solve for  $u^*$ . So,

$$u^{*2} = \left(\frac{2\gamma RT_0}{\gamma-1}\right) - \left(\frac{2\gamma RT^*}{\gamma-1}\right)$$

Now, what you will do? We will now simplify this  $u^{*2}$ . So, you bring this particular part we say;  $\gamma RT^* = a^{*2} = u^{*2}$ . When you bring this expression to the left side and solve for  $u^*$  we get  $u^* = \sqrt{\left(\frac{2\gamma}{\gamma+1}\right) RT_0}$ .

So, basically we have to say use this expression and bring this expression to the left side and then solve for  $u^*$ . So, ultimately  $u^* = \sqrt{\frac{2\gamma RT_0}{\gamma+1}}$ . So, further we can write this  $\gamma RT^* = a^{*2}$ . Now, since  $u^* = a^*$  we can also write  $a^* = \sqrt{\frac{2\gamma RT_0}{\gamma+1}}$ .

So, there is some mathematical jugglery or simplifications one has to make to arrive at this relation.

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**Characteristics Properties of Compressible Fluid**

Speed ratios

➤ Maximum speed, characteristics speed of sound, stagnation speed of sound

$$a_0 = \sqrt{\gamma RT_0}; u_{\max} = \sqrt{\frac{2\gamma RT_0}{\gamma-1}}; a^* = \sqrt{\frac{2\gamma RT_0}{\gamma+1}}$$

$$\frac{a^*}{a_0} = \sqrt{\frac{2}{\gamma+1}}; \frac{u_{\max}}{a_0} = \sqrt{\frac{2}{\gamma-1}}; \frac{u_{\max}}{a^*} = \sqrt{\frac{\gamma+1}{\gamma-1}}$$

For air  $\gamma = 1.4$ ,  $\frac{a^*}{a_0} = 0.92$ ;  $\frac{u_{\max}}{a_0} = 2.24$ ;  $\frac{u_{\max}}{a^*} = 2.45$

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So, whatever I told, I summarized here. We here will talk about the speed ratio. So, we defined the maximum speed, we defined the characteristic speed of sound, we defined the stagnation of the speed of the sound.

So, what it essentially means; for any given state that is which is arbitrary. For this arbitrary state, we can define an imaginary state and this imaginary state is achieved through the adiabatic process and conditions are defined through \*. But same arbitrary states, we define stagnation state. For these conditions, we defined as o.

So, when we try to make relations between this imaginary state and stagnation state this is how we define. That is first one is stagnation speed of sound, maximum speed of sound, characteristic speed of sound. Then we can find this ratio  $\frac{a^*}{a_0}$ ,  $\frac{u_{\max}}{a_0}$  and  $\frac{u_{\max}}{a^*}$ .

Now, here we can see that all these expressions are function of gamma. Now, for air with gamma is equal to 1.4 one can fix this ratio as follows that is  $\frac{a^*}{a_0} = 0.92$ ,  $\frac{u_{\max}}{a_0} = 2.24$  and

$$\frac{u_{\max}}{a^*} = 2.45.$$

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### Characteristics Properties of Compressible Fluid

Adiabatic Ellipse

- When there is a large relative speed between the body and the compressible fluid surrounding it, the variation of density with speed influences the properties in the flow field. So, Mach number is defined as the ratio of "directed kinetic energy" to the "random molecular kinetic energy".
- "Adiabatic Ellipse" is a graphical representation of speed of sound as function of speed of gases in a steady flow compressible medium.
- The locus of all the points turns out to be an ellipse and all the points on the ellipse have same total energy.
- Each point on the ellipse differs from other owing to the relative proportions of thermal and kinetic energy and corresponds to a different Mach number.

$M = \frac{u}{a}$

$\frac{u^2}{u_{\max}^2} + \frac{a^2}{a_0^2} = 1$

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Now, with this basic concepts and defining these properties, we will now visit the information through graphical manner. So, what does this mean? We will define a term

called as adiabatic ellipse because most of us known as what is a ellipse which is a very common mathematical term.

And here we will define an ellipse which is adiabatic ellipse. So, in a compressible flow field, this adiabatic ellipse is a graphical representation of speed of sound as a function of speed of gases in a steady flow medium.

So, as you know Mach number is equal to  $\frac{u}{a}$ . So, it is essentially the relative speed between  $u$  and  $a$  that defines this Mach number. Mach number becomes high when speed of the gas is high for a given conditions.

