Solid Mechanics
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Lecture - 30
Energy Methods (contd...)

Hello everyone! Welcome to Lecture 30! We will continue the discussion on energy methods.

1 Castigliano's First Theorem (start time: 00:26)

We had derived the expression for energy stored in a deformable body as

$$E = \sum_i \frac{1}{2} F_i \delta_i = \sum_i \frac{1}{2} F_i \Big(\sum_j k_{ij} F_j \Big) = \sum_i \sum_j \frac{1}{2} k_{ij} F_i F_j \,. \tag{1}$$

Here, F_i represents the ith generalized force (i.e., it can be a force or moment) and δ_i represents the corresponding generalized displacement (i.e., it can be a displacement or rotation). Let us take the first derivative of this expression with respect to the kth generalized force:

$$\begin{aligned} \frac{\partial E}{\partial F_k} &= \sum_i \sum_j \frac{1}{2} k_{ij} \left(\frac{\partial F_i}{\partial F_k} F_j + F_i \frac{\partial F_j}{\partial F_k} \right) \\ &= \sum_i \sum_j \frac{1}{2} k_{ij} (\delta_{ik} F_j + F_i \delta_{jk}) = \frac{1}{2} \Big(\sum_j k_{kj} F_j + \sum_i k_{ik} F_i \Big). \end{aligned} \tag{2}$$

Upon further replacing the summation index j with i, we get

$$\begin{aligned} \frac{\partial E}{\partial F_k} = \frac{1}{2} \sum_i (k_{ki} F_i + k_{ik} F_i) &= \sum_i k_{ki} F_i \text{ (using reciprocal relation)} \\ &= \delta_k . \end{aligned} \tag{3}$$

We have thus derived that the corresponding displacement δ_k is the derivative of total energy with respect to the corresponding force F_k. This is called the Castigliano's first theorem. If we can write the energy in terms of the applied forces/moments, we can apply this theorem to find the corresponding displacement/rotation. We need not solve any differential equation. Of course, we need to know all the influence coefficients.

2 Deriving expression for energy stored in a beam in terms of internal contact force and moment which acts in the beam's cross-section (start time: 06:33)

Writing the stored energy in terms of the externally applied generalized forces is crucial if we want to use the Castigliano's first theorem. Let us see how this can be achieved. Think of a general three-dimensional body which is clamped at some points as shown in Figure 1.

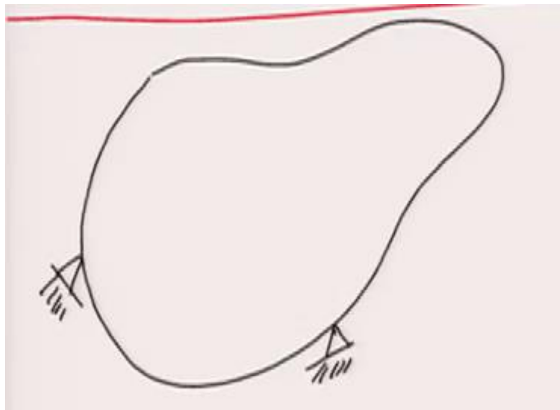

Figure 1: A general three-dimensional body clamped at two points

We want to know the energy stored in this body. For linear stress-strain relation, the energy stored becomes

$$E = \sum_i \sum_j \iiint_V \frac{1}{2}\sigma_{ij}\epsilon_{ij} dV. \tag{4}$$

Let us simplify this further in the context of beams which we will be mostly dealing with. Consider the beam shown in Figure 2 having cross-section Ω and length *L*.

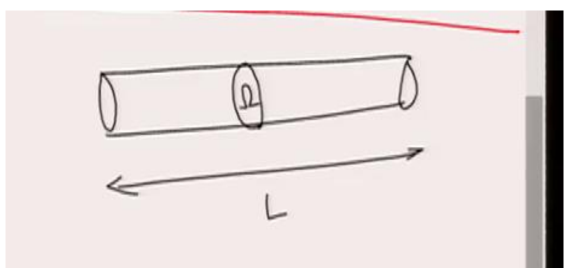

Figure 2: A beam having cross-section Ω and length *L*

Equation (4) can be rewritten as follows for a beam:

$$E = \int_0^L ds \Bigg(\sum_i \sum_j \iint_\Omega \frac{1}{2}\sigma_{ij}\epsilon_{ij}\, d\Omega \Bigg) \tag{5}$$

The integral in the parentheses above is called the cross-sectional strain energy as it is obtained by the integration of three-dimensional strain energy density over the beam's cross-section. It is further integrated over the length of the beam to obtain the total energy. A beam can deform in several modes: it can undergo bending, stretching, shearing, twisting etc. For each of these deformations, the energy stored in the beam's cross-section has different mathematical form. Let's discuss them one by one.

2.1 Axial extensional energy (start time: 09:43)

Consider a beam which is being stretched by the application of an axial load *P* as shown in Figure 3.

