Solid Mechanics
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Lecture - 18
Linear Momentum Balance in Cylindrical Coordinate System (Contd...)

Hello everyone! Welcome to Lecture 18! In this lecture, we will continue with the derivation of Linear Momentum Balance LMB in cylindrical coordinate system.

1 Recap (start time: 00:28)

In the last lecture, we were discussing LMB Formulation in cylindrical coordinate system. We had considered a cylindrical element as shown in Figure 1.

Figure 1: A generalized cylinder with its axis along the z axis. A cylindrical element is considered within the cylinder.

After a lengthy derivation, We had figured out the total force due to traction on $+z$ and $-z$ planes as given below:

$$\underline{F}^{+z} + \underline{F}^{-z} = \left(\frac{\partial \sigma_{zz}}{\partial z}\underline{e}_z + \frac{\partial \tau_{\theta z}}{\partial z}\underline{e}_\theta + \frac{\partial \tau_{rz}}{\partial z}\underline{e}_r\right)\Delta V + o(\Delta V) \tag{1}$$

We now present a simpler but approximate derivation to obtain the same result.

2 A simpler approach to find the traction force (start time: 01:58)

Let us consider the cylindrical element shown in Figure 2. Earlier, we had done the derivation by also considering the variation of traction at different points on $+z$ and $-z$ planes. If we assume the traction to be constant everywhere on the plane with its value being equal to the value at the center of the plane, we can directly multiply it with the area of the plane to get the total force. We illustrate it now.

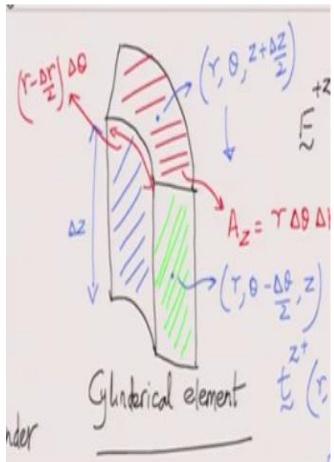

Figure 2: A cylindrical element with it's various dimensions

As the center of the cylindrical element has coordinates (r,θ,z), the center of the $+z$ plane will be at $(r,\theta,z+\frac{\Delta z}{2})$. Thus, the traction at the center of $+z$ plane will be

$$\underline{t}^{+z}\left(r,\theta,z+\frac{\Delta z}{2}\right)=\sigma_{zz}\left(r,\theta,z+\frac{\Delta z}{2}\right)\underline{e}_z+\tau_{\theta z}\left(r,\theta,z+\frac{\Delta z}{2}\right)\underline{e}_\theta+\tau_{rz}\left(r,\theta,z+\frac{\Delta z}{2}\right)\underline{e}_r \tag{2}$$

The three basis vectors in the above expression, when evaluated at the center of $+z$ plane will be the same as the one at the center of cylindrical element because the two points have the same θ coordinate. Similarly, traction on the $-z$ face at its center will be

$$\underline{t}^{-z}\left(r,\theta,z-\frac{\Delta z}{2}\right)=-\sigma_{zz}\left(r,\theta,z-\frac{\Delta z}{2}\right)\underline{e}_z-\tau_{\theta z}\left(r,\theta,z-\frac{\Delta z}{2}\right)\underline{e}_\theta-\tau_{rz}\left(r,\theta,z-\frac{\Delta z}{2}\right)\underline{e}_r. \tag{3}$$

All the traction components point in the negative direction as this is $-z$ face. The basis vectors at this point are also the same as those at the center of the cylindrical element. In the previous lecture, we had also found that the area of the z faces is:

$$A_z = r\Delta\theta\Delta r \tag{4}$$

Thus, the total force on the two z planes will be

$$\underline{F}^{+z} + \underline{F}^{-z} = \left[\underline{t}^{+z}\left(r, \theta, z + \frac{\Delta z}{2}\right) + \underline{t}^{-z}\left(r, \theta, z - \frac{\Delta z}{2}\right)\right] A_z \tag{5}$$

