Solid Mechanics
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Lecture - 8
Maximizing the Shear Component of Traction

Welcome to Lecture 8! In this lecture, we will find out the plane on which the shear component of traction is maximized/minimized. This is again important because one of the failure theories says that the body will fail if the shear component of traction reaches a critical value.

1 Shear component of traction on an arbitrary plane (start time: 00:45)

To maximize/minimize, we need to first find an expression for the shear traction on any plane. We consider a part of our body as shown in Figure 1. The plane shown has normal $\underline{n}$ and the traction on it is denoted by $\underline{t}$.

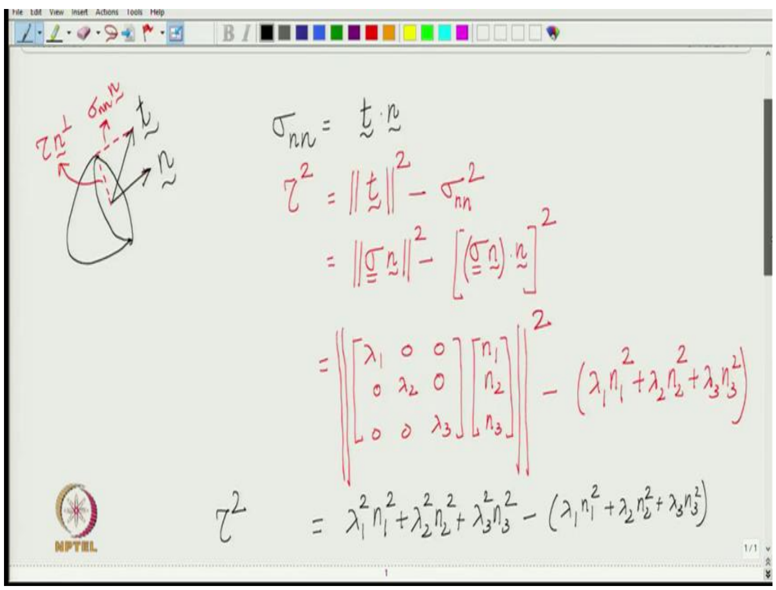

Figure 1: A section of the body with normal and shear component of traction shown

In the last lecture, we had seen that the normal component of this traction (σ_{nn}) is given by:

$$\sigma_{nn} = \underline{t} \cdot \underline{n} \tag{1}$$

To get the shear component of traction, we need to subtract this normal component from the total traction vectorially. In Figure 1, the traction $\underline{t}$ has been decomposed into two parts: normal and shear component. The projection of the traction along $\underline{n}$ corresponds to the normal component given by $\sigma_{nn}\underline{n}$. The remaining component (perpendicular to $\underline{n}$) is the shear part. It can be represented as $\tau\underline{n}^{\perp}$ where τ represents the magnitude and $\underline{n}^{\perp}$ represents the direction (has to be perpendicular to $\underline{n}$). Applying

Pythagoras theorem in the right angled triangle formed by traction and its components shown in Figure 1, τ^2 will be given by

$$\begin{aligned}\tau^2 &= ||\underline{t}||^2 - \sigma_{nn}^2 \\ &= ||\underline{\underline{\sigma}}\,\underline{n}||^2 - \left|(\underline{\underline{\sigma}}\,\underline{n})\cdot\underline{n}\right|^2\end{aligned} \qquad (2)$$

Note that two vertical bars on each side are used to denote the magnitude of a vector while a pair of single vertical bars is used to denote the magnitude of a scalar.