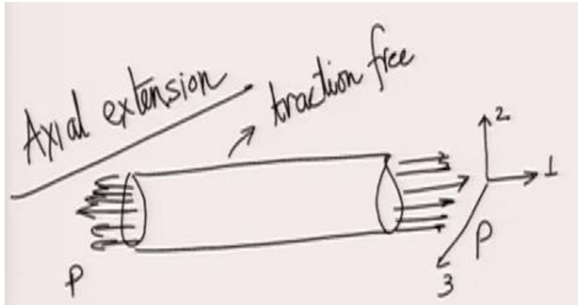

Figure 3: A beam being stretched by the application of load *P* on both its ends

The total load *P* can be assumed to be uniformly distributed over the cross-section Ω. To find the energy stored in the cross-section, we need to first find the stress and strain components. We can recall from our previous lecture that, in case of axial extension of a beam, the cross-section is allowed to relax completely. So, the stress matrix has only one non-zero stress component, i.e.,

$$[\underline{\sigma}] = \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{6}$$

Using three-dimensional Hooke's Law, we can then write

$$\epsilon_{11} = \frac{1}{E}(\sigma_{11} - \nu(\sigma_{22} + \sigma_{33})) = \frac{\sigma_{11}}{E}. \tag{7}$$

The cross-sectional energy will thus become

$$\sum_i \sum_j \iint_\Omega \frac{1}{2}\sigma_{ij}\epsilon_{ij} d\Omega \;=\; \iint_\Omega \frac{1}{2}\sigma_{11}\epsilon_{11} d\Omega = \iint_\Omega \frac{\sigma_{11}^2}{2E} d\Omega = \iint_\Omega \frac{P^2}{2EA^2} d\Omega\,. \tag{8}$$

In the last step, we assumed that the axial load *P* is uniformly distributed over the cross-section. Upon final integration over the cross-section, we get

$$E_{stretching} = \frac{P^2}{2EA}. \tag{9}$$

This can be further integrated over the length of the beam to get the total energy due to stretching of a beam.

2.2 Bending energy (start time: 15:05)

Figure 4 shows a beam which bends due to the application of bending moment *M*.

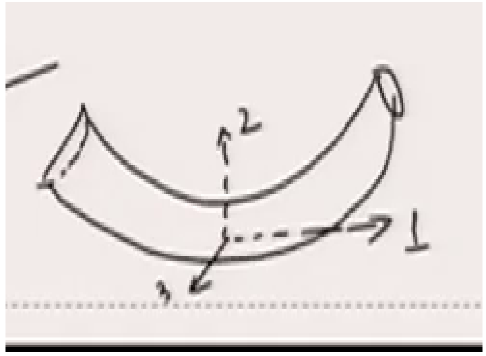

Figure 4: A beam under pure bending.

We can recall that the state of stress for such a case is given by

$$[\underline{\sigma}] = \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{10}$$

This is because the cross-section again relaxes freely in the transverse direction in case of pure bending. We had also derived that

$$\begin{aligned} \kappa &= \frac{M_z}{EI_{zz}}, \\ \epsilon_{11} &= -\kappa y = \frac{-M_z y}{EI_{zz}}, \\ \sigma_{11} &= -E\kappa y = \frac{-M_z y}{I_{zz}}. \end{aligned} \tag{11}$$

Here, κ denotes the bending curvature and y denotes distance from the neutral axis. Thus, the cross-sectional energy will be

$$\sum_i \sum_j \iint_\Omega \frac{1}{2}\sigma_{ij}\epsilon_{ij} d\Omega = \iint_\Omega \frac{1}{2}\sigma_{11}\epsilon_{11} d\Omega = \iint_\Omega \frac{M_z^2}{2EI_{zz}^2} y^2 d\Omega \,. \tag{12}$$

The final integration over the cross-section yields

$$E_{bending} = \frac{M_z^2}{2EI_{zz}} \,. \tag{13}$$

If we compare this form of bending energy with the axial strain energy form (13), we can see that the form of both the energies is exactly the same: the axial force *P* is replaced by bending moment *M* while the stretching stiffness *EA* is replaced by bending stiffness EI_{zz}.

2.3 Torsional energy (start time: 20:25)

In case of torsion, the state of stress in cylindrical coordinate system is

$$[\underline{\sigma}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \tau_{\theta z} \\ 0 & \tau_{\theta z} & 0 \end{bmatrix}. \tag{14}$$

We had also derived earlier that

$$\begin{aligned}\kappa &= \frac{T}{GJ},\\ \gamma_{\theta z} &= \kappa r = \frac{Tr}{GJ},\\ \tau_{\theta z} &= G\kappa r = \frac{Tr}{J}.\end{aligned} \tag{15}$$

Here *κ* denotes twist, *r* denotes radial coordinate relative to the cross-section's centroid and *T* denotes the internal torque or twisting moment. Thus, the cross-sectional energy will be

$$\sum_i \sum_j \iint_\Omega \frac{1}{2}\sigma_{ij}\epsilon_{ij} d\Omega = \iint_\Omega \frac{1}{2}[\tau_{\theta z}\epsilon_{\theta z} + \tau_{z\theta}\epsilon_{z\theta}] d\Omega = \iint_\Omega \frac{1}{2}\tau_{\theta z}\gamma_{\theta z} d\Omega = \iint_\Omega \frac{T^2}{2GJ^2} r^2 d\Omega. \tag{16}$$