which is obtained after assuming that traction does not vary over different points in a face of the cylindrical element. Now, we do the Taylor's expansion of the components of traction about the center of the cylindrical element. As only *z* coordinate has changed, the Taylor's expansion will only have terms corresponding to derivatives of *z*. Also, there will be no variation in the basis vectors. The first term for the tractions on +*z* and −*z* planes can be expanded as

$$\begin{aligned} \sigma_{zz}\left(r, \theta, z + \frac{\Delta z}{2}\right)\underline{e}_z &= \left[\sigma_{zz}(r, \theta, z) + \frac{\partial \sigma_{zz}}{\partial z}(r, \theta, z)\frac{\Delta z}{2}\right]\underline{e}_z, \\ -\sigma_{zz}\left(r, \theta, z - \frac{\Delta z}{2}\right)\underline{e}_z &= \left[-\sigma_{zz}(r, \theta, z) + \frac{\partial \sigma_{zz}}{\partial z}(r, \theta, z)\frac{\Delta z}{2}\right]\underline{e}_z \end{aligned} \tag{6}$$

When we add the above two equations, the σ_{zz} terms cancel while the partial derivative terms add. We can also expand the τ_{rz} and $\tau_{\theta z}$ terms in the same way. The total force on the *z* planes will thus become

$$\begin{aligned} \underline{F}^{+z} + \underline{F}^{-z} &= \left[\frac{\partial \sigma_{zz}}{\partial z}\underline{e}_z + \frac{\partial \tau_{\theta z}}{\partial z}\underline{e}_\theta + \frac{\partial \tau_{rz}}{\partial z}\underline{e}_r\right]\Delta z A_z + o(\Delta V) \\ &= \left[\frac{\partial \sigma_{zz}}{\partial z}\underline{e}_z + \frac{\partial \tau_{\theta z}}{\partial z}\underline{e}_\theta + \frac{\partial \tau_{rz}}{\partial z}\underline{e}_r\right]\Delta V + o(\Delta V). \end{aligned} \tag{7}$$

We have thus gotten the same result that when we had obtained while also considering variation of traction over the plane. This is because the variation of traction over the *z* planes only gives us smaller order terms and is captured in $o(\Delta V)$ term. This term vanishes when we divide by ΔV and shrink the volume of the cylindrical element to a point. Therefore, the variation of traction over the planes does not play any role.

3 Force on +*r* and −*r* planes (start time: 14:04)

For getting the total force on +*r* and −*r* planes, we will use the approximate method just discussed above as the exact derivation is much more tedious. We thus assume that traction on the +*r* plane (the convex plane) is constant and equal to the value at its center, i.e. at $(r + \frac{\Delta r}{2}, \theta, z)$ and also the traction on the −*r* plane is constant and equal to the value at $(r - \frac{\Delta r}{2}, \theta, z)$. We will denote the area of the +*r* plane by A_{r+} and the area of the −*r* plane by A_{r-}. From Figure 2, we can observe that the area of the *r* planes will be equal to the curved edge multiplied with the height. Thus:

$$A_{r-} = \underbrace{\left[\left(r - \frac{\Delta r}{2}\right)\Delta\theta\right]}_{\text{curved edge}} \underbrace{\Delta z}_{\text{height}}, \qquad A_{r+} = \underbrace{\left[\left(r + \frac{\Delta r}{2}\right)\Delta\theta\right]}_{\text{curved edge}} \underbrace{\Delta z}_{\text{height}} \tag{8}$$

The forces on both the planes can be simply obtained by multiplying the traction at the center of the planes with the area of the planes. So, the total force on $+r$ and $-r$ planes will be

$$\begin{aligned} \underline{F}^{+r} + \underline{F}^{-r} &= \underline{t}^{+r}\left(r + \frac{\Delta r}{2}, \theta, z\right) A_{r+} + \underline{t}^{-r}\left(r - \frac{\Delta r}{2}, \theta, z\right) A_{r-} \\ &= \underline{t}^{+r}\left(r + \frac{\Delta r}{2}, \theta, z\right)\left(r + \frac{\Delta r}{2}\right)\Delta\theta\Delta z + \underline{t}^{-r}\left(r - \frac{\Delta r}{2}, \theta, z\right)\left(r - \frac{\Delta r}{2}\right)\Delta\theta\Delta z \end{aligned} \tag{9}$$

