Solid Mechanics
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Lecture - 29
Energy Methods

Hello everyone! Welcome to Lecture 29! In this lecture, we will discuss energy methods which can be used to solve the deformation problem.

1 Introduction (start time: 00:20)

To obtain the deformation of a body, we first discussed how stress-equilibrium equation can be used. We then discussed beam theory for solving deformation of beams. There is also an alternate method to obtain deflection in an arbitrary body, which is the *energy method*. You might recall from first year mechanics course that we could solve for the motion of rigid bodies either by using Newton's laws of motion or by using the principal of minimum potential energy. The latter approach yields simpler equations in several cases and thus is attractive. Likewise, *energy methods* provide an alternate method for solving the deformation of bodies using simpler equations.

2 Linearity and superposition (start time: 01:52)

Consider an arbitrary body as shown in Figure 1.

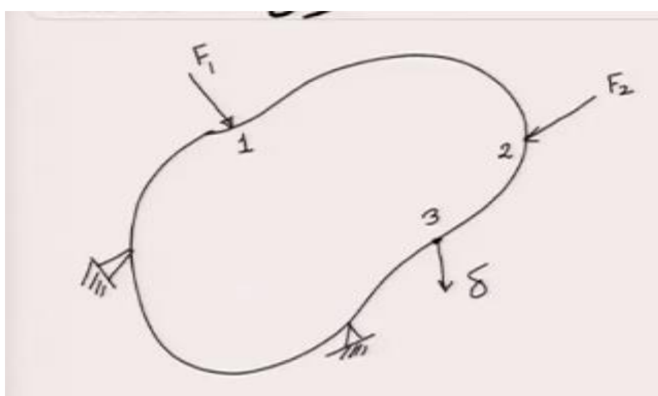

Figure 1: An arbitrary body clamped at two points and subjected to a point load at point 1

It is clamped at two points and subjected to a concentrated load F_1 at point 1. Let us say we want to measure the deflection δ at a different point (say point 3 in the figure) which can be either on the surface of the body or within the body. How is the deflection δ related to force F_1? To explore it, let us look at the stress equilibrium equations as given below:

$$\begin{aligned}
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + b_x &= 0, \\
\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + b_y &= 0, \\
\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + b_z &= 0.
\end{aligned} \tag{1}$$

These equations are supplemented with the following boundary conditions:

$$\underline{\underline{\sigma}}\,\underline{n} = \underline{t}_0, \quad \text{(traction boundary condition)}$$

$$\underline{u} = \underline{u}_0. \quad \text{(displacement boundary condition)} \tag{2}$$

Notice that the above equation is linear in stress components which, in turn, are linear in strain components (due to linear stress-strain relation) . Finally, strain components are also linear in displacement components.[1] Thus, the governing equations become linear in unknown displacement components. Similarly, the boundary conditions are also linear unknown displacement components. Due to this linearity of governing equations, the displacement of the body will be linearly related to externally applied load. To illustrate it more clearly, think of a spring-mass problem as shown in Figure 2.

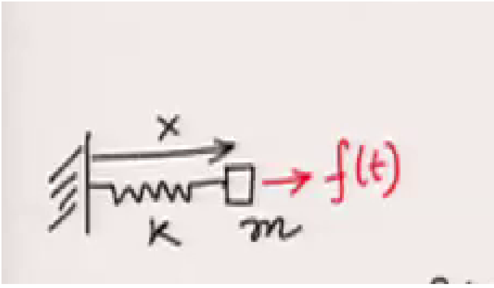

Figure 2: A mass tied to a spring is stretched by a force $f(t)$.

We know that the governing equation for the displacement x of the block of mass m is:

$$m\ddot{x} + kx = f(t). \tag{3}$$

Here, k represents the spring constant and $f(t)$ represents the externally applied force. The equation being linear, we know that the displacement x is linearly related to the forcing term $f(t)$. So, if we double the applied force, the displacement also gets doubled. Furthermore, if the displacement solutions for forces F_1 and F_2 are x_1 and x_2, respectively, then the solution when the applied force is F_1+F_2 will be x_1+x_2. This means that if the equation is linear in the unknown, then the unknown becomes a linear function of the forcing term and superposition also holds. The linearity and superposition property applies also to solid mechanics problems since the governing equations and boundary conditions are linear too in the unknown displacement. The external load here consists of the body force and boundary