So, it has also another implications that because  $u$  happens to be related to the kinetic energy of the gas and  $a$  is related to random motion in terms of the disturbance. And based on this particular concept we can now derive these equations for adiabatic ellipse

$$\text{as } \frac{u^2}{u_{\max}^2} + \frac{a^2}{a_0^2} = 1$$

So, as I mentioned we know what is  $u_{\max}$ ; that means, how a gas can attain maximum speed. And we also know what is the speed of sound speed of sound at the stagnation conditions for a given gas. So, this turns out to be the equation of an ellipse in terms of mathematics.

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**Characteristics Properties of Compressible Fluid**

**Adiabatic Ellipse**

$$\frac{u^2}{u_{\max}^2} + \frac{a^2}{a_0^2} = 1$$

Energy:

$$\frac{C_p T_0}{C} = C_p T + \frac{u^2}{2}$$

$$\gamma R T = \frac{u^2}{2} + \left( \frac{\gamma R}{\gamma - 1} \right) T = C_1$$

$$\Rightarrow u^2 + \left( \frac{2}{\gamma - 1} \right) a^2 = \frac{C_1}{\gamma}$$

$T \rightarrow 0$   $u \rightarrow u_{\max}$   $C_1 = \frac{u_{\max}^2}{2}$

$$a \rightarrow 0 \quad u^2 + \left( \frac{2}{\gamma - 1} \right) a^2 = u_{\max}^2$$

$$\Rightarrow \frac{u^2}{u_{\max}^2} + \left( \frac{2}{\gamma - 1} \right) \frac{a^2}{u_{\max}^2} = 1$$

$$\Rightarrow \frac{u^2}{u_{\max}^2} + \left( \frac{2}{\gamma - 1} \right) \left( \frac{\gamma - 1}{2} \right) \frac{a^2}{a_0^2} = 1$$

$$\frac{u^2}{u_{\max}^2} + \frac{a^2}{a_0^2} = 1$$

$u_{\max}^2 = \left( \frac{2}{\gamma - 1} \right) a_0^2$



And let us derive how we get these equations. So, to start these equations we again revisit the same energy equations that is  $C_p T_0 = C_p T + \frac{u^2}{2}$ .

Now, we can say, since we are using a mathematical term we say this happens to be constant so assign a term which is a constant  $C$ . So, what you do now? We can write this equations as  $\frac{u^2}{2} + \left( \frac{\gamma R}{\gamma - 1} \right) T = C_1$ . So,  $\gamma R T = a^2$ .

So, this will implies  $u^2 + \left( \frac{2}{\gamma - 1} \right) a^2 = C_1$ . So, let us find what is this constant  $C_1$ . So, here we will try to bring how we can get  $u_{\max}$  here. So, to get  $u_{\max}$ ; we have to say that when  $u$  goes to  $u_{\max}$  at what condition?

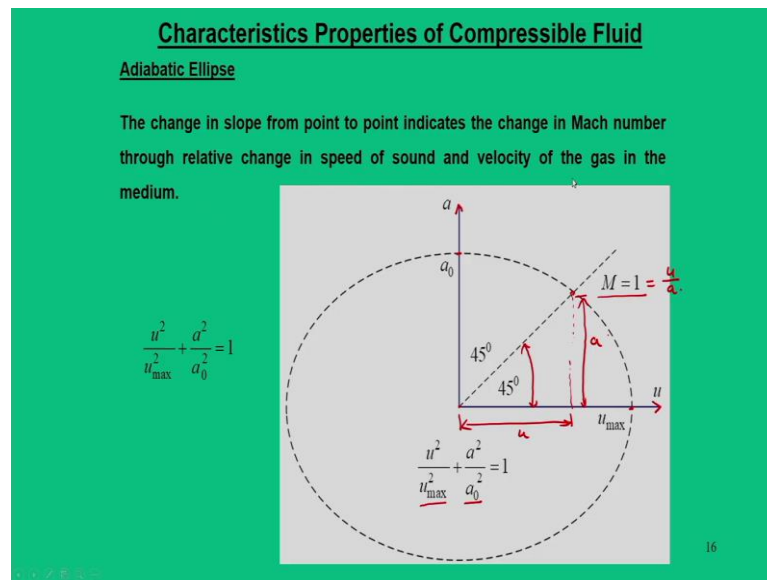
So, what it will do now this? So, we say  $T$  goes to 0;  $u$  goes to  $u_{\max}$ . So, we can write  $u^2 + \left( \frac{2}{\gamma - 1} \right) a^2 = u_{\max}^2$  because this  $C_1$  becomes  $u_{\max}^2$  where  $T$  is equal to 0 and  $a$  goes to 0 so  $C_1$  becomes  $u_{\max}^2$ . Then we bring what is this  $u_{\max}$  square here.

