Solid Mechanics
Prof. Ajeet Kumar
Deptt. of Applied Mechanics
IIT, Delhi
Lecture - 31
Theories of Failure

Hello everyone! Welcome to Lecture 31! In this lecture, we will discuss about different theories of failure.

1 Introduction (start time: 00:25)

Think of an arbitrary body clamped at some points on its surface as shown in Figure 1.

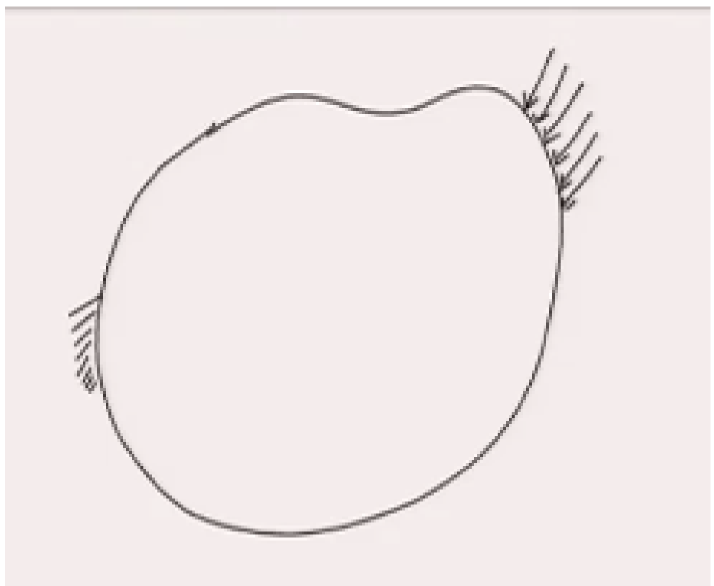

Figure 1: A body is clamped at some points on the boundary. An arbitrary traction is applied on the body.

It is further subjected to traction at some other parts of the surface. The body deforms due to the applied traction. As we increase the magnitude of applied traction, the body deforms more and more and at one point, the body will eventually fail. This failure can be in terms of the body suddenly getting deformed a lot (yielding or buckling) or due to crack developing in the body. The failure usually initiates at a point in the body and not at all points together (e.g., due to crack). At each point in the body, a different state of stress and strain exists. If, at a point, some function of stress or strain components reaches a critical value, failure occurs. The different theories of failure have been developed on the basis of the specific form of the failure function in terms of stress/ strain components. For example, failure could occur at a point due to principal stress component reaching a critical value or maximum shear stress reaching a critical value. In some cases, neither strain not stress but energy stored at a point might reach a critical value causing failure. The various theories of failure can be listed as follows:

1. Maximum principal stress theory
2. Maximum shear stress theory
3. Maximum normal strain theory
4. Maximum shear strain theory
5. Distortional energy theory
6. Octahedral shear stress theory

As there are multiple strain and stress components at a point, we therefore talk of maximum stress or strain reaching a critical value. But, the energy stored at a point being unique, there is nothing like maximum energy theory: note that we have considered only the distortional part of energy at a point and not the total elastic energy. Also, we will see that the distortional energy theory and octahedral shear stress theory are related. Let us discuss these theories one by one.

2 Maximum principal stress theory (start time: 05:18)

To obtain the critical value for failure, we usually do simple tests like simple tension test or torsion test. Let us assume that we are doing a simple tension test to find the critical value. Think of a bar which is being pulled from both the ends by a distributed tensile force σ_{11} as shown in Figure 2.

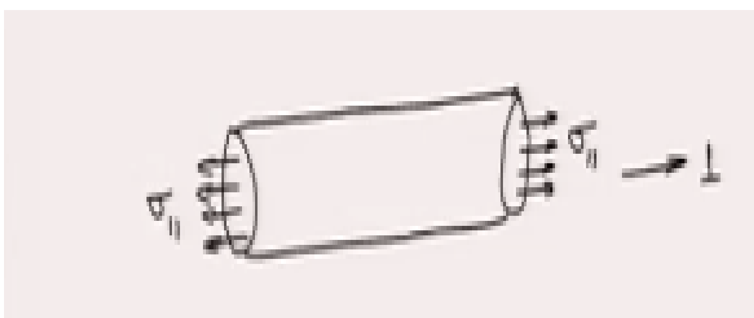

Figure 2: A bar is pulled from both ends by distributed force σ_{11}.