[1] If we consider large deformation problem, then stress-strain relation and strain-displacement relation will become nonlinear.

traction/point loads. Coming back to the problem shown in Figure 1, δ_3 will thus be linearly related to the applied load F_1:

$$\delta_3 \propto F_1 \Rightarrow \delta_3 = k_{31}F_1. \tag{4}$$

The proportionality constant k_{31} is also called influence coefficient. If we change the location of the force F_1, k_{31} will change. Similarly, if we change the point at which we are measuring the displacement, k_{31} will change. Hence, the influence coefficient depends on the location of the applied force as well as on the location of the point where displacement is being measured. It also depends on the direction of applied force but does not depend on the magnitude of force.[2] Finally, the influence coefficient also depends on the direction along which we are measuring the displacement component. For example, displacement being a vector quantity, $\underline{\delta}_3$ will have three components δ_3^x, δ_3^y and δ_3^z. For each of the components, we will have different influence coefficients relating to each component of force applied at point 1, i.e.,

$$\begin{aligned} &\delta_3^x = k_{31}^{xx}F_1^x, \quad \delta_3^x = k_{31}^{xy}F_1^y, \quad \delta_3^x = k_{31}^{xz}F_1^z,\\ &\delta_3^y = k_{31}^{yx}F_1^x, \quad \delta_3^y = k_{31}^{yy}F_1^y, \quad \delta_3^y = k_{31}^{yz}F_1^z,\\ &\delta_3^z = k_{31}^{zx}F_1^x, \quad \delta_3^z = k_{31}^{zy}F_1^y, \quad \delta_3^z = k_{31}^{zz}F_1^z. \end{aligned} \tag{5}$$

Each of the equations above resembles Hooke's law for a spring where force in the spring is proportional to the elongation of the spring. The above equations hold only if one (but any) of the components of force is present. In case all components are present, one can apply the principle of superposition as mentioned earlier which would yield

$$\begin{aligned} \delta_3^x &= k_{31}^{xx}F_1^x + k_{31}^{xy}F_1^y + k_{31}^{xz}F_1^z,\\ \delta_3^y &= k_{31}^{yx}F_1^x + k_{31}^{yy}F_1^y + k_{31}^{yz}F_1^z,\\ \delta_3^z &= k_{31}^{zx}F_1^x + k_{31}^{zy}F_1^y + k_{31}^{zz}F_1^z. \end{aligned} \tag{6}$$

Note that the influence coefficients are unaffected whether only one component of force is present or all components are present. The principle of superposition would hold similarly even if the force is acting at two (or more) different points in the body. For example, suppose we only apply a force F_1 at point 1 and no other force, the displacement at point 3 in a prescribed direction will be given by:

$$\delta_3 = k_{31}F_1. \tag{7}$$

Now, in another situation, if we apply a force F_2 at point 2 and no other force, the displacement at point 3 will be given by:

$$\delta_3 = k_{32}F_2. \tag{8}$$

In a third situation, if we apply force F_1 at point 1 and force F_2 at point 2 together, the displacement at point 3 will become

[2] If the influence coefficient were to depend also on the magnitude of applied load, the relationship between displacement and applied force will become nonlinear.

$$\delta_3 = k_{31}F_1 + k_{32}F_2. \quad (9)$$

3 Alternate way to prove superposition (start time: 19:54)

Consider an arbitrary body clamped at two locations. We apply a force F_1 at point 1 and measure vertical component of displacement at point 3 (see Figure 3).

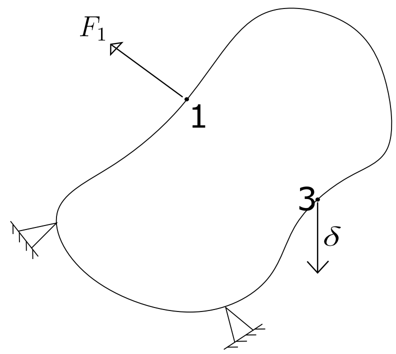

Figure 3: A force F_1 is applied on a body at a point 1 and vertical displacement is being measured at point 3.

