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**Lecture - 28**
**Theory of Beams (contd...) and Beam Buckling**

Hello everyone! Welcome to Lecture 28! In this lecture, we will discuss about Timeshenko beam theory.

## 1 Timoshenko Beam Theory (start time: 00:20)

### 1.1 Introduction (start time: 00:25)

In Euler-Bernouli beam theory, it is assumed that the centerline tangent and the cross-section normal are aligned as shown in Figure 1a. So, we have only one kinematic variable which is the deflection of the beam *y*: we can obtain the cross-section orientation from its derivative , i.e., $\frac{dy}{dX}$. However, in Timoshenko Beam Theory (TBT), we do not assume the centerline tangent and cross-section normal to be aligned aligned as shown in Figure 1b.

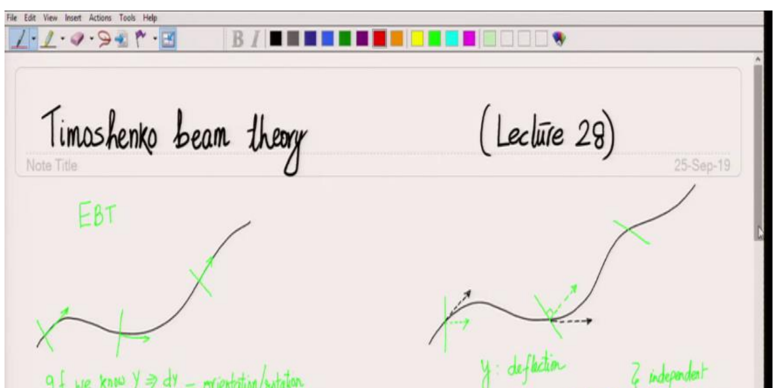


Figure 1: (a) Alignment of centerline tangent and cross-section normal in EBT (b) Non-alignment of centerline tangent and cross-section normal in TBT

Thus, this is a more general theory than EBT: we now have deflection *y* and cross-section rotation $\theta$ as two independent unknowns.

## 1.2 Governing equations (start time: 03:33)

Let us consider a beam which is initially straight as shown in Figure 2a and gets deformed due to some arbitrary loading. Let us focus on a typical cross-section of the beam in its deformed configuration as shown in Figure 2b.


Figure 2: (a) An initially straight beam clamped at one end (b) the cross-section rotation $\theta$ and centerline tangent rotation $\alpha$ of the deformed beam shown

The centerline tangent makes an angle $\alpha$ with the horizontal which implies

$$\begin{aligned} \alpha &= \tan^{-1}\left(\frac{dy}{dx}\right) \\ &\approx \frac{dy}{dx} \quad \text{(assuming the slope to be small enough)} \\ &\approx \frac{dy}{dX} \quad \text{(assuming the axial displacement to be small enough)} \end{aligned} \tag{1}$$

The deformed cross-section normal makes an angle $\theta$ with the horizontal. Thus, the angle between the cross-sectional line (shown as solid green line in Figure 2b: it should not be confused with cross-section normal) and the centerline tangent becomes

$$\frac{\pi}{2} + \theta - \alpha = \frac{\pi}{2} + \theta - \frac{dy}{dX}\,. \tag{2}$$

In the undeformed straight beam, the angle between the cross-sectional line (which lies along $\underline{e}_2$ axis) and the centerline tangent (which lies along $\underline{e}_1$ axis) was $\frac{\pi}{2}$ which, upon deformation, changes to $\frac{\pi}{2} + \theta - \frac{dy}{dx}$ as shown above. As shear strain measures the changes in angle (initial-final angle) between perpendicular line elements, we have

$$\gamma^0_{xy} = \frac{\pi}{2} - \left(\frac{\pi}{2} + \theta - \frac{dy}{dx}\right) = \frac{dy}{dX} - \theta. \tag{3}$$