The final integration over the cross-section yields

$$E_{twisting} = \frac{T^2}{2GJ}. \tag{17}$$

2.4 Shear energy due to transverse load (start time: 21:05)

We learnt earlier that the presence of transverse load leads to non-uniform bending of beams which also generates transverse shear stress in beams - this shear stress points in the direction of applied transverse load and is different from the one due to torsion. In torsion of beams, total shear force due to shear stress $\tau_{\theta z}$ is zero whereas the resultant shear force due to shear stress under transverse loading is obviously non-zero. Proceeding along the similar lines as as earlier, the total shear energy due to transverse load is

$$E_{shearing} = \frac{V^2}{2kGA}. \tag{18}$$

Here *V* represents shear force, *GA* shearing stiffness and *k* shear correction factor.

2.5 Total energy stored in the beam (start time: 23:00)

When we apply a general loading on a beam, force and moment both act on the cross-section (see Figure 5).

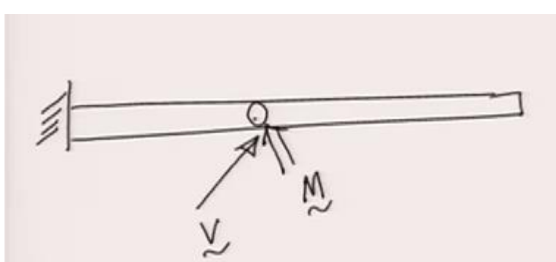

Figure 5: A beam clamped at one end and subjected to general loading in the cross-section

The internal force $\underline{V}$ in the cross-section can be decomposed into its three components as

$$\left[\underline{V}\right] = \left[V_x \quad V_y \quad V_z\right]. \tag{19}$$

The component along the axis V_x is axial force P. The other components are shear forces. We can similarly resolve the internal moment into three components as

$$\left[\underline{M}\right] = \left[M_x \quad M_y \quad M_z\right]. \tag{20}$$

The component along the axis M_x is the torque T. The other two components are bending moments. We derived energy due to each of the six components separately. As we are working in the regime of linear elasticity, the energy when all of these are present together can be obtained using principle of superposition. This means that the total energy stored in the beam in the general case can be written as

$$E = \int_0^L \left[\frac{P(x)^2}{2EA} + \frac{M_y(x)^2}{2EI_{zz}} + \frac{M_z(x)^2}{2EI_{zz}} + \frac{T(x)^2}{2GJ} + \frac{V_y(x)^2}{2kGA} + \frac{V_z(x)^2}{2kGA}\right] dx. \tag{21}$$

The first three terms here are due to normal strain while the last three terms are due to shear strain.

3 Verification of reciprocal relation (start time: 26:39)

Let us verify the reciprocal relation through an example. Think of a beam which is clamped at one end and is being acted upon by a force P at a distance a from the clamped end (see Figure 6).

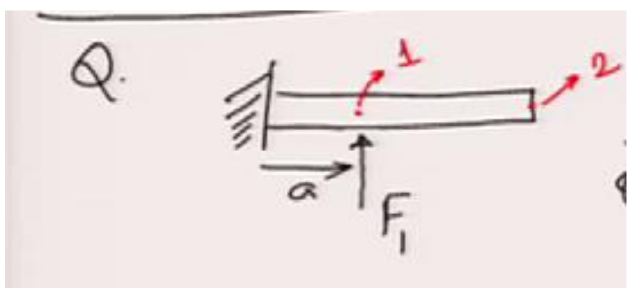

Figure 6: A force P acts on a beam at a distance a from the clamped end

Our goal is to find the rotation of the cross-section at $x = L$, i.e., the tip rotation $\theta(L)$. We can solve this problem using beam theory. Let us solve it by EBT. We denote the point at which P is acting as point 1 and the free end as point 2. According to EBT:

$$EI\frac{d^2y}{dx^2} = M(x). \tag{22}$$

The bending moment profile can be found by cutting a section in the beam and analyzing one of its portions. It turns out to be

$$M = 0 \qquad (x > a)$$

$$= F_1(a - x) \qquad (x < a) \tag{23}$$

Here we are writing the force P as F_1 since it is acting at point 1. Substituting the moment profile in equation (22) gives:

$$\begin{aligned} EI\frac{d^2y}{dx^2} &= 0 && (x > a) \\ &= F_1(a-x) && (x < a) \end{aligned} \tag{24}$$

Integrating the expression for $x < a$ twice gives

$$EIy = F_1 a\frac{x^2}{2} - F_1\frac{x^3}{6} + C_1 x + C_2 \, . \tag{25}$$

The boundary condition at the clamped end is

$$y(0) = 0, \quad \theta(0) = \frac{dy}{dx}(0) = 0 \tag{26}$$

which, when applied, gives

$$C_1 = C_2 = 0. \tag{27}$$

Thus, we get

$$y = \frac{F_1}{EI}\left(a\frac{x^2}{2} - \frac{x^3}{6}\right) \tag{28}$$

and its derivative gives cross-sectional rotation θ, i.e.,

$$\theta = \frac{dy}{dx} = \frac{F_1}{EI}\left(ax - \frac{x^2}{2}\right). \tag{29}$$