The displacement will be

$$\delta_3 = k_{31}F_1. \quad (10)$$

Due to the application of force, the system must have deformed and changed its shape and size (see blue dotted lines in Figure 4).

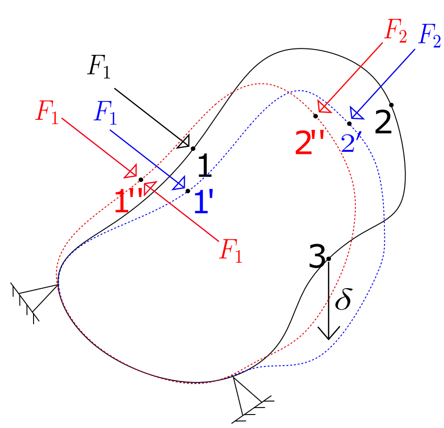

Figure 4: A force F_2 is applied on the deformed body (shown as blue dotted line) at point 2' (originally at 2)

In the next step, we apply force F_2 at point 2' (originally at 2) to the deformed body. So, both F_1 and F_2 are acting now. As F_2 starts to act after the body has deformed, so the influence coefficient k_{32} may have changed to k'_{32} which implies

$$\delta_3 = k_{31}F_1 + k'_{32}F_2. \tag{11}$$

In the third step, we apply a force $-F_1$ at point 1" (originally at 1) to the currently deformed body. Due to this, the total force acting at point 1 becomes zero and we are only left with a force F_2 at point 2. As the condition of the body has changed, the influence coefficient k_{31} may have changed to k'_{31}. So, the total vertical displacement at the same point 3 will be

$$\delta_3 = k_{31}F_1 + k'_{32}F_2 + k'_{31}(-F_1). \tag{12}$$

Alternately, as in the final configuration, we only have a force F_2 at point 2, the vertical displacement of point 3 should be

$$\delta_3 = k_{32}F_2. \tag{13}$$

Notice that we have used the original influence coefficient in the above equation as if only force F_2 were acting right from the beginning. We could say so because the displacement in an elastic body only depends on the final state of applied loads and does not depend on how those loads were applied. Equating equations (12) and (13) since the final state of loading is the same, we obtain

$$\begin{aligned} & k_{31}F_1 + k'_{32}F_2 + k'_{31}(-F_1) = k_{32}F_2 \\ \Rightarrow\quad & (k_{31} - k'_{31})F_1 + (k_{32} - k'_{32})F_2 = 0. \end{aligned} \tag{14}$$

Since this has to hold for arbitrary magnitudes of (F_1, F_2), their coefficients must vanish. Thus, we get

$$\begin{aligned} k_{31} &= k'_{31}, \\ k_{32} &= k'_{32}. \end{aligned} \tag{15}$$

This means that in step 2 when both F_1 and F_2 act, the displacement would simply be

$$\delta_3 = k_{31}F_1 + k_{32}F_2 \tag{16}$$

which proves that one can apply superposition principle.

4 Corresponding displacement/Work absorbing displacement (start time: 26:55)

The displacement of the body at the point of application of force and in the same direction as the applied force is called the corresponding displacement. As this component of displacement is also responsible for the actual work done by the force, it is therefore also called the work absorbing displacement. The perpendicular component of displacement does no work. Suppose we apply force at point 1 and measure the corresponding displacement δ_1, then

$$\delta_1 \propto F_1 \text{ or } \delta_1 = k_{11}F_1. \tag{17}$$

5 Energy stored in the body due to an applied force (start time: 30:24)

We now want to find the energy stored in a deformable body due to a force, say F_1. Would it simply be $F_1 \cdot \delta_1$ where δ_1 is the corresponding displacement? To find an answer to this, consider a spring having spring constant k and clamped at one end. A force F is applied at the other end as shown in Figure 5.

Figure 5: A spring clamped at one end is stretched by a force F.

We know that the energy stored in the spring due to this force is

$$\text{Energy stored} = \frac{1}{2}kx^2 = \frac{1}{2}kx \cdot x = \frac{1}{2}F \cdot x. \tag{18}$$

Here, x denotes the elongation in the spring or the displacement of other end which is also the corresponding displacement. To understand the origin of the factor $\frac{1}{2}$ in the above expression, we can think of pulling the spring very slowly. When the displacement of the spring is Δ, the force within the spring is just $k\Delta$. So, to ramp up the displacement from 0 to x, we need to ramp up the force from 0 to kx (= F). The corresponding force-displacement curve is shown in Figure 6.

