Solid Mechanics
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Lecture - 3
Stress Tensor and its Matrix Representation

In the previous lecture, we had learnt about traction vector and how we could find it on any arbitrary plane. In this lecture, we will learn about stress tensor, its matrix representation and its physical meaning.

1 Traction vector (start time: 00:35)

We had found the formula for traction at point $\underline{x}$ on an arbitrary plane with normal $\underline{n}$ (shown in Figure 1) as:

$$\underline{t}^n = \sum_{i=1}^{3} \underline{t}^i (\underline{n} \cdot \underline{e}_i) \tag{1}$$

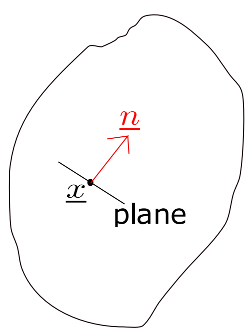

Figure 1: A plane with normal $\underline{n}$ at point $\underline{x}$

But physically, this $\underline{t}^n$, the traction on the plane with normal $\underline{n}$, has to be the same irrespective of what three planes were used to find out this traction, i.e.,

$$\underline{t}^n = \sum_{i=1}^{3} \underline{t}^i (\underline{n} \cdot \underline{e}_i) = \sum_{i=1}^{3} \underline{t}^{\hat{i}} (\underline{n} \cdot \underline{\hat{e}}_i) \tag{2}$$

In the first case, the planes used have normals along $\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$ whereas in the second case, the planes used have normals along $\underline{\hat{e}}_1$, $\underline{\hat{e}}_2$, $\underline{\hat{e}}_3$.

Result : The formula for $\underline{t}^n$ is independent of what three planes are used.

2 Stress Tensor (start time: 05:24)

We can write equation (2) in a slightly different way using the commutative property of dot product:

$$\underline{t}^n = \sum_{i=1}^{3} \underline{t}^i (\underline{n} \cdot \underline{e}_i) = \sum_{i=1}^{3} \underline{t}^i (\underline{e}_i \cdot \underline{n}) \tag{3}$$

A vector is represented as a column and dot product ($\underline{a} \cdot \underline{b}$) in matrix form is given as $[\underline{a}]^T[\underline{b}]$ (derived in lecture 1). So, writing a general term of this summation in matrix form:

$$\underset{\underset{\underline{t}_i}{\downarrow}}{\begin{bmatrix} \\ \\ \end{bmatrix}_{3\times 1}} \left(\underset{\underset{\underline{e}_i^T}{\downarrow}}{\begin{bmatrix} \quad \end{bmatrix}_{1\times 3}} \underset{\underset{\underline{n}}{\downarrow}}{\begin{bmatrix} \\ \\ \end{bmatrix}_{3\times 1}} \right) \tag{4}$$

As matrix multiplication is associative, we can also write this as :

$$\underbrace{\left(\underset{\underset{\underline{t}_i}{\downarrow}}{\begin{bmatrix} \\ \\ \end{bmatrix}_{3\times 1}} \underset{\underset{\underline{e}_i^T}{\downarrow}}{\begin{bmatrix} \quad \end{bmatrix}_{1\times 3}} \right)}_{\text{tensor product}} \underset{\underset{\underline{n}}{\downarrow}}{\begin{bmatrix} \\ \\ \end{bmatrix}_{3\times 1}} \tag{5}$$

We know that $[\underline{a}][\underline{b}]^T$ was the matrix representation for tensor product ($\underline{a} \otimes \underline{b}$). We can check that the dimensions of the overall product remain same as the tensor product gives a 3×3 matrix and that multiplied by a vector will again give back a vector (3×1). Going back to the vector notation again by using rearrangement given by (5) in summation terms of equation (3), we get:

$$\underline{t}^n = \underbrace{\sum_{i=1}^{3} (\underline{t}^i \otimes \underline{e}_i)}_{\text{stress tensor}} \underline{n} \tag{6}$$