The above two relations hold for $x \leq a$ only. In order to obtain expression for $x > a$, we set $M = 0$ in EBT theory, i.e.,

$$\frac{d^2y}{dx^2} = 0 \Rightarrow \frac{d\theta}{dx} = 0 \, . \tag{30}$$

Thus, θ is constant in the segment from $x = a$ to $x = L$ which implies

$$\theta(L) = \theta(a) = \frac{F_1 a^2}{2EI} . \tag{31}$$

The second equality is obtained upon using continuity of displacement and rotation at $x = a$ and setting $x = a$ in equation (29). The above equation gives us influence coefficient relating θ at point 2 with force at point 1, i.e.,

$$\theta_2 = \underbrace{\frac{a^2}{2EI}}_{k_{21}} F_1 \tag{32}$$

We can also find k_{12} to check whether it comes out to be equal to k_{21}. To obtain k_{12}, we apply a bending moment M_2 at point 2 and measure the deflection at point 1 (see Figure 7).

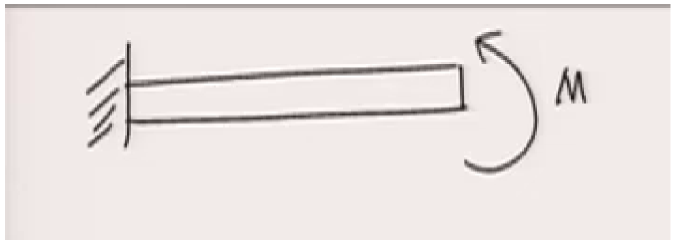

Figure 7: A bending moment *M* applied at point 2 instead of force *P* being applied at point 1

In this case, bending moment will be constant all along the length of the beam and equal the externally applied moment M_2. Substituting this in equation (22), we get

$$EI\frac{d^2y}{dx^2} = M(x) = M_2$$
$$\Rightarrow \quad y = \frac{1}{EI}\left[\frac{M_2\; x^2}{2} + C_1 x + C_2\right] \tag{33}$$

The boundary conditions at the clamped end are the same as before (see (26)), which again yield $C_1 = C_2 = 0$. Thus, we get

$$y = \frac{M_2\; x^2}{2EI} \tag{34}$$

using which the deflection at point 1 turns out to be

$$y_1 = y(a) = \underbrace{\frac{a^2}{2EI}}_{k_{12}} M_2\,. \tag{35}$$

We can see that k_{12} is the same as k_{21}. Thus, we are able to verify the reciprocal relation. As we can see, the pure bending problem is much easier to solve than the problem of finding the tip rotation. If we are asked to solve for tip rotation, we actually need k_{21}. But if we can find k_{12} instead in a simple way (as we saw above), we can then use reciprocal relation to set it equal to k_{21}. Thus, reciprocal relation turns out to be very useful in obtaining solution to a difficult problem by solving a simpler problem.

4 Example 1 (start time: 35:10)

Consider a beam clamped at one end. A transverse load V is applied at the tip of the beam as shown in Figure 8.

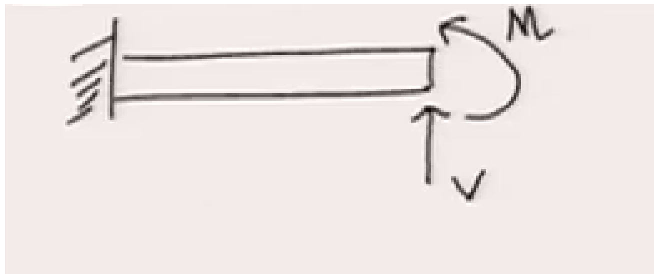

Figure 8: A cantilever beam subjected to a transverse load V and a dummy moment M at the free end

We want to find the rotation of the cross-section at the tip of the beam using energy method. The corresponding displacement of shear force is tip deflection. In order to find the rotation at the tip, a bending moment would also be needed at the tip: the unknown $\theta(L)$ is the corresponding displacement of bending moment $M(L)$. So, the steps needed to be followed are:

1. Apply a dummy bending moment M at the tip.

2. Obtain the energy of entire beam in terms of (V,M).

3. Apply Castigliano's first theorem, i.e., obtain the partial derivative $\frac{\partial E}{\partial M}$ and equate it to the θ_L.