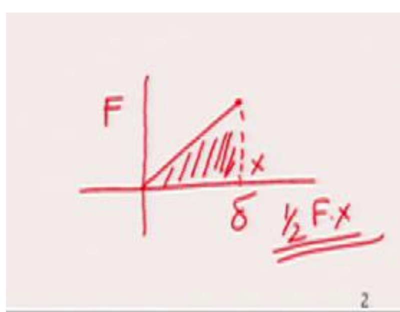

Figure 6: Force-displacement curve for a spring that is stretched quasi-statically from 0 to x.

The work done in this process will be area under the curve which is the area of the shaded triangle ($= \frac{1}{2} F \cdot x$). When we apply the load quasi-statically in this manner, the total energy stored is just $\frac{1}{2} F \cdot x$. However, the final energy stored is independent of the loading path taken in the process: it only depends on the final force/displacement. Thus, even if we apply the force instantly to reach the final stage, the energy stored in equilibrium would still be $\frac{1}{2}F \cdot x$. We should keep in mind that when we apply the force F instantly and the spring displaces by x, the work done by the external agent is $F \cdot x$. The energy stored in the spring is just half of this value - the other half of the energy either gets converted into kinetic energy or dissipates as heat. In the quasistatic path, the work done by the external agent exactly equals the energy stored in the spring, i.e. $\frac{1}{2}F \cdot x$ and no part of the work is dissipated. Taking the analogue of the spring, we can note that the energy stored in a deformable body due to the action of force F_1 is

$$\frac{1}{2}F_1\delta_1 = \frac{1}{2}k_{11}F_1^2 \quad \text{using (17)}. \tag{19}$$

This is the energy stored due to a single force. We can have multiple forces $F_1, F_2, \ldots F_n$ (with corresponding displacements $\delta_1, \delta_2, \ldots \delta_n$) acting on the deformable body as shown in Figure 7.

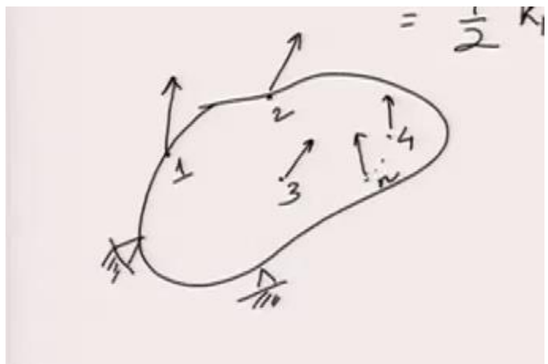

Figure 7: Multiple forces applied on a body

The total energy stored would be

$$\sum_{i=1}^{n} \frac{1}{2}F_i\delta_i\,. \tag{20}$$

In the above, we cannot substitute δ_i as $k_{ii}F_i$ because δ_i contains contributions from multiple forces. It would be the following:

$$\delta_i = \sum_{i=1}^{n} k_{ij}F_j. \tag{21}$$

Plugging this into equation (20), we get

$$\text{Total energy stored: } \sum_i \sum_j \frac{1}{2}k_{ij}F_iF_j\,. \tag{22}$$

This is the expression of total energy stored in the body in terms of the applied forces and influence coefficients.

6 Reciprocal relation (start time: 38:05)

The reciprocal relation allows us to obtain relationship between influence coefficients k_{ij} and k_{ji}. Let us work this out through energy considerations. We will consider two different processes. In the first process, think of a body with a force F_1 applied on it at point 1. The energy stored in the body will be

$$E = \frac{1}{2}F_1\delta_1 = \frac{1}{2}k_{11}F_1^2. \tag{23}$$

This action is shown in Figure 8 where due to this force, the body deforms from solid black configuration to dotted red configuration.

