Solid Mechanics
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Lecture - 21
Extension-Torsion-Inflation in a Hollow Cylinder (Contd.)

Hello everyone! Welcome to Lecture 21! We will continue with our discussion on Extension-Torsion-Inflation in a hollow cylinder. In the previous lecture, we had derived the reduced form of equilibrium equations which we need to solve to obtain the displacement.

1 Recap (start time: 00:27)

We had derived the following simplified form of equilibrium equations when the cylinder is in static equilibrium and not subjected to any body force:

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0, \tag{1}$$

$$\frac{\partial \sigma_{zz}}{\partial z} = 0. \tag{2}$$

We had also obtained the following expressions of stress components in terms of displacement components:

$$\sigma_{rr} = \lambda(u_r' + \frac{u_r}{r} + u_z') + 2\mu u_r', \tag{3}$$

$$\sigma_{\theta\theta} = \lambda(u_r' + \frac{u_r}{r} + u_z') + 2\mu \frac{u_r}{r}, \tag{4}$$

$$\sigma_{zz} = , \lambda(u_r' + \frac{u_r}{r} + u_z') + 2\mu u_z' \tag{5}$$

$$\tau_{r\theta} = 0, \tag{6}$$

$$\tau_{rz} = 0, \tag{7}$$

$$\tau_{\theta z} = G\frac{\Omega r}{L}. \tag{8}$$

2 Mathematical form of u_z (start time: 01:05)

As u_z and u_r are functions of z and r respectively, we can rewrite equation (5) as

$$\sigma_{zz} = \underbrace{(\lambda + 2\mu)u_z'}_{\text{depends on } z} + \underbrace{\lambda\left(u_r' + \frac{u_r}{r}\right)}_{\text{depends on } r}. \tag{9}$$

From equation (2), we can infer that σ_{zz} does not depend on z. Also, σ_{zz} does not have any term that has dependence on θ. Thus, it is a function of r only, i.e.,

$$\sigma_{zz} = \underbrace{(\lambda + 2\mu)u_z'}_{\text{depends on } z} + \underbrace{\lambda\left(u_r' + \frac{u_r}{r}\right)}_{\text{depends on } r} = f(r). \tag{10}$$

Accordingly, the term dependent on z in the LHS must be a constant or $u_z{}'$ must be a constant. As $u_z{}'$ denotes longitudinal strain in axial direction, we will denote it by ϵ, an unknown but constant parameter.

3 Mathematical form of u_r (start time: 05:04)

The expressions of σ_{rr} and $\sigma_{\vartheta\vartheta}$ given in (3) and (4) now become

$$\sigma_{rr} = \lambda(u_r' + \frac{u_r}{r} + \epsilon) + 2\mu u_r' \tag{11}$$

$$\sigma_{\theta\theta} = \lambda(u_r' + \frac{u_r}{r} + \epsilon) + 2\mu \frac{u_r}{r} \tag{12}$$

Subtracting them, we get

$$\sigma_{rr} - \sigma_{\theta\theta} = 2\mu\left(u_r' - \frac{u_r}{r}\right). \tag{13}$$

Likewise, taking the partial derivative of equation (11) with respect to r, we get

$$\frac{\partial \sigma_{rr}}{\partial r} = \lambda\left(u_r' + \frac{u_r}{r}\right)' + 2\mu u_r'' \qquad \text{(as } \epsilon \text{ is constant)} \tag{14}$$

Now, we can plug equations (13) and (14) into equilibrium equation (1) to get

$$\begin{aligned} &\lambda\left(u_r' + \frac{u_r}{r}\right)' + 2\mu u_r'' + 2\mu\left(u_r' - \frac{u_r}{r}\right) = 0 \\ \Rightarrow\ &\lambda\left(u_r' + \frac{u_r}{r}\right)' + 2\mu\left(u_r' + \frac{u_r}{r}\right)' = 0 \\ \Rightarrow\ &\left(u_r' + \frac{u_r}{r}\right)' = 0 \Rightarrow \left(\frac{1}{r}(u_r r)'\right)' = 0. \end{aligned} \tag{15}$$

We can infer from this that $\left(u_r' + \frac{u_r}{r}\right)$ is also a constant, say C. Further, from the definition of strain components, we know that $\epsilon_{rr} = u_r'$ and $\epsilon_{\theta\theta} = \frac{u_r}{r}$. Thus, we get

$$\boxed{u_r' + \frac{u_r}{r} = \epsilon_{rr} + \epsilon_{\theta\theta} = C} \tag{16}$$

Integrating (15) twice, we finally get

$$u_r = \frac{C}{2}r + \frac{D}{r} \tag{17}$$

where *C* and *D* are the unknown integrating constants.