4. Finally, set M = 0 in the expression of θ_L since in the original problem, no bending moment exists at the tip).

To obtain the beam's energy, we first need shear force and bending moment profiles along the length of the beam. If we cut a section in the beam at a distance x from the clamped end and balance forces and moments in the right part of the cut beam, we will find that

$$V_y(x) = V, \qquad M_z(x) = M + V(L - x). \tag{36}$$

The axial force P, shear force V_z, torque T and bending moment M_y are all zero here. Using these in equation (21), we get

$$E = \int_0^L \left[\frac{V_y^2}{2kGA} + \frac{M_z^2}{2EI_{zz}} \right] dx = \int_0^L \left[\frac{V^2}{2kGA} + \frac{(M + V(L-x))^2}{2EI_{zz}} \right] dx. \tag{37}$$

We have thus been able to write the energy stored in the beam in terms of the externally applied loads (V and M). To obtain θ_L, we now apply Castigliano's first theorem, i.e.,

$$\theta_L = \frac{\partial E}{\partial M} = \int_0^L \left[\frac{2V\frac{\partial V}{\partial M}}{2kGA} + \frac{2(M + V(L-x))\frac{\partial}{\partial M}(M + V(L-x))}{2EI_{zz}} \right] dx. \tag{38}$$

As V and M are independent, $\frac{\partial V}{\partial M}$ and $\frac{\partial M}{\partial V}$ would be zero which leads to

$$\theta_L = \int_0^L \frac{2(M + V(L-x))}{2EI_{zz}} dx \tag{39}$$

We can finally set M = 0 to get

$$\theta_L = \int_0^L \frac{2(0 + V(L-x))}{2EI_{zz}} dx = \frac{V}{EI_{zz}} \int_0^L (L-x) dx = \frac{V}{EI_{zz}} \left(Lx - \frac{x^2}{2} \right) \Bigg|_0^L = \frac{VL^2}{2EI_{zz}}. \tag{40}$$

5 Example 2 (start time: 44:08)

Consider a beam clamped at one end. The other end of the beam has a roller support.[1] A transverse force P is applied at a distance a from the clamped end as shown in Figure 9.

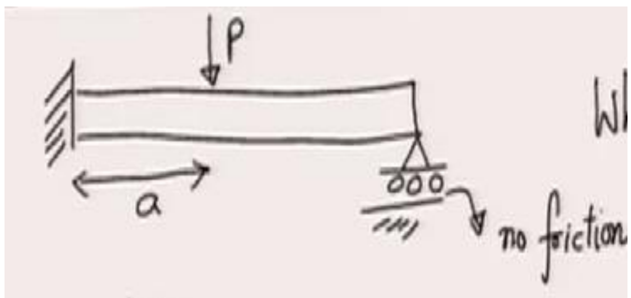

Figure 9: A transverse load is applied on a beam which is clamped at one end and having roller support at the other end.

We want to find the reaction from the roller support. The contact between the roller support and the beam is a line contact, so this support cannot exert any moment on the beam. Hence, the roller support only exerts a vertical reaction force (say R) on the beam which we need to obtain. We can solve this problem using EBT/TBT but let us see how energy methods can be applied. We first draw the beam with the unknown tip vertical force R replacing the roller support as shown in Figure 10.

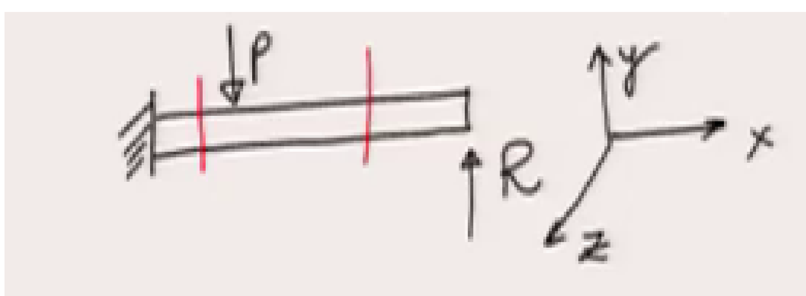

Figure 10: The roller support is replaced by a vertical reaction force

[1] A roller support can move freely in the horizontal direction and hence does not exert any horizontal force on the beam

To obtain the energy of the beam in terms of external forces (P and R), we need to first find the shear force and bending moment profiles. We cut two sections, one at $x > a$ and another at $x < a$. The free body diagrams of the cut portions of the beam are shown in Figure 11.

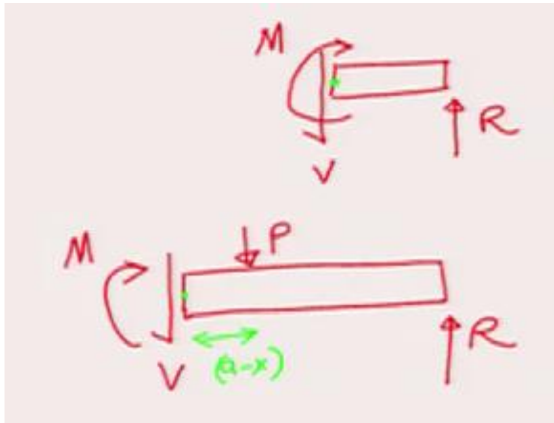

Figure 11: Free body diagrams of the cut parts of the beam

Force balance of the two parts yields

$$\begin{aligned} V_y(x) &= R \qquad (x > a) \\ &= R - P \qquad (x < a). \end{aligned} \tag{41}$$