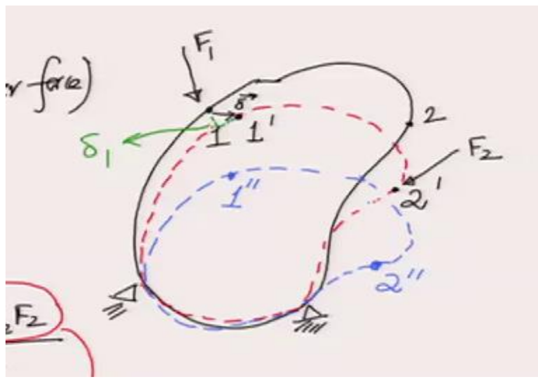

Figure 8: Subsequent application of forces F_1 and F_2 on a body.

We call this step 1 where the point 1 gets displaced to 1' during deformation. Then, we apply force F_2 at point 2' in the current configuration which was at point 2 in the original/undeformed body. The body gets further deformed to dashed blue configuration as shown in Figure 8. The point 1' goes to 1" and 2' goes to 2". This is step 2. The energy stored in the body in the final state will be

$$\begin{aligned} E &= \frac{1}{2}k_{11}F_1^2 + F_1(k_{12}F_2) + \frac{F_2 \cdot k_{22}F_2}{2} \\ &= \frac{1}{2}k_{11}F_1^2 + \frac{1}{2}k_{22}F_2^2 + k_{12}F_1F_2 \end{aligned} \tag{24}$$

Notice that the contribution to energy due to F_1 during step 2 is not divided by 2. This is because during step two, the full F_1 acts throughout the deformation: the work done will thus simply be the force F_1 times the extra displacement (1' → 1''). This additional displacement is nothing but the displacement due to F_2, i.e., $k_{12}F_2$. The work done due to the sudden application of force F_2 at its own point of application is same as the ramped up work. Thus, it is divided by 2. The corresponding displacement in this term represents movement of 2'→2" (and not of 2→2") as this part of displacement is the one due to the application of F_2 which can be written as $k_{22}F_2$.

We now think of the second process which is basically the first process only but the order of the two steps is reversed here. This means that we first apply force F_2 and then apply force F_1 in the second step. Due to application of F_2 in the first step, the energy stored in the body will be

$$E = \frac{1}{2}k_{22}F_2^2. \tag{25}$$

In the second step, F_1 is applied. As F_2 is acting fully in this step, the additional energy due to it in step 2 will not be divided by 2. The total energy stored in the body will thus be

$$\begin{aligned} E &= \frac{1}{2}k_{22}F_2^2 + \left(\underbrace{F_2 \cdot k_{21}F_1}_{\text{due to } F_2} + \underbrace{\frac{1}{2}F_1(k_{11}F_1)}_{\text{due to } F_1} \right) \\ &= \frac{1}{2}k_{22}F_2^2 + \frac{1}{2}k_{11}F_1^2 + K_{21}F_1F_2. \end{aligned} \tag{26}$$

Now, in both the processes, the final state is the same with both F_1 and F_2 acting. Thus, the final energy stored in the two processes must be the same. Hence, upon comparing equations (24) and (26), we get

$$k_{12} = k_{21}. \tag{27}$$

Generalizing this, we can write the reciprocal relation as

$$k_{ij} = k_{ji}. \tag{28}$$

7 Maxwell-Betti-Rayleigh reciprocal theorem (start time: 50:19)

Consider again a body which is clamped at some points. We think of two situations. In the first situation, multiple forces $F_1, F_2, ... F_n$ are applied on it as shown in Figure 7. The corresponding displacements for these forces are $\delta_1, \delta_2, ... \delta_n$. In the second situation, we apply forces $F_1', F_2', ... F_n'$ at the same locations and measure the corresponding displacements as $\delta_1', \delta_2', ... \delta_n'$. The Maxwell-Betti-Rayleigh reciprocal theorem tells us that

$$\sum_{i=1}^{n} F_i\delta_i' = \sum_{i=1}^{n} F_i'\delta_i. \tag{29}$$

This means that the work done by forces in situation 1 through the corresponding displacements of situation 2 equals the work done by forces in situation 2 through the corresponding displacements of situation 1. Let us try to prove this. Consider the LHS of equation (29):

$$\begin{aligned} \sum_i F_i\delta_i' &= \sum_i F_i \sum_j k_{ij}F_j' \\ &= F_1(k_{11}F_1' + k_{12}F_2' + ... + k_{1n}F_n') \\ &\quad +F_2(k_{21}F_1' + k_{22}F_2' + ... + k_{2n}F_n') \\ &\quad +... \\ &\quad +F_n(k_{n1}F_1' + k_{n2}F_2' + ... + k_{nn}F_n') \end{aligned} \tag{30}$$