This is just a different viewpoint and the nice thing about the final result here is that the orientation $\underline{n}$ has been separated. The tensor that is multiplied with $\underline{n}$ is called STRESS TENSOR. It is denoted by $\underline{\underline{\sigma}}$. So finally, we get :

$$\begin{aligned} \underline{t}^n &= \underline{\underline{\sigma}}\, \underline{n} \\ \Rightarrow \underline{t}^n(\underline{x}; \underline{n}) &= \underline{\underline{\sigma}}(\underline{x})\, \underline{n} \end{aligned} \tag{7}$$

We have thus found out the expression for the stress tensor (dependent on $\underline{x}$ alone) from equation (6) as:

$$\underline{\sigma}(\underline{x}) = \sum_{i=1}^{3} (\underline{t}^i \otimes \underline{e}_i) \tag{8}$$

Thus, to obtain a stress tensor, choose three independent planes at a point, find tractions on those planes, do their tensor product and sum! The stress tensor is independent of what three planes we choose! (Because this summation has to remain the same for any set of three planes we choose)

3 Representation of vectors and second order tensors in a coordinate system (start time: 14:40)

3.1 Representation of vectors (start time: 14:51)

We have a vector $\underline{v}$ in space (with magnitude v) and we first choose a coordinate system ($\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$) such that our vector is aligned along $\underline{e}_1$ as shown in Figure 2. Thus, representing the vector in this coordinate system, we get:

$$[\underline{v}]_{(\underline{e}_1,\underline{e}_2,\underline{e}_3)} = \begin{bmatrix} v \\ 0 \\ 0 \end{bmatrix} \tag{9}$$

Now, choose another coordinate system ($\underline{\hat{e}}_1$, $\underline{\hat{e}}_2$, $\underline{\hat{e}}_3$) as shown in Figure 2 in red. Here, $\underline{\hat{e}}_3$ is same as $\underline{e}_3$ and $\underline{\hat{e}}_1$ makes an angle of 45° with $\underline{e}_1$. Representation of $\underline{v}$ in this new coordinate system will be:

$$[\underline{v}]_{(\underline{\hat{e}}_1,\underline{\hat{e}}_2,\underline{\hat{e}}_3)} = \begin{bmatrix} v/\sqrt{2} \\ v/\sqrt{2} \\ 0 \end{bmatrix} \tag{10}$$

Thus, for vectors, their representation in different coordinate systems is different even though the vectors themselves do not change with coordinate system. This is true for n^{th}-order tensors in general.

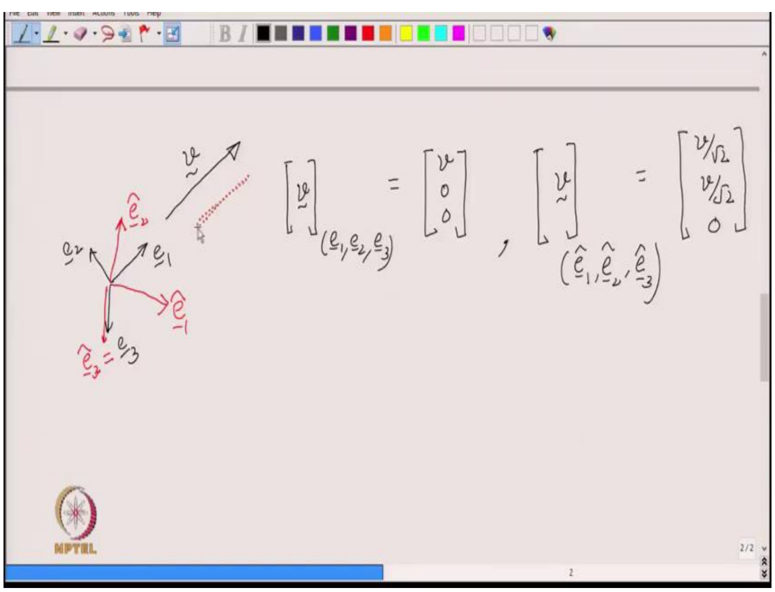

Figure 2: Two coordinate systems (black and red) with a vector $\underline{v}$

3.2 Representation of second order tensors (start time: 17:30)

Let us try to represent equation (8) in ($\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$) coordinate system. Now, as we know that stress tensor is a second order tensor, so it's representation is going to be a matrix.