We increase the value of σ_{11} slowly until the body fails, say at $\sigma_{11} = \sigma_y$. Now, suppose the body is obeying the maximum principal stress theory for failure. As we are doing a simple tension test, the state of stress at a general point in the body will be

$$\begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{1}$$

The principal stress components for such a state of stress can be found directly through inspection of the stress matrix, i.e.,

$$\lambda_1 = \sigma_{11}, \qquad \lambda_2 = \lambda_3 = 0. \tag{2}$$

Thus, the maximum principal stress is simply σ_{11} which should be less than or equal to the critical value of principal stress, i.e., (σ_y) which we obtained from direct measurement in simple tension test.

For a general deformation, the maximum principal stress may not be equal to σ_{11}. Even then, one first does simple tension test to obtain σ_y since it is easier to perform and that, one gets σ_y by directly comparing it with the applied load at failure. Then for the general loading scenario, one has to first obtain the distribution of stress in the body and then obtain maximum principal stress at every point in the body, say $\lambda_1(x)$. One further computes the maximum value of $\lambda_1(x)$ among every point in the body and compare with the critical value σ_y to check for failure, i.e.,

$$\max_x \quad \lambda_1(x) < \sigma_y. \tag{3}$$

3 Maximum shear stress theory (start time: 10:42)

We again do the simple tension test to find out the critical value of shear stress in the body. We increase the distributed load σ_{11} as shown in Figure 2. The load at which the body fails is σ_y. The state of stress is given in (1). The principal stress components for this state of stress are $\lambda_1 = \sigma_{11}$, $\lambda_2 = 0$ and $\lambda_3 = 0$. Thus, maximum shear stress which is obtained by $\frac{\lambda_1 - \lambda_3}{2}$ turns out to be $\frac{\sigma_{11}}{2}$.

This will become equal to the critical shear stress value τ_y when $\sigma_{11} = \sigma_y$ which implies

$$\tau^y = \frac{\sigma_y}{2}. \tag{4}$$

Once τ_y is obtained through simple tension test, it holds even for general loading scenario. The maximum shear stress for a general loading case with principal stress components λ_1, λ_2 and λ_3, is given by $\frac{\lambda_1 - \lambda_3}{2}$. So, to avoid failure, the following condition should be satisfied at every point in the body:

$$\frac{|\lambda_1 - \lambda_3|}{2} < \frac{\sigma_y}{2}. \tag{5}$$

We can also obtain the critical value of shear stress τ_y from torsion test instead of tension test. Figure 3 shows a circular beam subjected to equal and opposite torques at the ends.

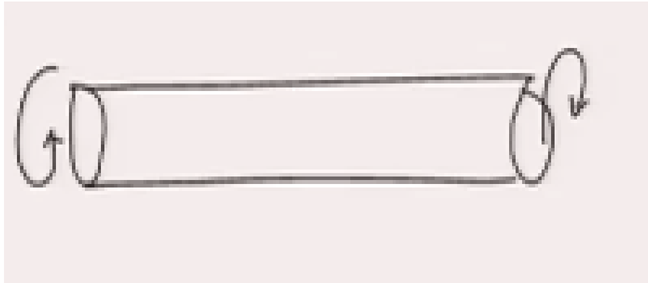

Figure 3: A circular beam undergoing torsion.

We had found the state of stress for such a case earlier. In cylindrical coordinate system, the stress matrix can be written as

$$[\underline{\sigma}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \tau_{\theta z} \\ 0 & \tau_{\theta z} & 0 \end{bmatrix}. \tag{6}$$

We can find the maximum shear stress for this state of stress by drawing the Mohr's circle. As e_r is a principal axis, we can draw the Mohr's circle directly. As $\sigma_{\theta\theta}$ and σ_{zz} are zero, the center of the circle will be at the origin while the radius of the circle will be the length from origin to $\tau_{\theta z}$. Figure 4 shows the Mohr's circle.