The superscript ($^0$) here denotes the fact that it is the shear strain at the centroid of the cross-section since one of the perpendicular lines, i.e., the centerline passes through the centroid of the cross-section. Let us relate this centroidal shear strain to the total shear force acting in the cross-section. Suppose $V$

represents the shear force acting in the cross-section and $A$ the cross-sectional area. The average shear stress $\tau_{xy}$ in the cross-section will then be

$$\tau_{xy}^{avg} = \frac{V}{A} \,. \tag{4}$$

If we further assume rectangular cross-section for the beam, we can find the exact value of shear stress at the centroid. In a previous lecture, we had found the variation of shear stress in rectangular cross-section which is also shown in Figure 3.

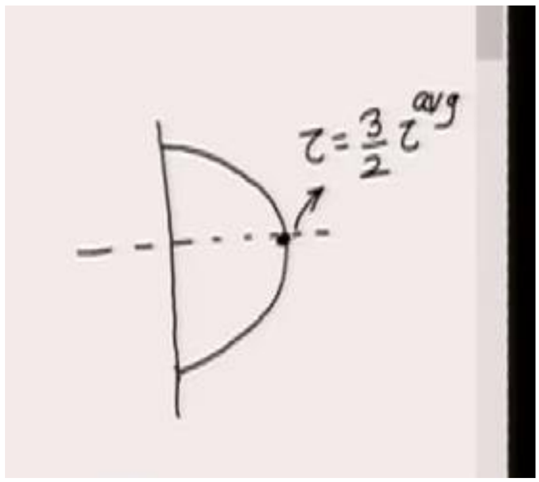


Figure 3: Variation of shear stress in a rectangular cross-section

We had found the value of shear stress at the centroidal line to be $\frac{3}{2}\tau^{avg}$, i.e.,

$$\tau_{xy}^{0} = \frac{V}{\frac{2}{3}A} = \frac{V}{kA}. \tag{5}$$

The factor $\frac{2}{3}$ is the shear correction factor $k$ for rectangular beams which is different for different cross-section shapes. Further, using linearly isotropic stress-stress relationship, we get

$$\gamma_{xy}^{0} = \frac{\tau_{xy}^{0}}{G} = \frac{V}{kGA} \,. \tag{6}$$

Substituting this in equation (3), we finally get

$$\boxed{\frac{dy}{dX} - \theta = \frac{V}{kGA}} \tag{7}$$

This is one of the equations of TBT. We need one more equation which comes from moment-curvature relation, i.e.,

$$EI\,\kappa = M. \tag{8}$$

In EBT, we could write the bending curvature as the curvature of the centerline. But in TBT, bending curvature is no more equal to the curvature of the centerline since the centerline is also undergoing shear, i.e.,

$$\kappa \neq \frac{d^2y}{dX^2}. \tag{9}$$

In TBT, we can relate the bending curvature to cross-section rotation $\theta$. Figure 4 shows a bent beam into an arc of a circle of radius *R*.