1.1 Representation in terms of principal planes (start time: 04:08)

If we work in the coordinate system of principal directions, our stress matrix will be a diagonal matrix and that will greatly simplify the calculation. We thus represent everything in this coordinate system which yields

$$\begin{aligned}\tau^2 &= \left|\left|\begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix}\begin{bmatrix}n_1\\ n_2\\ n_3\end{bmatrix}\right|\right|^2 - (\lambda_1 n_1^2 + \lambda_2 n_2^2 + \lambda_3 n_3^2)^2 \\ &= \left|\left|\begin{bmatrix}\lambda_1 n_1\\ \lambda_2 n_2\\ \lambda_3 n_3\end{bmatrix}\right|\right|^2 - (\lambda_1 n_1^2 + \lambda_2 n_2^2 + \lambda_3 n_3^2)^2 \\ &= \lambda_1^2 n_1^2 + \lambda_2^2 n_2^2 + \lambda_3^2 n_3^2 - (\lambda_1 n_1^2 + \lambda_2 n_2^2 + \lambda_3 n_3^2)^2\end{aligned} \qquad (3)$$

This gives us the formula for square of the magnitude of total shear component of traction on an arbitrary plane. Note that this total/resultant shear would be acting in a direction on the plane given by the vector sum of both shear components on that plane. For example, on the $\underline{e}_1$ plane, we have both τ_{21} and τ_{31}. The vector resultant of these two ($\tau_{21}^2 + \tau_{31}^2$) will give us the square of the magnitude of total shear component of traction as given by equation (3).

2 Maximization/Minimization using Lagrange Multipliers (start time: 07:34)

Having obtained the expression for the total shear component of traction, this needs to be maximized/minimized. As n_1, n_2 and n_3 are not independent of each other, we again use the method of Lagrange multipliers. So, we define a function V as given below:

$$V = \sum_i \lambda_i^2 n_i^2 - \Big(\sum_i \lambda_i n_i^2\Big)^2 + \alpha\Big(\sum_i n_i n_i - 1\Big) \qquad (4)$$

We're using α to denote the lagrange multiplier here because we already have λ's for principal stress components. This is the function that has to be maximized/minimized with respect to the 4 unknowns: n_1,n_2,n_3 and α. Let us take the derivative of V with respect to these four unknowns starting with the k^{th} component of normal vector (n_k):

$$\begin{aligned}
\frac{\partial V}{\partial n_k} &= 2\sum_i \lambda_i^2 n_i \delta_{ik} - 2\Big(\sum_i \lambda_i n_i^2\Big)\cdot 2\sum_i \lambda_i n_i \delta_{ik} + 2\alpha \sum_i n_i \delta_{ik} = 0 \\
&= 2\lambda_k^2 n_k - 2\Big(\sum_i \lambda_i n_i^2\Big)\cdot 2\lambda_k n_k + 2\alpha n_k = 0 \\
&= n_k\left[\lambda_k^2 - 2\Big(\sum_i \lambda_i n_i^2\Big)\lambda_k + \alpha\right] = 0, \qquad k = 1,2,3
\end{aligned} \tag{5}$$

We have used the fact that the derivative of one component of normal vector with another will give us Kronecker delta function. Then, using the Kronecker delta property, we have removed one of the summations from each of the terms. The fourth equation is now obtained by taking derivative with respect to α and equating to zero, i.e.,

$$\sum_i n_i n_i - 1 \;=\; 0. \tag{6}$$

This is our constraint itself (i.e. magnitude of $\underline{n}$ has to be unity). Writing equation (5) for each k separately, we get:

$$n_1\left[\lambda_1^2 - 2\Big(\sum_i \lambda_i n_i^2\Big)\lambda_1 + \alpha\right] \;=\; 0 \tag{7}$$

$$n_2\left[\lambda_2^2 - 2\Big(\sum_i \lambda_i n_i^2\Big)\lambda_2 + \alpha\right] \;=\; 0 \tag{8}$$

$$n_3\left[\lambda_3^2 - 2\Big(\sum_i \lambda_i n_i^2\Big)\lambda_3 + \alpha\right] \;=\; 0 \tag{9}$$

In each of these equations, one among the two terms multiplied has to be zero for the product to be zero. There are multiple solutions to this problem and all solutions can be found by considering different cases. If suppose $n_1 = 0$ in the first equation, $n_2 = 0$ in the second and $n_3 = 0$ in the third, we get a trivial solution for n_1, n_2 and n_3 but that will not give us a valid direction as the magnitude of the direction vector will not be 1. So, we take the first term in the first equation and the second terms in the other two equations to be zero. So, we have the following equations at hand now:

$$\begin{aligned} n_1 &= 0 && (10)\\ \lambda_2^2 - 2(\lambda_2 n_2^2 + \lambda_3 n_3^2)\lambda_2 + \alpha &= 0 && (11)\\ \lambda_3^2 - 2(\lambda_2 n_2^2 + \lambda_3 n_3^2)\lambda_3 + \alpha &= 0 && (12) \end{aligned}$$