The traction vectors can be written as

$$\begin{aligned} \underline{t}^{+r}\left(r + \frac{\Delta r}{2}, \theta, z\right) &= \sigma_{rr}\left(r + \frac{\Delta r}{2}, \theta, z\right)\underline{e}_r + \tau_{\theta r}\left(r + \frac{\Delta r}{2}, \theta, z\right)\underline{e}_\theta + \tau_{zr}\left(r + \frac{\Delta r}{2}, \theta, z\right)\underline{e}_z, \\ \underline{t}^{-r}\left(r - \frac{\Delta r}{2}, \theta, z\right) &= -\sigma_{rr}\left(r - \frac{\Delta r}{2}, \theta, z\right)\underline{e}_r - \tau_{\theta r}\left(r - \frac{\Delta r}{2}, \theta, z\right)\underline{e}_\theta - \tau_{zr}\left(r - \frac{\Delta r}{2}, \theta, z\right)\underline{e}_z \end{aligned} \tag{10}$$

The basis vectors at the centers of the r planes are the same as those at the center of the cylindrical element as they have the same θ coordinate. Thus, we would not have to use Taylor's expansion for basis vectors. Let's consider the first term in the total force expression:

$$\sigma_{rr}\left(r + \frac{\Delta r}{2}, \theta, z\right)\underline{e}_r\left(r + \frac{\Delta r}{2}\right)\Delta\theta\Delta z. \tag{11}$$

We can now rearrange the terms to bring all the terms dependent on r and use the Taylor's expansion about the center of the cylindrical element. We know that not only σ_{rr} changes with r but the r term itself also changes with r. So, instead of doing the Taylor's expansion of σ_{rr} alone, we will consider our function to be $\sigma_{rr}r$ and do its Taylor's expansion, i.e.,

$$\begin{aligned} \sigma_{rr}\left(r + \frac{\Delta r}{2}, \theta, z\right)\underline{e}_r\left(r + \frac{\Delta r}{2}\right)\Delta\theta\Delta z &= \left[\sigma_{rr}\left(r + \frac{\Delta r}{2}, \theta, z\right)\left(r + \frac{\Delta r}{2}\right)\right]\underline{e}_r\Delta\theta\Delta z \\ &= \left[\sigma_{rr}r + \frac{\partial}{\partial r}(\sigma_{rr}r)\frac{\Delta r}{2} + \ldots\right]\underline{e}_r\Delta\theta\Delta z \\ &= \left[\sigma_{rr}r + \left(\frac{\partial\sigma_{rr}}{\partial r}r + \sigma_{rr}\right)\frac{\Delta r}{2} + \ldots\right]\underline{e}_r\Delta\theta\Delta z. \end{aligned} \tag{12}$$

The corresponding term for the $-r$ plane can be simplified in a similar manner, i.e.,

$$
\begin{aligned}
-\sigma_{rr}\left(r-\frac{\Delta r}{2},\theta,z\right)\underline{e}_r\left(r-\frac{\Delta r}{2}\right)\Delta\theta\Delta z &= -\left[\sigma_{rr}\left(r-\frac{\Delta r}{2},\theta,z\right)\left(r-\frac{\Delta r}{2}\right)\right]\underline{e}_r\Delta\theta\Delta z \\
&= -\left[\sigma_{rr}r-\frac{\partial}{\partial r}(\sigma_{rr}r)\frac{\Delta r}{2}\right]\underline{e}_r\Delta\theta\Delta z \\
&= \left[-\sigma_{rr}r+\left(\frac{\partial\sigma_{rr}}{\partial r}r+\sigma_{rr}\right)\frac{\Delta r}{2}\right]\underline{e}_r\Delta\theta\Delta z
\end{aligned} \tag{13}
$$