Moment balance of the first part (see Figure 11) about the centroid of its leftmost cross-section yields

$$-M_z + R(L - x) = 0 \Rightarrow M_z = R(L - x) \qquad (x > a). \tag{42}$$

Similarly, moment balance of the second part in Figure 11 about the centroid of its leftmost cross-section yields

$$-M_z + R(L - x) - P(a - x) = 0 \Rightarrow M_z = R(L - x) - P(a - x) \qquad (x < a). \tag{43}$$

We can now write an expression for the energy of the beam as follows:

$$\begin{aligned} E &= \int_0^L \left[\frac{V_y^2}{2kGA} + \frac{M_z^2}{2EI_{zz}}\right] dx \\ &= \int_0^a \left[\frac{V_y^2}{2kGA} + \frac{M_z^2}{2EI_{zz}}\right] dx + \int_a^L \left[\frac{V_y^2}{2kGA} + \frac{M_z^2}{2EI_{zz}}\right] dx \\ &= \int_0^a \left[\frac{(R-P)^2}{2kGA} + \frac{[R(L-x) - P(a-x)]^2}{2EI_{zz}}\right] dx + \int_a^L \left[\frac{R^2}{2kGA} + \frac{R^2(L-x)^2}{2EI_{zz}}\right] dx. \end{aligned} \tag{44}$$

Notice that the corresponding displacement of the unknown *R* is tip displacement in *y* direction. This displacement must be zero because the roller support restricts any vertical movement. So, we can use Castigliano's first theorem to write

$$\frac{\partial E}{\partial R} = 0. \tag{45}$$

Taking the derivative of the energy expression (44) with respect to *R* yields

$$\begin{aligned} &\int_0^a \left[\frac{2(R-P)}{2kGA} + \frac{2[R(L-x)-P(a-x)](L-x)}{2EI_{zz}}\right]dx + \int_a^L \left[\frac{2R}{2kGA} + \frac{2R(L-x)^2}{2EI_{zz}}\right]dx = 0 \\ &\Rightarrow R\left[\frac{L}{kGA} + \frac{1}{EI_{zz}}\int_0^L (L-x)^2 dx\right] = P\left[\frac{a}{kGA} + \frac{1}{EI_{zz}}\int_0^a (a-x)(L-x)dx\right] = 0. \end{aligned} \tag{46}$$

We can solve the above equation for the unknown reaction force *R*.

6 Example 3 (start time: 57:41)

We will now discuss a problem which is difficult to solve using EBT/TBT. Consider a ring of radius *R* which is subjected to equal and opposite forces along its vertical diameter line as shown in Figure 12a.

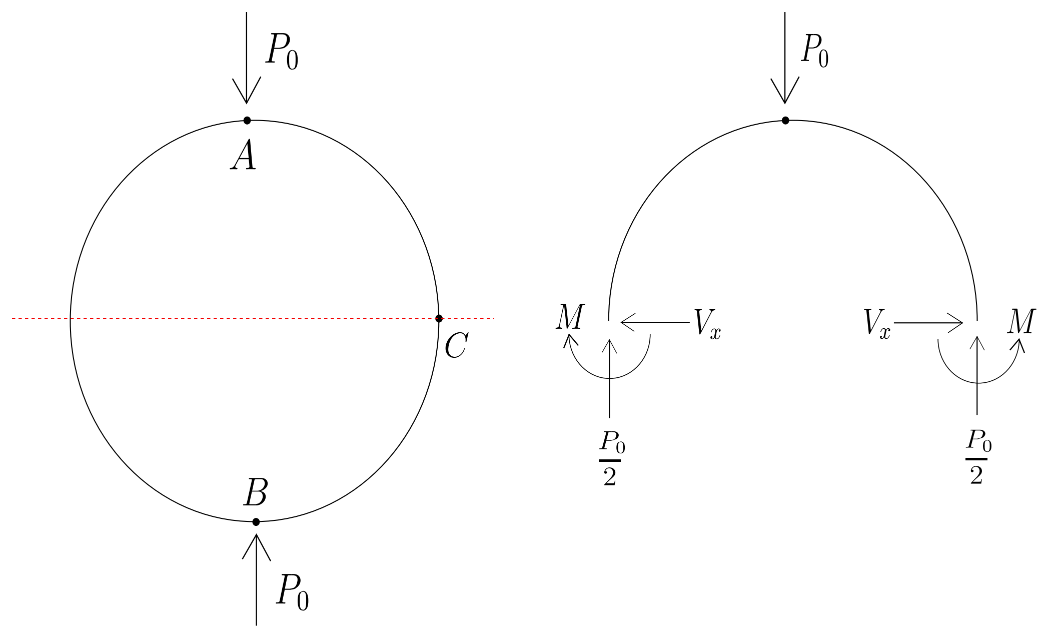

Figure 12: (a) Equal and opposite forces of magnitude P_0 are applied on a circular ring along its diameter (b) Free body diagram of the upper half of the ring

The magnitude of the force applied is P_0. The points at which the forces act are labelled *A* and *B*. Another point *C* is marked as shown in Figure 12a. We need to find answers to the following three questions:

- By how much do points A and B get closer?
- By how much does point C move outward?
- What is the internal bending moment at point C?