We can rearrange these terms by collecting the first terms within each bracket, second terms within each bracket and so on, i.e.,

$$\begin{aligned}
\sum_i F_i\delta_i' &= F_1'(k_{11}F_1 + k_{21}F_2 + ... + k_{n1}F_n) \\
&\quad +F_2'(k_{12}F_1 + k_{22}F_2 + ... + k_{n2}F_n) \\
&\quad +... \\
&\quad +F_n'(k_{1n}F_1 + k_{2n}F_2 + ... + k_{nn}F_n) \\
&= F_1'(k_{11}F_1 + k_{12}F_2 + ... + k_{1n}F_n) \\
&\quad +F_2'(k_{21}F_1 + k_{22}F_2 + ... + k_{2n}F_n) \\
&\quad +... \\
&\quad +F_n'(k_{n1}F_1 + k_{n2}F_2 + ... + k_{nn}F_n) \quad \text{(using reciprocal relation)} \\
&= \sum_i F_i'\delta_i. \qquad (31)
\end{aligned}$$

This completes the proof of reciprocal theorem.

8 Generalized forces and generalized displacements (start time: 58:35)

Till now, we have only discussed about forces and displacements. The above relations can be extended to moments and rotations as well. Just like the work analog of force is displacement, the work analog of moment is rotation. Consider an arbitrary body on which several forces and moments act (see Figure 9).

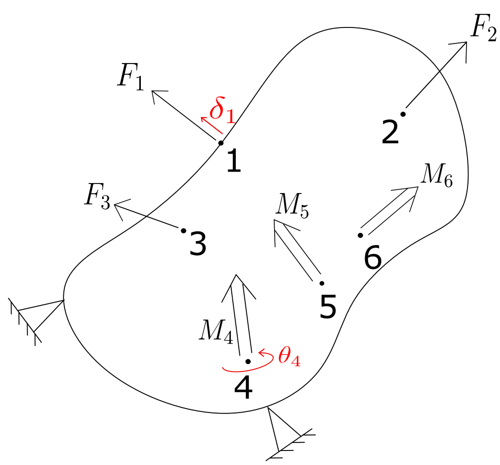

Figure 9: A body on which several forces and moments act

We can measure the corresponding displacements and corresponding rotations. A corresponding rotation is the local rotation of the deformable body in the direction of the applied moment at the same point. Moments and rotations can be seen as generalized forces and generalized displacements, respectively. If we think in these terms, the relations derived above do not differentiate between forces and moments or between displacements and rotations. Thus, all above relations will hold for moments and rotations also. For example, the corresponding rotation (θ_4) for M_4 will be given by

$$\theta_4 = k_{41}F_1 + k_{42}F_2 + k_{43}F_3 + k_{44}M_4 + k_{45}M_5 + k_{46}M_6 \qquad (32)$$

The dimensions of the influence coefficients relating force and rotation and those relating moment and rotation are different though. For example, the dimension of k_{41} is $\frac{\theta}{force}$ or $\frac{1}{force}$ while that of k_{44} is $\frac{\theta}{moment}$ or $\frac{1}{moment}$.[3] Similarly, the corresponding displacement for force F_1 can be written as

$$\delta_1 = k_{11}F_1 + k_{12}F_2 + k_{13}F_3 + k_{14}M_4 + k_{15}M_5 + k_{16}M_6 \quad (33)$$

The dimension of k_{11} is of $\frac{displacement}{force}$ while that of k_{14} is of $\frac{displacement}{moment}$. As the dimension of moment equals that of force × displacement, the dimension of k_{14} can also be written as that of $\frac{1}{force}$. This means that the dimension of k_{14} and k_{41} are same. So, the reciprocal relation can be easily extended to mixed combinations (i.e., displacement-moment or rotation-force) without any dimensional constraints and we can write

$$k_{14} = k_{41}. \quad (34)$$

[3] Note that rotation is dimensionless.