So, what you do here to use this equation we can say  $\frac{u^2}{u_{\max}^2} + \left( \frac{2}{\gamma - 1} \right) \frac{a^2}{u_{\max}^2} = 1$ . So, from this  $u_{\max}^2$  you bring these relations from the earlier one i.e.  $u_{\max}^2 = \left( \frac{2}{\gamma - 1} \right) a_0^2$ . So, we put  $u_{\max}$  square from this then we rewrite equations as  $\frac{u^2}{u_{\max}^2} + \left( \frac{2}{\gamma - 1} \right) \left( \frac{\gamma - 1}{2} \right) \frac{a^2}{a_0^2} = 1$ .

So, this terms will get cancelled ultimately this leads to this expression  $\frac{u^2}{u_{\max}^2} + \frac{a^2}{a_0^2} = 1$ .

So, this is how we derive these equations for adiabatic ellipse starting with energy equations.

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So, now let me see that when you take this adiabatic ellipse equation, how you represent it graphically. So, this is the equation of the ellipse. So we are plotting this graphically where x axis represent u, y axis represent a. We have a gas can attain maximum velocity that is  $u_{\max}$  which is given by the abscissa of this ellipse or that is half of the major axis.

And the stagnation speed of sound  $a_0$  which is defined by this point which is the half of the minor axis of the ellipse. So, at every point on this ellipse we can define about the relative importance between u and a. So, when your Mach number is equal to 1 which is

$$\frac{u}{a}$$

That means, essentially u and a are same; so we can drop a slope from this point. This point can be located that is from origin we can draw a line  $45^\circ$  and wherever it touches the ellipse at that point we will have Mach number is equal to 1 because this u and a will be same that is slope remains same.

So, this is how the physical significance of this adiabatic ellipse. So, what it means that change in the slope from point to point indicates the change in the Mach number through relative change in the speed of sound and velocity of the gas in the medium.

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**Characteristics Properties of Compressible Fluid**

**Adiabatic Ellipse**

The change in slope from point to point indicates the change in Mach number through relative change in speed of sound and velocity of the gas in the medium.

$$\frac{u^2}{u_{\max}^2} + \frac{a^2}{a_0^2} = 1; \quad M = -\left(\frac{2}{\gamma-1}\right)\left(\frac{da}{du}\right) \quad \frac{a_0^2}{u_{\max}^2} = \frac{\gamma-1}{2}$$

*Sensitive analysis*

$$\frac{a^2}{a_0^2} = 1 - \frac{u^2}{u_{\max}^2}$$

$$\Rightarrow a^2 = a_0^2 - \frac{1}{u_{\max}^2} a_0^2 (u^2)$$

*Differentiate*

$$2a \frac{da}{du} = -\left(\frac{a_0^2}{u_{\max}^2}\right) 2u$$

$$\frac{da}{du} = -\left(\frac{\gamma-1}{2}\right) M$$

$$M = -\left(\frac{2}{\gamma-1}\right) \frac{da}{du}$$

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So, here we will now talk about something more what is called as a sensitive analysis. What we essentially trying to say that how we can relate this Mach number with the slope of this curve or ellipse. So, for that we want to find out this particular expression that is  $\frac{da}{du}$ .

So, to find these equations we start this basic equations  $\frac{u^2}{u_{\max}^2} + \frac{a^2}{a_0^2} = 1$  and from these

basic equations you try to find out this  $\frac{da}{du}$ . So, how do you do it? So, we separate this

equation as  $\frac{a^2}{a_0^2} = 1 - \frac{u^2}{u_{\max}^2}$ .

Then what you do; simplify  $a^2 = a_0^2 - \frac{1}{u_{\max}^2} a_0^2 (u^2)$ . Then you find differentiate that is