4 Solution for σ_{rr} and $\sigma_{\theta\theta}$ (start time: 09:51)

4.1 Mathematical form (start time: 09:51)

Let us add equations (11) and (12):

$$\begin{aligned}\sigma_{rr} + \sigma_{\theta\theta} &= 2(\lambda+\mu)\left(u_r' + \frac{u_r}{r}\right) + 2\lambda\epsilon \\ &= 2(\lambda+\mu)C + 2\lambda\epsilon = A \quad \text{(a constant)}\end{aligned} \tag{18}$$

Thus, the sum of radial and hoop stresses turns out to be a constant through the thickness of the tube, however they individually vary through the tube's thickness. Let us now solve the equilibrium equation (1) directly in terms of stress components as follows:

$$\begin{aligned}\frac{\partial\sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} &= 0 \\ \Rightarrow \frac{\partial\sigma_{rr}}{\partial r} + 2\frac{\sigma_{rr}}{r} - \frac{\sigma_{rr} + \sigma_{\theta\theta}}{r} &= 0 \\ \Rightarrow \frac{\partial\sigma_{rr}}{\partial r} + 2\frac{\sigma_{rr}}{r} &= \frac{A}{r} \quad \text{(using (18))}.\end{aligned} \tag{19}$$

From equation (11), we know that σ_{rr} is a function of *r* alone. Thus, the partial derivative with respect to *r* becomes total derivative, i.e.,

$$\begin{aligned}&\frac{d\sigma_{rr}}{dr} + 2\frac{\sigma_{rr}}{r} = \frac{A}{r} \\ &\Rightarrow \frac{1}{r^2}\frac{d}{dr}(\sigma_{rr}r^2) = \frac{A}{r} \\ &\Rightarrow (\sigma_{rr}r^2)' = Ar \\ &\Rightarrow \sigma_{rr}r^2 = \frac{Ar^2}{2} + B \\ &\text{or, } \sigma_{rr} = \frac{A}{2} + \frac{B}{r^2} \text{ and } \sigma_{\theta\theta} = \frac{A}{2} - \frac{B}{r^2} \quad \text{(using (18))}\end{aligned} \tag{20}$$

4.2 Application of boundary conditions (start time: 14:48)

Whenever we solve a differential equation, we get unknown integrating constants. To obtain those constants, one has to apply boundary condition. Similarly, we need to identify the boundary conditions for our deformation problem. Figure 1 shows our cylinder which is subjected to pressure on its inner surface. The pressure acts as an externally applied known traction and thus can be used as one of the boundary conditions. Similarly, there is zero traction on the outer surface of the cylinder which provides the other boundary condition. We had seen earlier that the internal traction due to stress at the surface point equals the externally applied traction ($\underline{t}^{app}$) through the following relation:

$$\underline{\underline{\sigma}}\,\underline{n} = \underline{t}^{app} \tag{21}$$

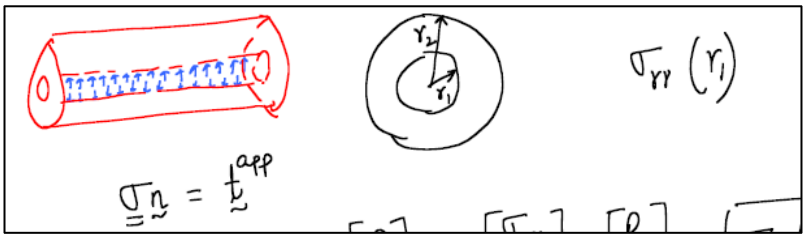

Figure 1: Pressure acts on the inner surface of our hollow cylinder. The cross section of the cylinder is shown on the right.