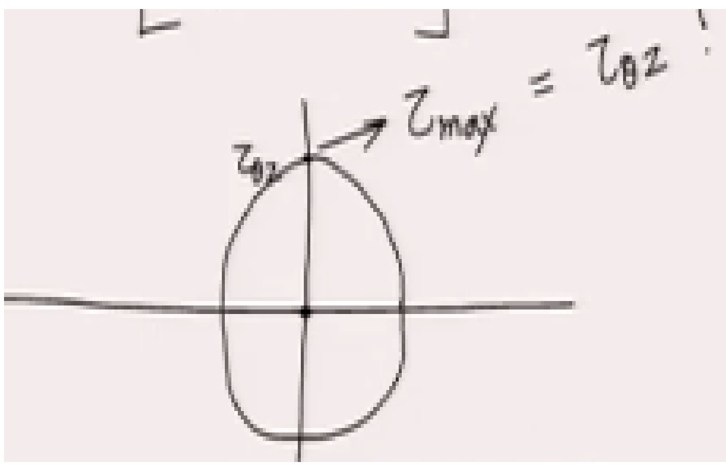

Figure 4: Mohr's circle for the state of stress given in (6).

Therefore, the maximum shear stress τ_{max} is simply $\tau_{\theta z}$, the value of shear stress in the cross-sectional plane. We can write this shear stress $\tau_{\theta z}$ in terms of the applied torque T as follows

$$\tau_{\theta z} = \frac{Tr}{J}. \tag{7}$$

This happens to be the maximum shear stress at any point for the case of pure torsion. As it changes with radial coordinate r, its maximum value for the entire body is attained when r is maximum, i.e., on the outer surface of the cylinder. Thus, the critical value of shear stress τ^y can be written as

$$\tau_y = \tau^*_{\theta z} = \frac{T^* R}{J}. \tag{8}$$

where T^* is the critical torque value at failure. Once τ_y is obtained from torsion test, for the general loading scenario, one can write

$$\frac{|\lambda_1 - \lambda_3|}{2} < \tau_y = \tau^*_{\theta z} = \frac{T^* R}{J} \tag{9}$$

We can also formulate the maximum normal strain and maximum shear strain theories similarly. Let us now look at the distortional energy theory.

4 Distortional energy theory (start time: 19:14)

Consider a body under general loading. Thus, we have general state of stress at every point in the body. There will be three principal components of stress at every point which we denote by σ_1, σ_2 and σ_3. Let us now obtain the strain components in the principal coordinate system. There is no shear stress in the principal coordinate system and thus the shear strains will also be zero using linear stress-strain relation for isotropic bodies. We only have normal strains ϵ_1, ϵ_2 and ϵ_3 which can be written as follows in terms of principal stress components:

$$\begin{aligned}
\epsilon_1 &= \frac{1}{E}(\sigma_1 - \nu(\sigma_2 + \sigma_3)) \\
\epsilon_2 &= \frac{1}{E}(\sigma_2 - \nu(\sigma_1 + \sigma_3)) \\
\epsilon_3 &= \frac{1}{E}(\sigma_3 - \nu(\sigma_1 + \sigma_2)).
\end{aligned} \tag{10}$$

The total elastic energy per unit volume can then be written as

$$\begin{aligned}
\text{strain energy/vol} = & \sum_i \sum_j \frac{1}{2}\sigma_{ij}\epsilon_{ij} \\
= & \frac{1}{2}(\sigma_1\epsilon_1 + \sigma_2\epsilon_2 + \sigma_3\epsilon_3) \\
= & \frac{1}{2}\left[\frac{1}{E}(\sigma_1 - \nu(\sigma_2 + \sigma_3)) + \frac{1}{E}(\sigma_2 - \nu(\sigma_1 + \sigma_3)) + \frac{1}{E}(\sigma_3 - \nu(\sigma_1 + \sigma_2))\right] \\
= & \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right].
\end{aligned} \tag{11}$$

We have to extract the distortional part of energy from this. We had decomposed the stress and strain matrices into hydrostatic and deviatoric (or distortional) parts in a previous lecture. We can use the hydrostatic part of the stress and strain to get the volumetric part of energy, which just causes change in the volume of the body. We can write

$$\text{Volumetric strain energy density} = \frac{1}{2}\sigma_{vol}\epsilon_{vol} \tag{12}$$