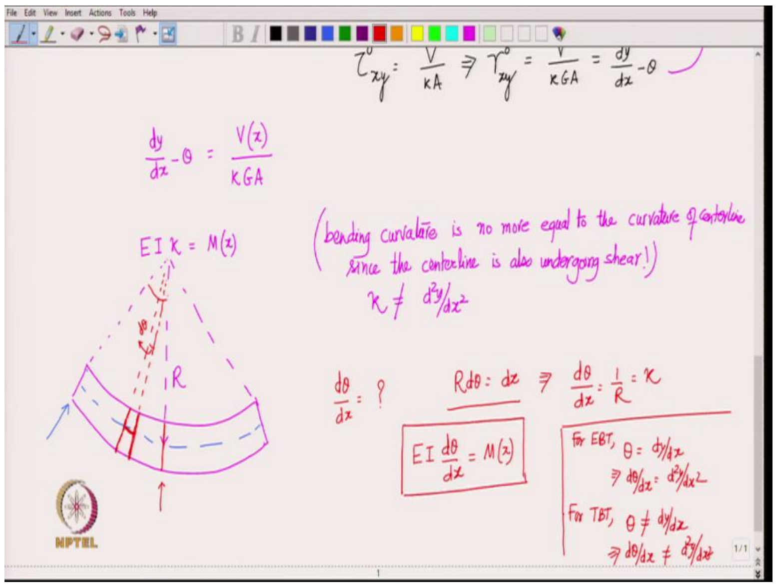


Figure 4: Bending of a beam into an arc of radius *R*

Let us focus on two cross-sections located very close to each other (shown by solid red lines in the Figure). The relative rotation between the two cross-sections is the same as the angle subtended by the arc joining the two cross-sections at the center (shown by $d\theta$). As the neutral axis does not undergo any stretching, we can write

$$Rd\theta = dX \Rightarrow \frac{d\theta}{dX} = \frac{1}{R} = \kappa. \tag{10}$$

In fact, the above relation for bending curvature holds even when the centerline undergoes stretching which of course is not considered in either EBT or TBT. Plugging the above form for bending curvature into equation (8), we get

$$\boxed{EI\,\frac{d\theta}{dX} = M\,(X)} \tag{11}$$

This is a more general equation than the one used in Euler-Bernouli beam theory. We can compare the two theories as follows:

$$\text{For EBT:} \qquad \theta = \frac{dy}{dX} \Rightarrow \frac{d\theta}{dX} = \frac{d^2y}{dX^2}$$

$$\text{For TBT:} \qquad \theta \neq \frac{dy}{dX} \Rightarrow \frac{d\theta}{dx} \neq \frac{d^2y}{dX^2}. \tag{12}$$

We have thus obtained the two equations for TBT (equations (7) and (11)) which form a system of coupled first order linear differential equations. Two boundary conditions will be needed to solve this system. In fact, more boundary conditions will be needed if shear force $V(x)$ and bending moment $M(x)$ contain extra unknown parameters.

### 1.3 Example (start time: 22:17)

Consider the problem that we had solved using EBT in the previous lecture. A transverse force *P* is applied at the free end of a beam which is clamped at the other one end as shown in Figure 5.

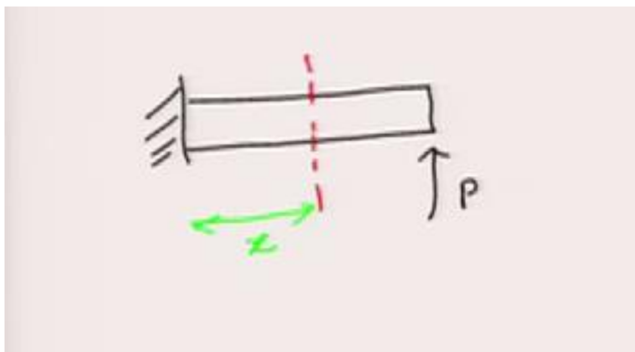


Figure 5: Bending of a beam subjected to a transverse load *P* at the free end

This problem is also called a Cantilever Problem. We would like to solve for the deformation using TBT and compare the result with those of EBT. The first step in solving is finding the shear force and bending moment profile in the beam. For that, we cut a section in the beam at a distance *X* from the clamped end and draw the free body diagram of the right portion of the beam as shown in Figure 6.

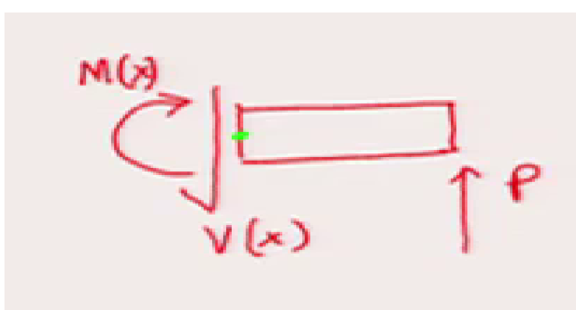


Figure 6: Free body diagram of the right section of the beam shown in Figure 5.