The outward surface normal of the inner curved surface (at $r = r_1$) points in $-r$ direction, i.e.,

$$\left[\underline{n}\right]_{(r,\theta,z)} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \tag{22}$$

Similarly, the external traction acts radially outward there, i.e., in the $+r$ direction. So

$$\left[\underline{t}^{\text{app}}\right]_{(r,\theta,z)} = \begin{bmatrix} P \\ 0 \\ 0 \end{bmatrix} \tag{23}$$

Upon writing equation (21) in cylindrical coordinate system and further substituing the above two results, we get:

$$\begin{bmatrix} \sigma_{rr} & \tau_{r\theta} & \tau_{rz} \\ \tau_{\theta r} & \sigma_{\theta\theta} & \tau_{\theta z} \\ \tau_{zr} & \tau_{z\theta} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} P \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -\sigma_{rr} \\ -\tau_{\theta r} \\ -\tau_{zr} \end{bmatrix} = \begin{bmatrix} P \\ 0 \\ 0 \end{bmatrix}$$

$$\text{or, } \sigma_{rr}(r_1) = -P, \qquad \tau_{\theta r}(r_1)0, \qquad \tau_{zr}(r_1) = 0. \tag{24}$$

Similar analysis for the outer surface where no external traction is present yields

$$\sigma_{rr}(r_2) = 0, \qquad \tau_{\theta r}(r_2), \qquad \tau_{zr}(r_2) = 0. \tag{25}$$

We have thus obtained the following two boundary conditions for σ_{rr} to obtain the unknown integrating constants in its expression (20):

$$\sigma_{rr}(r_1) = -P, \qquad \sigma_{rr}(r_2) = 0. \tag{26}$$

4.3 Final Solution (start time: 21:02)

Upon plugging the boundary condition (26) in equation (20), we get the following set of equations:

$$\frac{A}{2} + \frac{B}{r_1^2} = -P, \quad \frac{A}{2} + \frac{B}{r_2^2} = 0 \tag{27}$$

solving which we get

$$A = 2P\frac{r_1^2}{r_2^2 - r_1^2}, \quad \cdot\, B = -P\frac{r_1^2 r_2^2}{r_2^2 - r_1^2} \tag{28}$$

For a positive internal pressure P, we always have $A > 0$ and $B < 0$. Using (20), we can now plot the variation in both σ_{rr} and $\sigma_{\theta\theta}$ through the tube's thickness as shown in Figure 2. The red curve shows the variation of $\sigma_{\theta\theta}$ while the black curve shows the variation of σ_{rr}.