The volumetric stress is simply the first invariant of stress matrix while the volumetric strain equals the trace of the strain matrix. Thus

$$\begin{aligned}
\text{Volumetric strain energy} &= \frac{1}{2}\sigma_{vol}\epsilon_{vol} \\
&= \frac{1}{2}\left(\frac{\sigma_1+\sigma_2+\sigma_3}{3}\right) \times \frac{1}{E}\left(\epsilon_1+\epsilon_2+\epsilon_3\right) \\
&= \frac{1}{2}\left(\frac{\sigma_1+\sigma_2+\sigma_3}{3}\right) \times \frac{1}{E}\left(\sigma_1+\sigma_2+\sigma_3-2\nu(\sigma_1+\sigma_2+\sigma_3)\right) \\
&= \frac{1}{6E}(\sigma_1+\sigma_2+\sigma_3)(\sigma_1+\sigma_2+\sigma_3)(1-2\nu). \qquad (13)
\end{aligned}$$

Finally, the distortional energy becomes

$$\begin{aligned}
\text{Distortional energy} &= \text{total energy - volumetric strain energy} \\
&= \frac{1+\nu}{3E}(\sigma_1^2+\sigma_2^2+\sigma_3^2-\sigma_1\sigma_2-\sigma_2\sigma_3-\sigma_3\sigma_1)\,. \qquad (14)
\end{aligned}$$

One can obtain the critical value of distortion energy from simple tests like tension test or torsion test. In a simple tension test, σ_2 and σ_3 are zero and we only have σ_1. Thus, the distortional energy for a simple tension test equals

$$\text{Distortional energy} = \frac{1+\nu}{3E}\sigma_{11}^2 \qquad (15)$$

Thus, the critical value of distortional energy will correspond to the point where $\sigma_{11}=\sigma_{11}^*=\sigma_y$ (the tensile stress value when the body fails), i.e.,

$$\text{Critical distortional energy} = \frac{1+\nu}{3E}\sigma_y^2 \qquad (16)$$

For a general loading scenario, the following condition must be satisfied at each point in the body to avoid failure:

$$\frac{1+\nu}{3E}(\sigma_1^2+\sigma_2^2+\sigma_3^2-\sigma_1\sigma_2-\sigma_2\sigma_3-\sigma_3\sigma_1) < \frac{1+\nu}{3E}\sigma_y^2\,. \qquad (17)$$

We can similarly do a torsion test to get the critical distortional energy. The stress matrix for a torsion test is given in equation (6) and the Mohr's circle for this state of stress is drawn in Figure 4. From this, we can find the critical principal stress components as

$$\begin{aligned}
\lambda_1 &= \tau_{\theta z}^*, \\
\lambda_2 &= 0, \\
\lambda_3 &= -\tau_{\theta z}^*. \qquad (18)
\end{aligned}$$

The critical distortional energy from this simple torsion test thus becomes

$$
\begin{aligned}
&\frac{1+\nu}{3E}(\sigma_1^2+\sigma_2^2+\sigma_3^2-\sigma_1\sigma_2-\sigma_2\sigma_3-\sigma_3\sigma_1)\\
=&\ \frac{1+\nu}{3E}(\tau_{\theta z}^{*2}+(-\tau_{\theta z}^{*})^2-(-\tau_{\theta z}^{*})(\tau_{\theta z}^{*}))\\
=&\ \frac{1+\nu}{E}\tau_{\theta z}^{*2}
\end{aligned}
\quad (19)
$$

Thus, for a general loading scenario, the following condition must be satisfied at each point in the body to avoid failure:

$$
\frac{1+\nu}{3E}(\sigma_1^2+\sigma_2^2+\sigma_3^2-\sigma_1\sigma_2-\sigma_2\sigma_3-\sigma_3\sigma_1) < \frac{1+\nu}{E}\tau_{\theta z}^{*2} = \frac{1+\nu}{E}\tau_y^2. \quad (20)
$$

We can notice that the expression in the parentheses is proportional to the square of octahedral shear stress τ_{oct}. Therefore, the distortional energy theory and the octahedral shear stress theory are related.