The transverse load $P$ acts on the free end while a shear force and moment acts on its left end. Moment balance about the centroid of the left-end yields

$$-M(X) + P(L - X) = 0 \Rightarrow M(X) = P(L - X) \tag{13}$$

whereas force balance yields

$$V(X) = P. \tag{14}$$

Plugging them into equations of TBT (equations (7) and (11)), we get

$$EI\frac{d\theta}{dX} = P(L - X), \tag{15}$$

$$\frac{dy}{dX} - \theta = \frac{P}{kGA} \tag{16}$$

We have two boundary conditions both at the clamped end. The displacement $y$ and the cross-section rotation $\theta$ must be zero here, i.e.,

$$y(0) = 0, \tag{17}$$

$$\theta(0) = 0. \tag{18}$$

We should keep in mind that we cannot set $\frac{dy}{dX} = 0$ as $\frac{dy}{dX}$ no longer denotes rotation of the cross-section. Integrating equation (15), we get

$$\theta = \frac{P}{EI}\left(LX - \frac{X^2}{2}\right) + C_1 \tag{19}$$

Using equation (18) in it, we get $C_1 = 0$. Now, we can plug the above expression for $\theta$ in equation (16) to get

$$\frac{dy}{dX} = \frac{P}{EI}\left(LX - \frac{X^2}{2}\right) + \frac{P}{kGA} \tag{20}$$

whose integration yields

$$\begin{aligned} y &= \frac{P}{EI}\left(L\frac{X^2}{2} - \frac{X^3}{6}\right) + \frac{PX}{kGA} + C_2 \\ &= \frac{P}{EI}\left(L\frac{X^2}{2} - \frac{X^3}{6}\right) + \frac{PX}{kGA}. \quad \text{(using (17))} \end{aligned} \tag{21}$$

This is the final expression for deflection $y$ obtained by TBT. We get an extra term $\frac{PX}{kGA}$ when we compare the result with the one from EBT.

## 1.4 When to use EBT/TBT? (start time: 27:04)

Let us now explore which of the two theories to use in a given scenario. If the result from TBT is very close to the one from EBT, then we can simply apply EBT and neglect the effect of shear. The tip deflection from TBT can be found by substituting *X* = *L* in equation (21) which yields

$$y^{L,TBT} = \frac{PL^3}{3EI} + \frac{PL}{kGA} \tag{22}$$

whereas the tip deflection from EBT was

$$y^{L,EBT} = \frac{PL^3}{3EI}\,. \tag{23}$$

Assuming the result from TBT to be more accurate, the relative error in EBT solution ($\epsilon_r^{EBT}$) will be

$$\epsilon_r^{EBT} = \frac{y^{L,TBT} - y^{L,EBT}}{y^{L,TBT}} = \frac{\frac{PL}{kGA}}{\frac{PL^3}{3EI}} = \frac{3EI_{zz}}{kGAL^2}\,. \tag{24}$$

We can write the moment of area $I_{zz}$ in terms of the cross-sectional area and radius of gyration of the cross-section ($R_G$). The radius of gyration $R_G$ also depends on the shape and size of the cross-section. For a rectangular cross-section of height *h* and width *b*, e.g.,

$$I_{zz} = \frac{1}{12}bh^3 = bh\,\frac{h^2}{12} = A\left(\underbrace{\frac{h}{\sqrt{12}}}_{R_G}\right)^2. \tag{25}$$

Plugging this into equation (24), we get

$$\epsilon_r^{EBT} = \frac{3EAR_G^2}{kGAL^2} = \frac{3ER_G^2}{kGL^2}. \tag{26}$$

For EBT to be applicable, the relative error must be very small, i.e.,

$$\begin{aligned} \left(\frac{3E}{kG}\right)\left(\frac{R_G}{L}\right)^2 &\ll 1 \\ \Rightarrow \quad \frac{L}{R_G} &\gg \sqrt{\frac{3E}{kG}} = \sqrt{\frac{6(1+\nu)}{k}} \end{aligned} \tag{27}$$