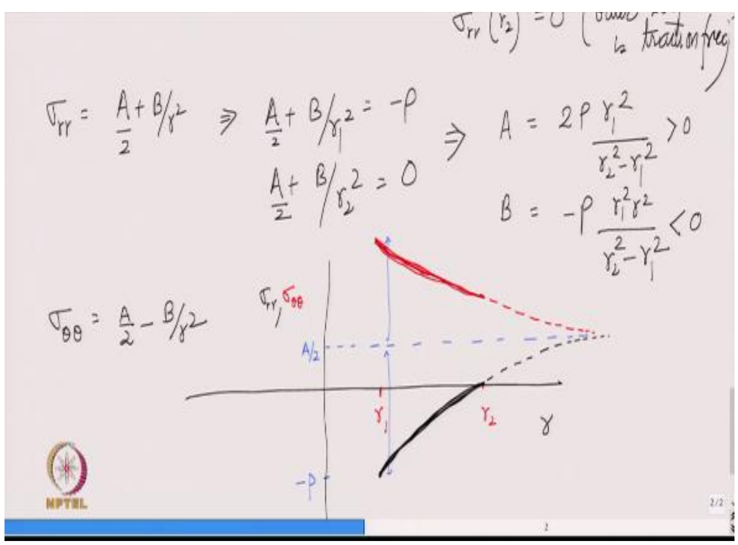

Figure 2: Plot showing variation of σ_{rr} and $\sigma_{\theta\theta}$ with r

As $r \to \infty$, both σ_{rr} and $\sigma_{\theta\theta}$ approach $\frac{A}{2}$ which is a positive number for positive pressure P although r is feasible only between r_1 and r_2. From the boundary condition, we also know directly from boundary condition that σ_{rr} is $-P$ at $r = r_1$ and 0 at $r = r_2$. As the sum of σ_{rr} and $\sigma_{\theta\theta}$ always has to be A, the curve for $\sigma_{\theta\theta}$ is the mirror image of σ_{rr} about the blue dashed line. Furthermore, σ_{rr} and $\sigma_{\theta\theta}$ depend only on P and the radii as can be seen from equations (28) and (20). If there is no internal pressure, σ_{rr} and $\sigma_{\theta\theta}$ will simply vanish even if axial force and twisting moment are present. Thus, we infer that extension and torsion cannot generate σ_{rr} and $\sigma_{\theta\theta}$. This happens because the cross section is free to relax during extension and torsion of a circular cylinder. For cross-sections of irregular shape however, we can have non-zero σ_{rr} and $\sigma_{\theta\theta}$ even due to extension and torsion.

5 Final solution for u_r (start time: 29:27)

We have to find the constants (C,D) in the expression (17) to obtain complete solution of u_r. The constant C can be found using (18) as follows:

$$
\begin{aligned}
2(\lambda+\mu)C + 2\lambda\epsilon &= A = 2P\frac{r_1^2}{r_2^2 - r_1^2} \\
\Rightarrow C &= \frac{-\lambda}{\lambda+\mu}\epsilon + \frac{P}{\lambda+\mu}\frac{r_1^2}{r_2^2 - r_1^2}
\end{aligned}
\tag{29}
$$

To get D, we can use equation (11) as shown below:

$$
\begin{aligned}
\sigma_{rr} &= \lambda\left(u_r' + \frac{u_r}{r} + \epsilon\right) + 2\mu u_r' \\
&= \lambda(C+\epsilon) + 2\mu\left(\frac{C}{2} - \frac{D}{r^2}\right) \qquad \text{(using (17))} \\
&= (\lambda+\mu)C + \lambda\epsilon - 2\mu\frac{D}{r^2} \\
&= \frac{A}{2} - 2\mu\frac{D}{r^2} \qquad \text{(using (18))}
\end{aligned}
\tag{30}
$$

Comparing this with equation (20), we get:

$$
\begin{aligned}
\frac{B}{r^2} &= -2\mu\frac{D}{r^2} \\
\Rightarrow D &= \frac{-B}{2\mu} = \frac{P}{2\mu}\frac{r_1^2 r_2^2}{r_2^2 - r_1^2} \qquad \text{(using (28))}
\end{aligned}
\tag{31}
$$

Thus, u_r finally becomes

$$
\boxed{u_r = \left(\frac{-\lambda}{2(\lambda+\mu)}\epsilon + \frac{P}{2(\lambda+\mu)}\frac{r_1^2}{r_2^2 - r_1^2}\right)r + \frac{P}{2\mu r}\frac{r_1^2 r_2^2}{r_2^2 - r_1^2}}.
\tag{32}
$$

Upon setting P=0 in (32), we get

$$u_r = \frac{-\lambda}{2(\lambda+\mu)}\epsilon r. \tag{33}$$

When we relate Lame's constants (λ and μ) with Young's modulus(E), Poisson's ratio(v) and shear modulus(G), it turns out that

$$\nu = \frac{\lambda}{2(\lambda+\mu)} \tag{34}$$

substituting which in the above expression for u_r, we get

$$u_r = -\nu\epsilon r \tag{35}$$

The radial longitudinal strain for P = 0 then turns out to be

$$\epsilon_{rr} = u_r' = -\nu\epsilon. \tag{36}$$

This expression is exactly what we expect - radial displacement has arisen due to Poisson's effect even in the absence of pressure.