5 Example (start time: 32:09)

Let us now discuss an example which demonstrates how we can use these theories to design a beam. Consider a circular beam which is subjected to equal and opposite torques *T* and bending moments *M* at its ends, as shown in Figure 5.

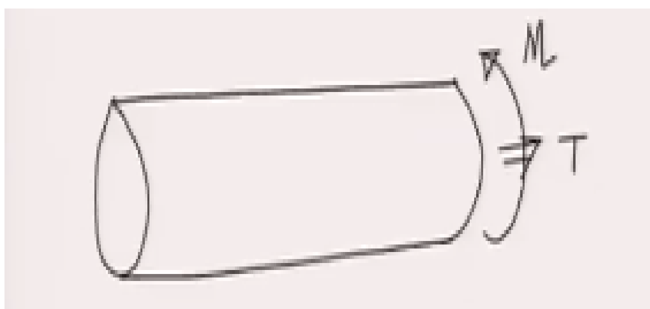

Figure 5: A circular beam, with its axis along *z* axis, is subjected to equal and opposite torque and bending moment at its ends.

We want to first find the state of stress corresponding to this loading in the cylindrical coordinate system. We can consider the effects of bending and torsion separately and then superimpose them. As the bending will be pure bending, we will only get non-zero σ_{zz} due it as

$$
\sigma_{zz} = \frac{-My}{I}. \quad (21)
$$

Due to torsion, only $\tau_{\theta z}$ and $\tau_{z\theta}$ will be non-zero and will be given by

$$
\tau_{\theta z} = \tau_{z\theta} = \frac{Tr}{J}. \quad (22)
$$

We now look at the cross-section of the beam as shown in Figure 6.

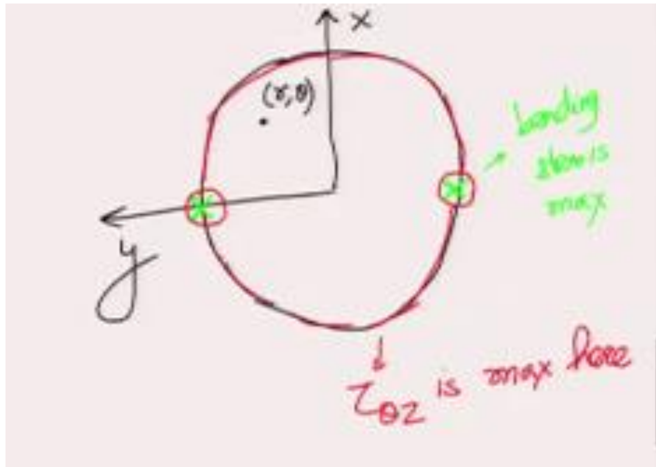

Figure 6: Cross-section of the beam shown in Figure 5.

The coordinates of a general point in the cross-section can be written as (r,θ). Thus, y coordinate of any point equals $r\sin\theta$. Substituting this in equation (21), we get

$$\sigma_{zz} = \frac{-M_x\, r\sin\theta}{I_{xx}}\,. \tag{23}$$

The state of stress obtained by superposition can be written as

$$[\underline{\underline{\sigma}}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{Tr}{J} \\ 0 & \frac{Tr}{J} & \frac{-M r\sin\theta}{I} \end{bmatrix}. \tag{24}$$

We can superpose the two phenomena because we are working in linear elasticity regime. We had also discussed that for circular cross-sections:

$$\begin{aligned} J &= I_{xx} + I_{yy} && \text{(using perpendicular axis theorem)} \\ &= 2I_{xx} && (\because I_{xx} = I_{yy}) \end{aligned} \tag{25}$$

To get the condition for failure, we can apply a suitable theory of failure. Let us think of applying maximum shear stress theory. We can find the maximum shear stress by drawing the Mohr's circle for the given state of stress. This can be done easily because $\underline{e}_r$ is already a principal plane. This Mohr's circle is shown in Figure 7.