In the above expression, we have separated the geometric and material parameters: the geometric term on the left is the aspect ratio of the beam whereas the material parameter *ν* on the right is the material's Poisson's ratio. For illustration, consider a case where the Poisson's ratio *ν* is 0.3 and the cross-section of the beam is rectangular. Thus, $k = \frac{2}{3}$. Putting these values into the RHS of the above relation, we get:

$$\frac{L}{R_G} \gg \sqrt{\frac{6 \times 1.3}{\frac{2}{3}}} \approx 3.4 \tag{28}$$

So, if the ratio $\frac{L}{R_G}$ is much greater than 3.4, (e.g., 10), then we can use Euler-Bernouli Beam theory safely. For shorter beams having low aspect ratio, one would have to use Timoshenko beam theory.

## 2 Buckling of Beams (start time: 35:43)

### 2.1 Introduction (start time: 35:53)

When we try to compress a stick of a broom (or any long and thin rod), it initially remains straight as shown in Figure 7 but as we increase the compressive force, it suddenly bends.

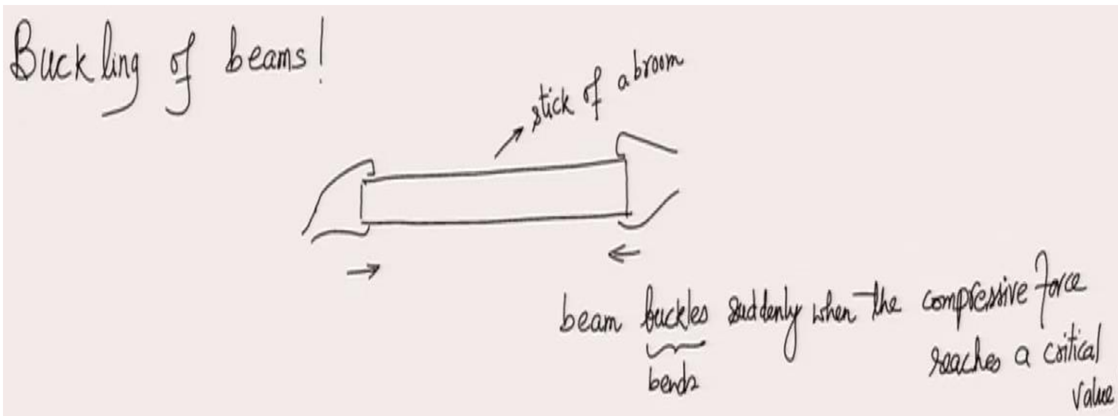


Figure 7: A representative beam being compressed from both its ends

Initially, there is no bending but when we apply the compressive force above a critical value, the stick/beam bends instantly. This is called buckling and the critical value of compressive load is called buckling load. This is an interesting phenomena. Whenever we design a machine having beam like element and it has to hold compressive load, we have to make sure that the operative compressive load is less than the buckling load. Otherwise, the beam element will buckle leading to failure of the machine.

### 2.2 Finding buckling load (start time: 38:55)

Let us see how to obtain buckling load. We will use EBT to model the beam. This means that we are neglecting the effect of shear which, as derived earlier, is a good assumption for long enough beams (aspect ratio > 10). We will consider the case of column buckling by which we mean that we have a column clamped at one end and is subjected to an axial compressive force at the other end (see Figure 8).