$$2a \frac{da}{du} = -\left(\frac{a_0^2}{u_{\max}^2}\right) 2u$$

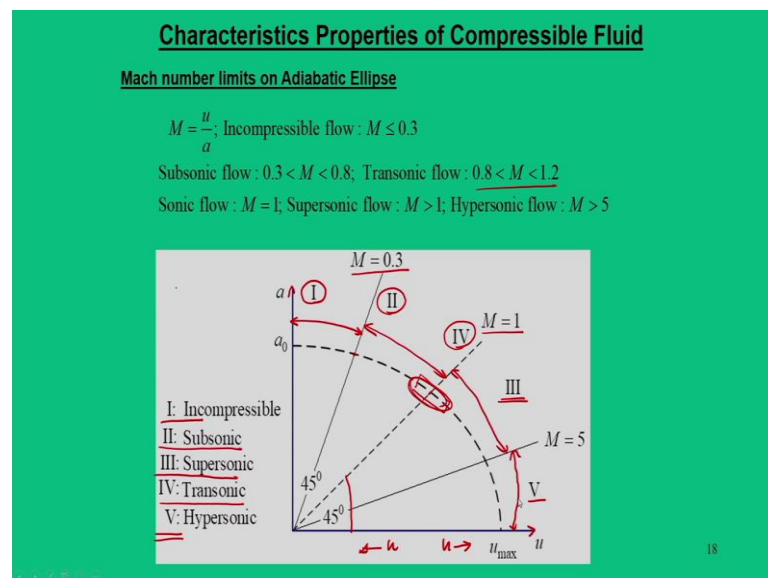
So, now you try to find out  $\frac{a_0^2}{u_{\max}^2}$  from this equation  $\frac{a_0^2}{u_{\max}^2} = \frac{\gamma-1}{2}$ , substitute here. Then

when you substitute then we say  $\frac{da}{du} = -\left(\frac{\gamma-1}{2}\right)M \cdot M$  you get because the left hand side

a goes to right hand side we get this Mach number. And  $M = -\left(\frac{2}{\gamma-1}\right)\frac{da}{du}$ .

So, through the sensitive analysis, we get Mach number which is related the slope at any point on the ellipse.

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So, now let us revisit what does this mean. So, what we have drawn here to study the effect of sensitivity of Mach number with respect to slope. We have taken only the one fourth portion of the ellipse; that is the first quadrant.

So, just to see this relative importance, we are now trying to define different flow regimes. So, just looking at this curve one can make out that when we our  $u$  tends to 0; when you move towards left that means we are going closer to 0 although we have speed of sound as maximum as  $a_0$ , but we are in the situation that the speed has no effect in the medium.

So, for that reasons we say that we are moving towards incompressible medium when  $u$  decreases. But as and when  $u$  increases; that means, when we are going towards right

hand side then  $u$  keeps on increasing. So, relative importance of  $u$  increases more and more as we move towards this.

Now, coming back to this figure here we have defined essentially 5 regimes of flow. So, first thing is that the first regime when you say the flow to be incompressible that is defined as I where the flow is incompressible.

That means, we are in the domain of incompressible region and you choose that point in such a way that that particular point your Mach number becomes 0.3. So, that means, as long as we are in this domain, the flow is incompressible in nature. That means, the flow is incompressible flow and the role of Mach number does not come into picture.

Now, as we move towards the region 2 which is this; the Mach number keeps on increases. So, till Mach number is equal to 1 which is shown through a line drawn  $45^\circ$  drawn from the origin to that point where Mach number is 1. The flow is defined to be sub sonic that is second category.

So, as and when keep on progressively increasing. So, we are moving towards the 3rd regions which is flow is happens to be supersonic. And in some dominance region like flow is neither in the vicinity of subsonic, but trying to going to supersonic region. Or in other words flow is initially supersonic and tried to goes to subsonic regions.

So, those regimes are defined in certain Mach number range that is 0.8 to 1.2. So, that is denoted by domain IV that is 4th region that is transonic flow. And the last part which is flow happens to be hypersonic so we are in the last part that is 5th region, flow is happens to be hypersonic. So, in this domain we say that we are moving flow toward the hypersonic regime.

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**Characteristics Properties of Compressible Fluid**

**Inferences from Adiabatic Ellipse**

- The change of slope from point to point indicates the change in Mach number related to the changes in the speed of sound and velocity of the gas. So, it gives a direct comparison of the relative magnitudes of thermal and kinetic energies in a steady flow compressible medium.
- At low Mach number flows, the changes in the property parameters are mainly associated with the change in velocity of the gas.
- At high Mach number flows, the changes in the property parameters are mainly associated with the change in speed of sound.
- For limiting case of Mach number in the range of 0.3 or less, the flow is treated to be incompressible.

$$\frac{u^2}{u_{\max}^2} + \frac{a^2}{a_0^2} = 1$$

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So, the last part of these things whatever I have explained which we talk as inferences from adiabatic ellipse. That is change in the slope from point to point indicates the change in the Mach number relative to change in the speed of sound and velocity of the gases. So, it gives the direct comparison of relative magnitude of thermal and kinetic energy in the steady flow compressible medium.

So, what it shows from the adiabatic ellipse that; at low Mach numbers flows the changes in the property parameter are mainly associated with the change in the velocity of the gases. And at high Mach number flows the changes in the property parameter are mainly associated with the change in the speed of sound. And in the limiting case when Mach number is in the range of 0.3 or less the flow is treated to be incompressible. These are the important inferences that we get from this adiabatic ellipse. So, with this I conclude my talk.

Thank you for your attention.