$$\left[\underline{\underline{\sigma}}\right]_{(\underline{e}_1,\underline{e}_2,\underline{e}_3)} = \sum_{i=1}^{3} \left[\underline{t}^i\right] \left[\underline{e}_i\right]^T$$

We have to represent $\underline{t}^i$ and $\underline{e}_i$ also in ($\underline{e}_1$,$\underline{e}_2$,$\underline{e}_3$) coordinate system:

$$\begin{aligned}
\Rightarrow \left[\underline{\underline{\sigma}}\right]_{(\underline{e}_1,\underline{e}_2,\underline{e}_3)} &= \left[\underline{t}^1\right]\begin{bmatrix}1 & 0 & 0\end{bmatrix} + \left[\underline{t}^2\right]\begin{bmatrix}0 & 1 & 0\end{bmatrix} + \left[\underline{t}^3\right]\begin{bmatrix}0 & 0 & 1\end{bmatrix} \\
&= \begin{bmatrix} t^1_1 & 0 & 0 \\ t^1_2 & 0 & 0 \\ t^1_3 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & t^2_1 & 0 \\ 0 & t^2_2 & 0 \\ 0 & t^2_3 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & t^3_1 \\ 0 & 0 & t^3_2 \\ 0 & 0 & t^3_3 \end{bmatrix} \\
&= \begin{bmatrix} t^1_1 & t^2_1 & t^3_1 \\ t^1_2 & t^2_2 & t^3_2 \\ t^1_3 & t^2_3 & t^3_3 \end{bmatrix}
\end{aligned} \tag{11}$$

Here, a general traction component signifies the following:

$$t^i_j = \underline{t}^i \cdot \underline{e}_j \tag{12}$$

So, t^i_j represents the component of traction on '*i*' plane along j^{th} direction. Thus, if we want to write down the stress matrix in ($\underline{e}_1$,$\underline{e}_2$,$\underline{e}_3$) coordinate system, then the first column has to be the representation of traction on plane whose normal is along the first coordinate axis (which is $\underline{e}_1$ here). Similarly, the second column has to be the representation of traction on plane with normal along second coordinate axis and the third column has to be the representation of traction on plane with normal along third coordinate axis. Often, a slightly different notation is used for stress matrix, i.e.,

$$\left[\underline{\underline{\sigma}}\right]_{(\underline{e}_1,\underline{e}_2,\underline{e}_3)} = \begin{bmatrix} \sigma_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \sigma_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \sigma_{33} \end{bmatrix} \tag{13}$$

Off diagonal elements are represented by τ and diagonal components are denoted by σ. So, if we consider a component τ_{ij} or σ_{ii}, 'j' denotes the plane normal and 'i' denotes the component of traction (i.e. the direction of component). So, τ_{ij} represents traction on j^{th} plane and its component along i^{th} direction. σ_{ii} is trivial and denotes traction on i^{th} plane and its component in the i^{th} direction itself.

3.2.1 Representation of stress tensor in Cartesian coordinate system (start time: 30:38)

Cartesian coordinate system means that our coordinate axes are $\underline{e}_1$,$\underline{e}_2$ and $\underline{e}_3$ (perpendicular axes). We want to know the stress tensor at a given point $\underline{x}$ in our body and we want to represent the stress matrix in the Cartesian coordinate system. Think of a cuboid around the point $\underline{x}$ as shown in Figure 3 on the left. It is centered at $\underline{x}$. Its six faces are along $\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$, -$\underline{e}_1$, -$\underline{e}_2$, -$\underline{e}_3$ respectively. The traction that acts on $\underline{e}_1$ plane is $\underline{t}^1$. It has three components as shown in Figure 3. The component along $\underline{e}_1$ is σ_{11}, the component along $\underline{e}_2$ is τ_{21} and the component along third direction ($\underline{e}_3$) is τ_{31}. We can conclude that σ_{11} is normal to the plane whereas τ_{21} and τ_{31} are in the plane. Thus, σ_{11} is called the normal component of traction and τ_{21} and τ_{31} are called as the shear components of traction.