As the beam is curved and the beam theories developed in this course are applicable only for initially straight beams, we cannot use them directly here. Let us divide the ring into two equal parts about a diameter line passing through C and draw the free body diagram of the upper half as shown in Figure 12b. The force P_0 acts at the top. From symmetry, we can easily see that the vertical force that the bottom part applies on the top part is $\frac{P_0}{2}$ on both the ends. There will also be force in the horizontal direction V_x and bending moment M at the two ends. If we apply force balance and moment balance, we will not be able to solve for V_x and M. Here, energy method turns out to be a very useful and powerful tool. First of all, let us think of the shape that the deformed ring would take (see Figure 13).

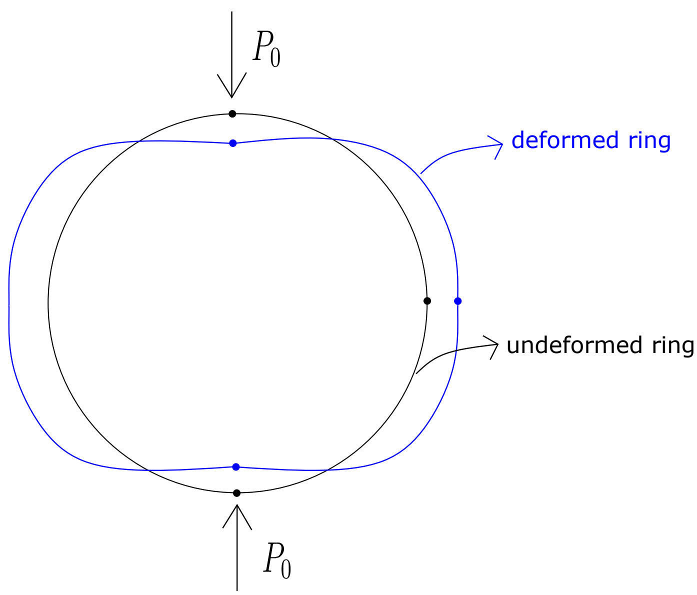

Figure 13: The undeformed and deformed configurations of the ring

We can note that due to symmetry, the cross-section at point A will not rotate. The point A also does not displace horizontally. Similarly, the cross-section at point C does not rotate. Neither does it displace vertically.

Let us focus on a quarter of the ring (from A to C: see Figure 14) with point A being clamped.

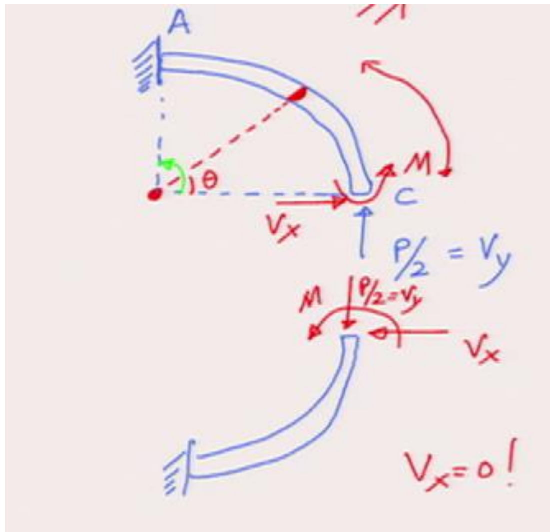

Figure 14: Two quarters of the ring being analyzed with the top and bottom points assumed as being clamped

The clamping of point A implies that all the displacements will be relative to the frame of point A: point C, e.g., will now have vertical displacement equal and opposite to that of point A in the original full ring problem whereas horizontal displacement of point C will be unaffected. We have also drawn the lower quarter similarly with point B being clamped. The upper quarter has a shear force at C acting towards the right while the lower part has a shear force at C acting towards the left (by Newton's third law). But, due to symmetry, V_x should act either towards the center or away from the center for both the quarters otherwise the deformation of the two quarters will be different violating symmetry. The only way both symmetry and Newton's third law are satisfied is when $V_x = 0$. Also note that the displacement of point *C* in the outward direction is the corresponding displacement of V_x. Thus, it will be equal to $\frac{\partial E}{\partial V_x}$ using Castigliano's first theorem. We also need to find bending moment at point *C* whose corresponding displacement is the rotation of the cross-section at point *C*, which is zero. So, we can again apply Castigliano's first theorem to get the following equation:

$$\frac{\partial E}{\partial M} = 0 \tag{47}$$

solving which we can obtain the unknown bending moment *M*.

Let us now find the energy of the ring's quarter for which we need the bending moment, shear force and axial force profile. For this purpose, we cut a section in the upper quarter of the ring at an angular distance θ from point *C* and draw the free body diagram of its lower cut part as shown in Figure 15.