6 Solution for u_z and u_θ (start time: 36:30)

As u_z is only a function of z and axial strain u_z' is a constant, integrating axial strain leads to

$$u_z = \epsilon z. \tag{37}$$

Here we assumed that u_z vanishes when z = 0 because the axial displacement of the cylindrical mid-section (z = 0) is zero by symmetry. We had also derived the expression for u_θ in the previous lecture as

$$u_\theta = \frac{\Omega}{L} rz \tag{38}$$

We need to finally obtain axial strain ϵ and end-to-end rotation Ω in terms of prescribed quantities (applied axial force F and Torque T).

6.1 Relating torque and end-to-end rotation (start time: 37:37)

A typical cross-section of the hollow cylinder is shown in Figure 3. The cross-section normal points in the z direction. So, σ_{zz}, $\tau_{\theta z}$ and τ_{rz} act on it.

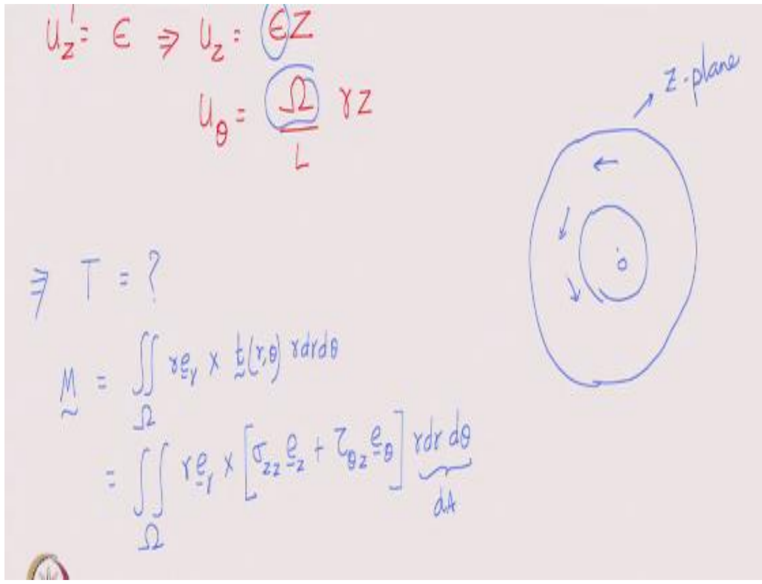

Figure 3: A typical cross-section of the hollow cylinder: $\tau_{\theta z}$ acts in θ direction and contributes to torque.

We had found τ_{rz} to be zero. Thus, we need to analyze σ_{zz} and $\tau_{\theta z}$ only. If the traction on any point on the cross-section is represented by $\underline{t}$, then the moment due to this traction about the cross-section center 'O' will be given by the integration of $\underline{r} \times \underline{t}$ for each small area element in the cross section ($\underline{r}$ represents the position vector of the area element from the center). If Ω_0 denotes the area of the cross section, the moment M will be given by

$$\begin{aligned} \underline{M} &= \iint_{\Omega_0} r\underline{e}_r \times \underline{t}(r,\theta) dA \\ &= \iint_{\Omega_0} r\underline{e}_r \times \underline{t}(r,\theta) r dr d\theta \\ &= \iint_{\Omega_0} r\underline{e}_r \times \left[\sigma_{zz}\underline{e}_z + \tau_{\theta z}\underline{e}_\theta\right] dA \end{aligned} \tag{39}$$

The torque T is simply the component of moment along the axis, i.e.,

$$\begin{aligned} T = \underline{M} \cdot \underline{e}_z &= \iint_{\Omega_0} \left(r\underline{e}_r \times \left[\sigma_{zz}\underline{e}_z + \tau_{\theta z}\underline{e}_\theta\right]\right) \cdot \underline{e}_z \, dA \\ &= \iint_{\Omega_0} r\tau_{\theta z} \, dA = \iint_{\Omega_0} rG\frac{\Omega r}{L} \, dA \qquad \text{(using (8))} \\ &= G\frac{\Omega}{L} \iint_{\Omega_0} r^2 dA. \end{aligned} \tag{40}$$