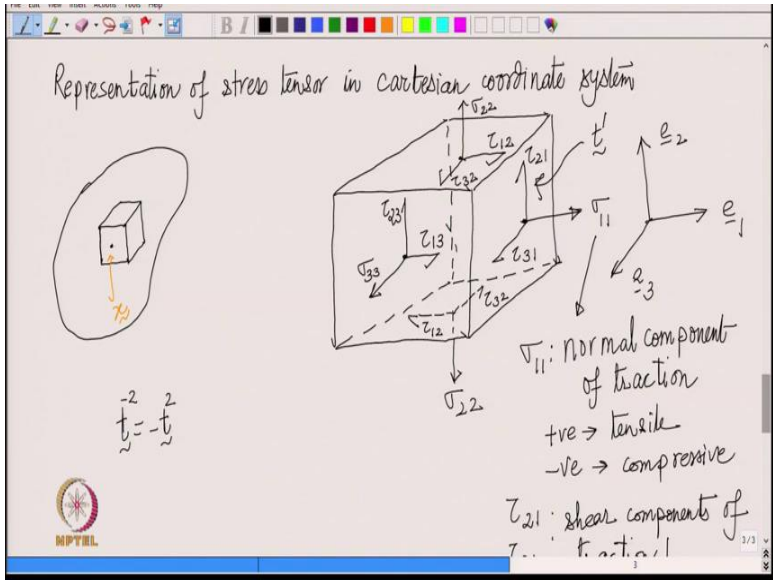

Figure 3: The coordinate axes are shown on the right. A cuboid is considered at point $\underline{x}$ and traction components are shown on this cuboid.

The shear components of the traction are in the plane and they try to shear the body. To visualize this, think of two planes in the body which are very close to each other and have their normals parallel to $\underline{e}_1$. Between those planes, if we have a traction with a component along the plane, then that will try to shear the plane (i.e. displace the two planes along the plane itself). Similarly, σ_{11} is perpendicular to both the planes considered. If σ_{11} is positive, the nature of the traction will be tensile and it will try to pull the two planes apart. So, on the first plane, the second plane will apply traction so as to pull it towards itself and vice versa. And for shear components, if there is some traction component in the $\underline{e}_2$ direction, then the shearing is going to happen along $\underline{e}_2$ direction and if there is some component along $\underline{e}_3$ direction then the shearing is going to happen along $\underline{e}_3$ direction. If we have a component in both these directions, then the shear will happen along some other resultant direction.

Remark: Positive σ_{11} implies it is tensile in nature and negative σ_{11} means it is compressive in nature. Shearing means sliding between two planes along the plane itself.

Going back to our cuboid, the components of traction on top and front face also are drawn in Figure 3. Remember that second index denotes the plane normal and the first index denotes the direction. For the bottom face however, plane normal is along $-\underline{e}_2$. So, we need to find $\underline{t}^{-2}$. We have already seen in previous lectures that:

$$\underline{t}^{-2} = -\underline{t}^{2} \tag{14}$$

So, on the bottom face, the traction components will point in the opposite direction of those of the top face. The magnitude of the traction on the bottom face remains same as that for the top face but the direction gets reversed. In this way, traction components on all the planes can be drawn.

Note : We should keep in mind that we wanted to find out the stress matrix at the point $\underline{x}$ and the six planes that we have here are not passing through the point $\underline{x}$. However, if the planes are not passing through point $\underline{x}$, then we do not have tractions at that point. We should understand that the point $\underline{x}$ is drawn at the center of the cuboid just for visualization. These six faces actually pass through the point $\underline{x}$ when we shrink this cuboid to the point $\underline{x}$. Only then, we have these traction components at this point itself which we can then relate with the stress matrix at the same point.