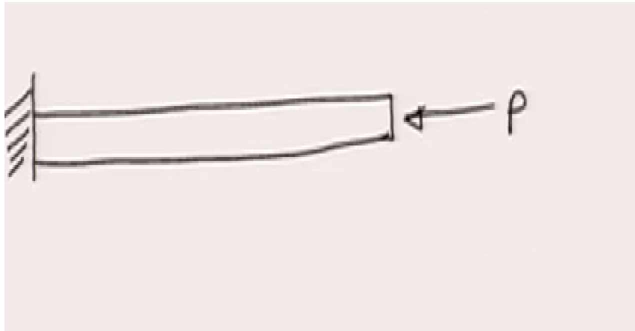


Figure 8: A compressive force being applied at the free end of a column

When the compressive load *P* reaches the critical value, the beam/column will bend as shown in Figure 9 even though we are applying axial compressive load here.

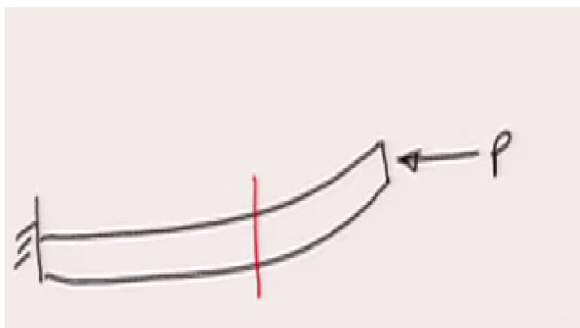


Figure 9: Buckling of a column as the compressive load *P* increases to buckling load

We want to find this critical load. Let us rewrite the equations of EBT:

$$EI\frac{d^2y}{dX^2} = M(X) \tag{29}$$

We need to first find the bending moment profile. For this, we cut a section in the beam at a distance *X* from the clamped end (see Figure 9) and draw the free body diagram of the right portion of the beam as shown in Figure 10.

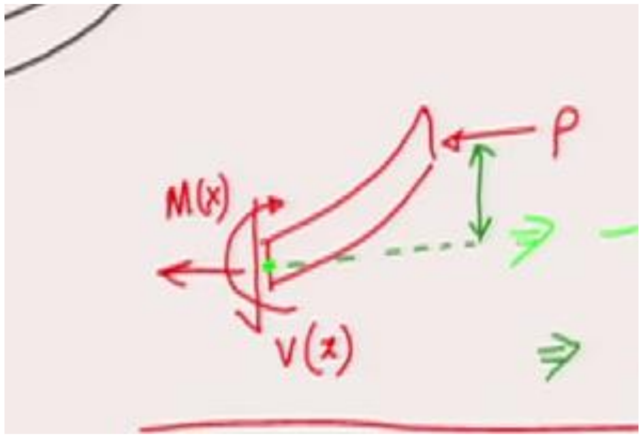


Figure 10: Free body diagram of the right portion of the column shown in Figure 9

The applied load $P$ acts on its right-end while shear force $V$ ($X$) and bending moment $M$($X$) acts on its left end. The $y$ coordinate of the left and right ends are $y$($X$) and $y^L$, respectively. Moment balance about the centroid of the left-end cross-section gives

$$-M(X) + P\Big(y^L - y(X)\Big) = 0 \Rightarrow M(X) = P\Big(y^L - y(X)\Big) \tag{30}$$

Upon substituting this in equation (29), we get

$$EI\frac{d^2y}{dX^2} + Py = Py^L \,. \tag{31}$$

This is a second order non-homogeneous equation. To find its general solution, we need to first solve the corresponding homogeneous equation and then add particular integral to it. The corresponding homogeneous equation is

$$EI\frac{d^2y}{dX^2} + Py = 0 \,. \tag{32}$$

We can note that $P$ is positive as it is always compressive in nature. If $P$ were a tensile load, it would become negative. So, we can substitute $y = e^{\lambda X}$ in the equation to get the complementary function.

$$\begin{aligned} &\Rightarrow \quad \lambda^2 e^{\lambda X} + \frac{P}{EI} e^{\lambda X} = 0 \\ &\Rightarrow \quad \lambda^2 + \frac{P}{EI} = 0 \\ &\text{Or,} \quad \lambda = \pm i\sqrt{\frac{P}{EI}} \end{aligned} \tag{33}$$