The term $\iint_{\Omega_0} r^2 dA$ is the polar moment of area (a geometrical quantity) and is denoted by J. Thus, we finally get

$$T = GJ\frac{\Omega}{L} \Rightarrow \Omega = \frac{TL}{GJ} \tag{41}$$

6.2 Relating axial force and axial strain (start time: 44:23)

To find ϵ, let us obtain the axial force in the cross-section through the integration of σ_{zz}, i.e.,

$$\begin{aligned} F &= \iint_{\Omega_0} \sigma_{zz} dA \\ &= \iint_{\Omega_0} \left[\lambda\left(u_r' + \frac{u_r}{r} + u_z'\right) + 2\mu u_z'\right] dA \qquad \text{(using (5))} \\ &= \iint_{\Omega_0} \left[\lambda C + (\lambda + 2\mu) u_z'\right] dA \qquad \text{(using (16))} \\ &= \iint_{\Omega_0} \left[\lambda C + (\lambda + 2\mu)\epsilon\right] dA = \lambda C A + (\lambda + 2\mu)\epsilon A. \end{aligned} \tag{42}$$

Here, A denotes the cross-sectional area. As we know the value of C from equation (29), we are finally able to relate axial force F with axial strain ϵ. In the special case when $P = 0$, the expression of C becomes simpler which yields

$$F = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} A\epsilon = EA\,\epsilon \tag{43}$$

where E is the Young's modulus of elasticity. In case of zero pressure, ϵ can also be obtained in a simpler way. We know that when $P = 0$, we get $\sigma_{rr} = \sigma_{\theta\theta} = 0$. Using three-dimensional Hooke's law, we can then write

$$\begin{aligned} \epsilon_{zz} &= \frac{1}{E}(\sigma_{zz} - \nu(\sigma_{rr} + \sigma_{\theta\theta})) = \frac{1}{E}(\sigma_{zz} - \nu \times 0) = \frac{\sigma_{zz}}{E} \\ \Rightarrow \sigma_{zz} &= E\epsilon. \end{aligned} \tag{44}$$

The axial force F for such a situation is then

$$F = \iint_{\Omega_0} \sigma_{zz}\, dA = EA\,\epsilon. \tag{45}$$

We have finally derived the relationship between Ω and T as well as the relationship between ϵ and F. The constant EA is called stretching stiffness while the constant GJ is called torsional stiffness, i.e.,

$$\begin{aligned} \underbrace{F}_{\text{axial force}} &= \underbrace{EA}_{\text{stretching stiffness}} \underbrace{\epsilon}_{\text{axial strain}} \\ \underbrace{T}_{\text{torque}} &= \underbrace{GJ}_{\text{torsional stiffness}} \underbrace{\frac{\Omega}{L}}_{\text{twist}}. \end{aligned}$$

The axial strain and twist are constant in the cross section of the cylinder.

7 Variation of $\gamma_{\theta z}$ and $\tau_{\theta z}$ in the cross section (start time: 50:18)

We know

$$\gamma_{\theta z} = \frac{\Omega}{L} r, \quad \tau_{\theta z} = G\gamma_{\theta z} = G\frac{\Omega}{L} r. \tag{46}$$

Note that both the quantities vary linearly with r. This means that whenever we twist a cylinder/bar, shear strain and shear stress vary linearly in the cross-section as we go outwards from the center (see Figure 4).