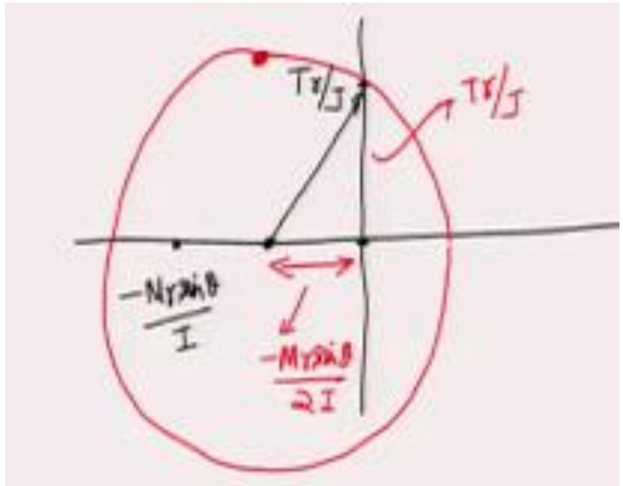

Figure 7: Mohr's circle for the state of stress given in equation (24).

The stress components $\sigma_{zz} = \frac{-Mr\sin\theta}{I}$ and $\sigma_{\theta\theta} = 0$ are marked on σ axis. The center of the circle is in the middle of these two. Similarly, $\tau_{\theta z} = \frac{Tr}{J}$ is marked on τ axis. The radius is the line joining this point and the center. With the center and radius known, the circle is drawn. From the Mohr's circle, we can see that the value of maximum shear stress is equal to the radius of the circle, i.e.,

$$\tau_{max} = \sqrt{\left(\frac{Tr}{J}\right)^2 + \left(\frac{Mr\sin\theta}{2I}\right)^2}$$

This value must be less than the critical value of shear stress (τ_y) to avoid failure in the beam. The body will fail first at the points where the shear stress is maximum. Both the terms in the square root are proportional to the radius. So, the shear stress will be maximum on the outer periphery (lateral surface) of the cylinder where both these terms attain their maximum values. So, we can substitute $r = R$ in both the terms. Also, the maximum value of sinθ will be 1 at $\theta = 90^\circ$. This is true for points on the y axis. When we find the points satisfying both the condition, i.e., lying on the periphery as well as the y axis, we get two points. These points are shown in green in Figure 8.

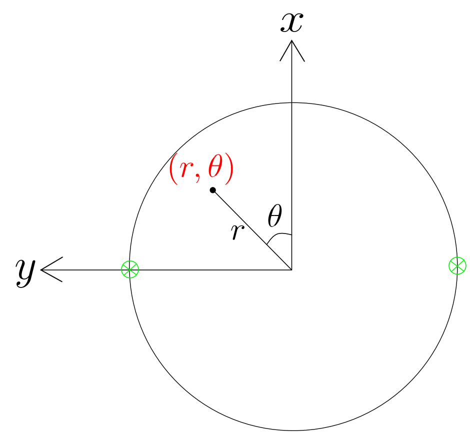

Figure 8: The cross-section of the cylinder with the points experiencing maximum shear stress shown in green

If we can ensure shear stress at these points is less than the critical value, the body will not fail. Thus, we can replace $2I = J$, $r = R$ and $\sin\theta = 1$ to get the following condition:

$$\sqrt{\left(\frac{TR}{J}\right)^2 + \left(\frac{MR}{J}\right)^2} < \tau_y \tag{26}$$

Any combination of bending moment M and torque T for which the LHS of the above equation is less than τ_y is a safe loading condition for operation. But, in reality, we always design considering some factor of safety for incorporating unexpected circumstances. Suppose the factor of safety to be used in designing is N which can take any value like 2, 3 or 2.5 but it has to be greater than 1. If the body fails at critical torque T^* and critical bending moment M^*, the operating torque T and operating bending moment M should be such that

$$\begin{aligned} TN &\leq T^* \\ MN &\leq M^* \end{aligned} \tag{27}$$

Thus, substituting T by TN and M by MN in equation (26), we get

$$\begin{aligned} &\sqrt{\frac{T^2N^2R^2}{J^2} + \frac{M^2N^2R^2}{J^2}} < \tau_y \\ \text{or,} \quad &\sqrt{\left(\frac{TR}{J}\right)^2 + \left(\frac{MR}{J}\right)^2} < \frac{\tau_y}{N} \end{aligned} \tag{28}$$

The factor of safety finally reflects on the RHS in this particular case which limits our operational value of torque and moment while designing as we don't want the operational load to reach anywhere near the critical load.