Equations (11) and (12) have been obtained by substitution of $n_1 = 0$. We need to find n_2 and n_3 now. To eliminate α, we subtract (12) from (11) to get

$$(\lambda_2^2 - \lambda_3^2) - 2(\lambda_2 - \lambda_3)(\lambda_2 n_2^2 + \lambda_3 n_3^2) = 0 \tag{13}$$

Using identity $a^2 + b^2 = (a + b)(a - b)$, we can cancel $(\lambda_2 - \lambda_3)$.

$$\begin{aligned} \Rightarrow (\lambda_2 + \lambda_3) - 2(\lambda_2 n_2^2 + \lambda_3 n_3^2) &= 0 \\ \Rightarrow \lambda_2(1 - 2n_2^2) + \lambda_3(1 - 2n_3^2) &= 0 && (14) \end{aligned}$$

As λ_2 and λ_3 are principal stress components, they are fixed for a given point in space and do not depend on what plane we are considering. Also, as this analysis holds for an arbitrary stress matrix at the point of interest, λ_2 and λ_3 can be assumed to be arbitrary. Thus, equation (14) should hold for all λ_2 and λ_3 implying that the coefficients of λ_2 and λ_3 must vanish independently. Thus, we get the values of n_1, n_2 and n_3 as

$$n_1 = 0, \quad n_2 = \pm\frac{1}{\sqrt{2}}, \quad n_3 = \pm\frac{1}{\sqrt{2}} \tag{15}$$

The above equation gives us four solutions for the direction as n_2 and n_3 both can take two values each independently. But these are not the only solutions. In equations (7), (8) and (9), if we would have assumed either n_2=0 or n_3 = 0 instead of n_1 = 0, our analysis would essentially have remained exactly similar leading to four solutions each given by:

$$n_1 = \pm\frac{1}{\sqrt{2}}, \quad n_2 = 0, \quad n_3 = \pm\frac{1}{\sqrt{2}} \tag{16}$$

$$n_1 = \pm\frac{1}{\sqrt{2}}, \quad n_2 = \pm\frac{1}{\sqrt{2}}, \quad n_3 = 0 \tag{17}$$

So, we have 12 sets of solutions in total given by equations (15), (16) and (17). Keep a note that these directions are with respect to the normals of principal planes. For example, if we choose one of the solutions: $n_1 = \frac{1}{\sqrt{2}}$, $n_2 = \frac{1}{\sqrt{2}}$, $n_3 = 0$, this tells us that the direction $\underline{n}$ has to be perpendicular to the third principal plane's normal. And at the same time, it makes an equal angle of 45° with the first and second principal axes. If we look at all the 12 solutions, at least one component of $\underline{n}$ is zero for each of the solutions. So, each of these directions are perpendicular to at least one of the principal axes. The first set (15) is perpendicular to the first principal axis, the second set (16) is perpendicular to the second principal axis and the third set (17) is perpendicular to the third principal axis.

3 Magnitude of traction components on planes having maximum shear (start time: 25:41)

We also want to know the value of the shear component of traction on these planes. To find this, we just need to plug in the solution for $\underline{n}$ in equation (3). For, the set of solutions given by equation (17), we get:

$$\begin{aligned}\tau^2 &= \frac{\lambda_2^2}{2} + \frac{\lambda_1^2}{2} - \left(\frac{\lambda_1+\lambda_2}{2}\right)^2 \\ &= \frac{2\lambda_1^2 + 2\lambda_2^2 - \lambda_1^2 - \lambda_2^2 - 2\lambda_1\lambda_2}{4} \\ &= \frac{(\lambda_1-\lambda_2)^2}{4} \\ \Rightarrow |\tau| &= \left|\frac{\lambda_1-\lambda_2}{2}\right|\end{aligned} \tag{18}$$