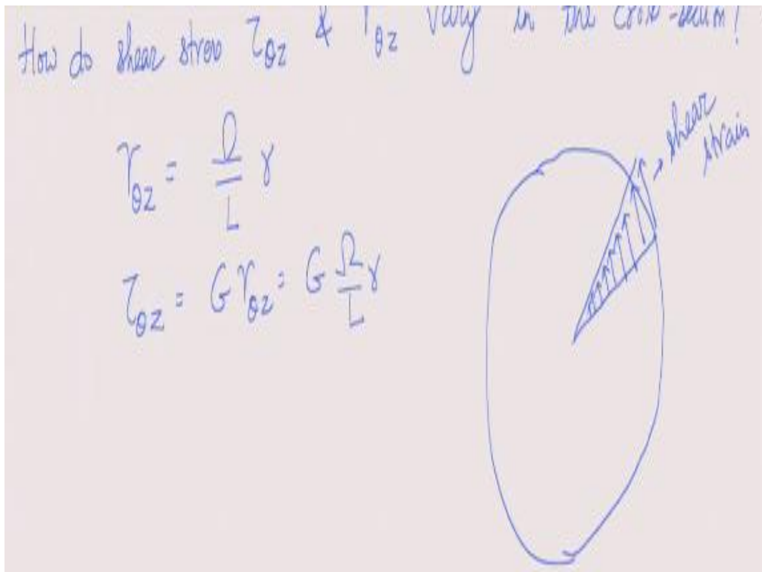

Figure 4: Variation of shear strain and shear stress in the cross section of the cylinder.

7.1 Special case: composite cylinder (start time: 52:10)

We can also think of a composite cylinder made up of two different materials as shown in Figure 5. Supposing the inner part (upto radius r_1) is made up of Aluminium and the outer part is made up of Steel. The materials are glued together.

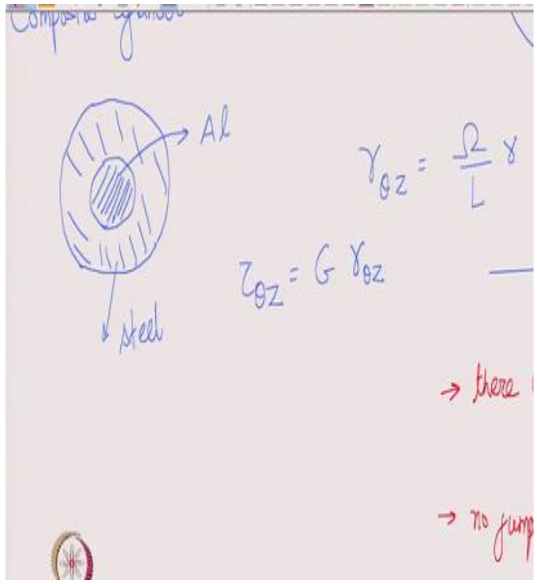

Figure 5: The cross section of a composite cylinder made up of Aluminium and Steel.

If we now twist the cylinder, the cross section is again going to rotate. The shear strain $\gamma_{\theta z}$ will be the same as earlier since it is completely prescribed by applied deformation and thus varies continuously. However, when we calculate $\tau_{\theta z}$ by equation (46), we will have different shear modulus for aluminium and steel. Thus, there will be a discontinuity in the shear stress at $r = r_1$. The plot of variation of $\gamma_{\theta z}$ and $\tau_{\theta z}$ in the cross-section is shown in Figure 6.

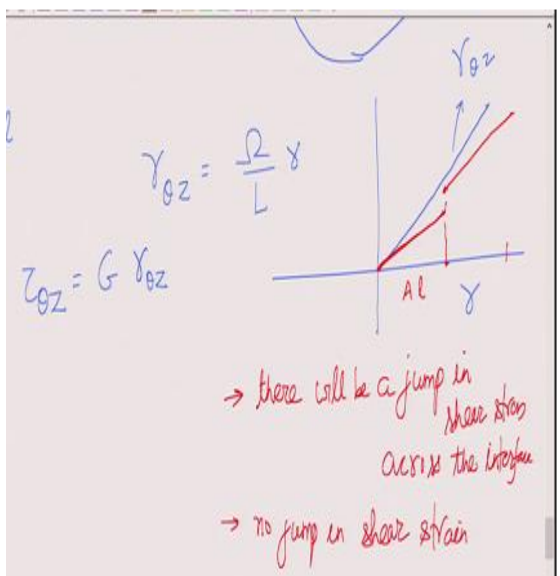

Figure 6: Variation of shear strain and shear stress in the cross section of the composite cylinder.

The variation of $\gamma_{\theta z}$ is shown by the continuous blue line while the variation of $\tau_{\theta z}$ is shown by the discontinuous red line (which is piecewise linear): the slopes of the two straight lines are different and proportional to the shear modulus of the corresponding material. There is no jump in shear strain because the steel and aluminum regions together act as a single body as they are assumed to be attached rigidly.