Here, $i$ represents the imaginary number $\sqrt{-1}$. As $\lambda$ is imaginary, the solution has cosine and sine parts. The complementary function can thus be written as

$$y = C_1 \cos\left(\sqrt{\frac{P}{EI}}X\right) + C_2 \sin\left(\sqrt{\frac{P}{EI}}X\right) \tag{34}$$

We can get the particular integral just by observation in this case. When we substitute $y = y^L$ in equation (31), we see that it satisfies the equation. As any solution/function that satisfies the differential equation can be considered as a particular integral, we can consider $y = y^L$ as the particular integral. Thus, the general solution of the non-homogeneous differential equation becomes

$$y = C_1 \cos\left(\sqrt{\frac{P}{EI}}X\right) + C_2 \sin\left(\sqrt{\frac{P}{EI}}X\right) + y^L. \tag{35}$$

To find the integration constants $C_1$ and $C_2$, we need to use boundary conditions. At the clamped end, both deflection and cross-section rotation are zero, i.e.,

$$y(0) = 0, \tag{36}$$

$$\theta(0) = \frac{dy}{dX}(0) = 0 \tag{37}$$

Differentiating equation (35), we get

$$y' = -C_1 \sin\left(\sqrt{\frac{P}{EI}}X\right) + C_2 \cos\left(\sqrt{\frac{P}{EI}}X\right) \tag{38}$$

Substituting $X = 0$ in this equation, we get

$$\begin{aligned} &y'(0) = C_2\sqrt{\frac{P}{EI}} = 0 \quad \text{(using (37))} \\ &\Rightarrow C_2 = 0. \end{aligned} \tag{39}$$

Similarly, substituting $X = 0$ in equation (35), we get

$$y(0) = C_1 + y^L = 0 \text{ (using (36))}$$

$$\Rightarrow C_1 = -y^L. \tag{40}$$

Thus, the general solution becomes

$$y(X) = y^L\left(1 - \cos\left(\sqrt{\frac{P}{EI}}X\right)\right). \tag{41}$$

To obtain $P$, we can use the fact that the above expression must be equal to $y^L$ if we substitute $X = L$ in it, i.e.,

$$\begin{aligned} y^L = y^L\left(1 - \cos\left(\sqrt{\frac{P}{EI}}L\right)\right) \Rightarrow 1 = 1 - \cos\left(\sqrt{\frac{P}{EI}}L\right) \\ \Rightarrow \quad \cos\left(\sqrt{\frac{P}{EI}}L\right) = 0 \quad \text{or,} \quad P = \left((2n+1)\frac{\pi}{2}\right)^2 \frac{EI}{L^2}. \end{aligned} \tag{42}$$

This is the expression for the buckling load. We can get multiple buckling loads by setting n=0,1,2,3 and so on. The smallest buckling load will be obtained for $n = 0$. This value is the critical buckling load that we wanted to find. Thus

$$P^{cr} = \frac{\pi^2 EI}{4L^2}. \tag{43}$$

This is the buckling load for column buckling. If the compressive force is greater than the above critical value, the beam will bend (buckle), otherwise it will simply remain straight. The critical buckling load is proportional to bending stiffness *EI* and inversely proportional to square of the beam's length. So, a longer beam requires lesser force to buckle than a shorter beam. This can be experienced easily in real life. The buckling solution was obtained by substituting the boundary conditions in general solution (35). If we apply different sets of boundary conditions, we would obtain different expression for buckling load. For example, for a beam clamped at both the ends as shown in Figure 11,

Figure 11: Buckling of a clamped-clamped beam under compressive force

the buckling load turns out to be

$$P^{cr} = \frac{4\pi^2 EI}{L^2}. \tag{44}$$

In fact, the buckled shape of the beam is also different (see the red dashed buckled solution in Figure 11). However, the buckling load is again inversely proportional to square of the beam's length. With this, we close our discussion on beam theory.