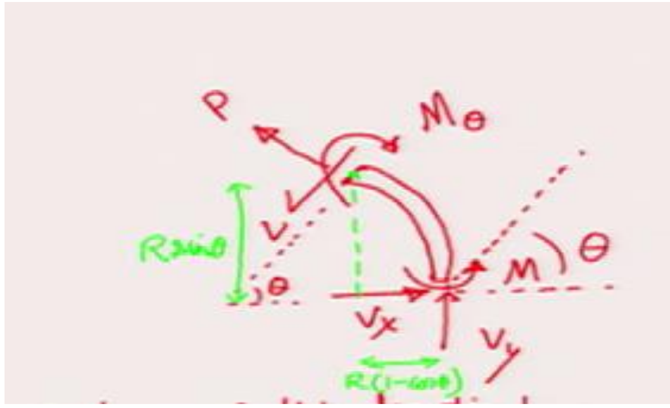

Figure 15: Free body diagram of the lower part of the ring's quarter (A→C)

The shear force (drawn parallel to the cross-section) and axial force (along the cross-section normal) at the cut section are inclined because the cross-section there is inclined. Let us denote the axial force by P which is our generic symbol to denote axial force: it should not be confused with the external force P_0 that is applied on the full ring at points A and B. Force balance of the cut part along the direction parallel to shear force V gives

$$-V + V_x \cos\theta + V_y \sin\theta = 0 \Rightarrow V = V_x \cos\theta + V_y \sin\theta. \quad (48)$$

Similarly, force balance along the direction parallel to axial force P gives

$$P - V_x \sin\theta + V_y \cos\theta = 0 \Rightarrow P = V_x \sin\theta - V_y \cos\theta. \quad (49)$$

Finally, moment balance about the centroid of the cut section gives

$$-M_\theta + M + V_x R \sin\theta + V_y R(1 - \cos\theta) = 0 \Rightarrow M_\theta = M + V_x R \sin\theta + V_y R(1 - \cos\theta). \quad (50)$$

Using the above expressions, the energy stored in the ring's quarter becomes

$$\begin{aligned} E &= \int_0^L \left[\frac{P(x)^2}{2EA} + \frac{M_z(x)^2}{2EI_{zz}} + \frac{V_y(x)^2}{2kGA}\right] dx \\ &= \int_0^L \left[\frac{[V_x \sin\theta - V_y \cos\theta]^2}{2EA} + \frac{[M + V_x R \sin\theta + V_y R(1-\cos\theta)]^2}{2EI} + \frac{[V_x \cos\theta + V_y \sin\theta]^2}{2kGA}\right] dx \end{aligned} \quad (51)$$

which can also be converted to θ integral with limits going from $\theta = 0$ to $\theta = \frac{\pi}{2}$. The length dx will simply be $Rd\theta$. Thus, we get

$$E = \int_0^{\frac{\pi}{2}} \left[\frac{[V_x \sin\theta - V_y \cos\theta]^2}{2EA} + \frac{[M + V_x R \sin\theta + V_y R(1-\cos\theta)]^2}{2EI} + \frac{[V_x \cos\theta + V_y \sin\theta]^2}{2kGA}\right] R d\theta \quad (52)$$

To get the bending moment at C, we can write

$$\frac{\partial E}{\partial M} = 0 \Rightarrow \int_0^{\frac{\pi}{2}} \left[\frac{2[M + V_x R \sin\theta + V_y R(1 - \cos\theta)]1}{2EI} \right] R d\theta = 0 \,. \tag{53}$$

We have already found that $V_x = 0$ and $V_y = \frac{P_0}{2}$ substituting which in the above equation, we get

$$\int_0^{\frac{\pi}{2}} \frac{M + \frac{P_0 R}{2}(1 - \cos\theta)}{EI} R d\theta = 0 \Rightarrow M = -\frac{P_0 R}{2}\left(1 - \frac{2}{\pi}\right). \tag{54}$$

Similarly, the horizontal displacement of point *C* is given by

$$\delta_C = \frac{\partial E}{\partial V_x} \,. \tag{55}$$

We can make an approximation here that the energy contribution due to shear force and axial force is insignificant when compared to contributions due to bending moment and torque. This is a good approximation whenever the beam is slender (i.e., the aspect ratio is large). Thus, we can drop the axial force and shear force energies from the expression of energy which leads to

$$\delta_C = \int_0^{\frac{\pi}{2}} \left[\frac{2[M + V_x R \sin\theta + V_y R(1 - \cos\theta)] R \sin\theta}{2EI} \right] R d\theta \tag{56}$$

Finally, we have to substitute $V_x = 0,\, V_y = -\frac{P_0}{2}$ and bending moment *M* from equation (54) in the above equation to obtain δ_C. We also need to get the deflection of points *A* and *B*. We can observe that the amount by which *A* and *B* get closer is double the amount by which *A* and *C* get closer along the vertical line. Also remember, we had clamped the quarter ring at point *A*. The vertical displacement of *C* in our reference frame will be $\frac{\partial E}{\partial V_y}$. The amount by which *A* and *B* get closer will just be double of this value. Thus, we have been able to solve the complete problem using energy methods demonstrating its usefulness.

This closes our discussion on energy methods and we will start with the theories of failure in the next lecture.