Therefore, the maximum value of shear traction is half of the difference of principal stress components. The value of the normal component of traction on this plane will be obtained by substituting (17) in expression of σ_{nn}, i.e.,

$$\begin{aligned}\sigma_{nn} &= \lambda_1 n_1^2 + \lambda_2 n_2^2 + \lambda_3 n_3^2 \\ &= \frac{\lambda_1+\lambda_2}{2}\end{aligned} \tag{19}$$

This is for one set of solution of $\underline{n}$. Similarly, we can find τ and σ_{nn} for other sets of solutions also. When we work it out, we find that for the solution set (15), we get

$$|\tau_{max}| = \left|\frac{\lambda_2-\lambda_3}{2}\right|, \quad \sigma_{nn} = \frac{\lambda_2+\lambda_3}{2} \tag{20}$$

and for the solutions set (16), we get

$$|\tau_{max}| = \left|\frac{\lambda_1-\lambda_3}{2}\right|, \quad \sigma_{nn} = \frac{\lambda_1+\lambda_3}{2} \tag{21}$$

4 Visualizing results (start time: 30:38)

To visualize this result, we draw a cuboid at the point of interest with their faces being principal planes as shown in Figure 2. Since the faces are principal planes, they have only got normal component of traction. We want to draw the planes corresponding to maximum shear. First, consider the set where the second normal component $n_2 = 0$, i.e., given by (16). The planes corresponding to this set of normal vectors are drawn in green in Figure 2. We now extract this green cuboid out and look at it in isolation as shown in Figure 3. For the front face of this cuboid, the normal is such that its second component is zero. The first and third components will both be $\frac{1}{\sqrt{2}}$. This normal makes equal angles with the first and third principal axes and is perpendicular to the second principal axis. Similarly, for the left plane, its first component will be negative and the third will be positive both with magnitude $\frac{1}{\sqrt{2}}$. We also know the shear and normal components of traction on these planes. For example, on the front face, normal component (σ_{nn}) will be $\frac{\lambda_1 + \lambda_2}{2}$ and the shear component (τ) will be $\left|\frac{\lambda_1 - \lambda_3}{2}\right|$. The top face of this green cuboid still has only λ_2 as it is still a principal plane. We can observe that the planes having maximum shear component of traction are at 45° relative to two of the principal axes. Also note that when we were maximizing the normal component of traction, shear component of traction on those planes turned out to be zero. But here, when we maximize the shear component of traction, the normal component of traction on these planes are not zero.

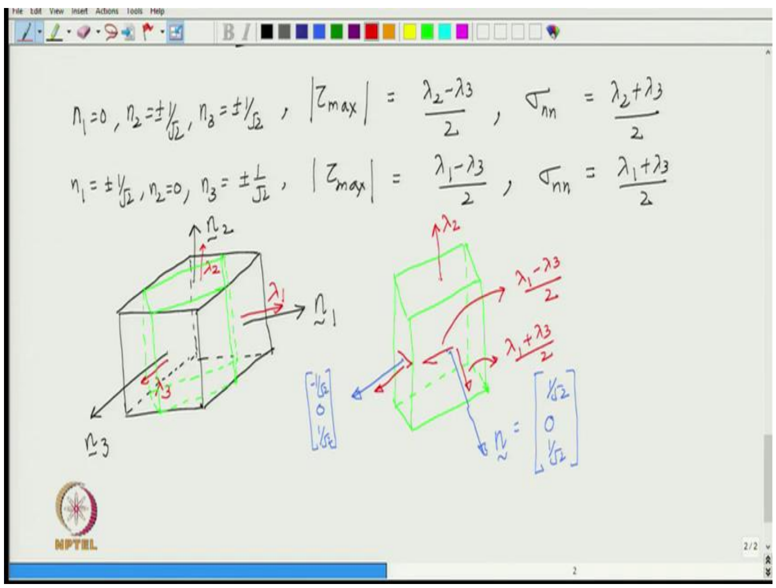

Figure 2: The black cuboid is centered at the point of interest with its faces having normals along principal directions. The planes shown in green are the planes where shear component of traction is maximized