When we add these contributions from the $+r$ and $-r$ planes in the total force expression, σ_{rr} terms cancel and the partial derivative terms add up. A similar analysis for the other terms in the total force expression can be done to obtain

$$
\begin{aligned}
\underline{F}^{+r}+\underline{F}^{-r} &= \left(\frac{\partial\sigma_{rr}}{\partial r}r+\sigma_{rr}\right)\underline{e}_r\Delta r\Delta\theta\Delta z+\left(\frac{\partial\tau_{\theta r}}{\partial r}r+\tau_{\theta r}\right)\underline{e}_\theta\Delta r\Delta\theta\Delta z \\
&\quad +\left(\frac{\partial\tau_{zr}}{\partial r}r+\tau_{zr}\right)\underline{e}_z\Delta r\Delta\theta\Delta z.
\end{aligned} \tag{14}
$$

As the volume of the cylindrical element is $r\Delta r\Delta\theta\Delta z$, the above expression can also be written as

$$
\Rightarrow \underline{F}^{+r}+\underline{F}^{-r}=\left[\left(\frac{\partial\sigma_{rr}}{\partial r}+\frac{\sigma_{rr}}{r}\right)\underline{e}_r+\left(\frac{\partial\tau_{\theta r}}{\partial r}+\frac{\tau_{\theta r}}{r}\right)\underline{e}_\theta+\left(\frac{\partial\tau_{zr}}{\partial r}+\frac{\tau_{zr}}{r}\right)\underline{e}_z\right]\Delta V. \tag{15}
$$

We get some extra terms $\frac{\sigma_{rr}}{r}$, $\frac{\tau_{\theta r}}{r}$ and $\frac{\tau_{zr}}{r}$ which are not present in LMB for Cartesian coordinate system. This happened because, in the cylindrical coordinate system, the areas of the $+r$ plane and $-r$ planes are not the same. For a cuboid element however, the areas of all the opposite faces are the same. We did not get similar terms for $+z$ and $-z$ planes in the cylindrical coordinate system because the areas of those planes are the same.

4 Force on $+\theta$ and $-\theta$ planes (start time: 29:42)

We will again use the approximate method. The coordinates of the center of the $+\theta$ and $-\theta$ planes are $(r,\theta+\frac{\Delta\theta}{2},z)$ and $(r,\theta-\frac{\Delta\theta}{2},z)$ respectively. The areas of the $+\theta$ and $-\theta$ planes are the same but we will still get extra terms here because of difference in basis vectors on the two planes: the two planes have different θ coordinates. From Figure 2, we can note the areas of the θ faces (A_θ) is:

$$A_\theta = \Delta r\Delta z \tag{16}$$

The total force on $+\theta$ and $-\theta$ planes will therefore be

$$\underline{F}^{+\theta} + \underline{F}^{-\theta} = \left[\underline{t}^{+\theta}\left(r, \theta + \frac{\Delta\theta}{2}, z\right) + \underline{t}^{-\theta}\left(r, \theta - \frac{\Delta\theta}{2}, z\right)\right] A_\theta \tag{17}$$

The traction on the $+\theta$ plane will be

$$\begin{aligned} \underline{t}^{+\theta}\left(r, \theta + \frac{\Delta\theta}{2}, z\right) &= \sigma_{\theta\theta}\left(r, \theta + \frac{\Delta\theta}{2}, z\right)\underline{e}_\theta\left(r, \theta + \frac{\Delta\theta}{2}, z\right) \\ &+ \tau_{r\theta}\left(r, \theta + \frac{\Delta\theta}{2}, z\right)\underline{e}_r\left(r, \theta + \frac{\Delta\theta}{2}, z\right) \\ &+ \tau_{z\theta}\left(r, \theta + \frac{\Delta\theta}{2}, z\right)\underline{e}_z\left(r, \theta + \frac{\Delta\theta}{2}, z\right) \end{aligned} \tag{18}$$

Now, $\underline{e}_r$ and $\underline{e}_\theta$ also have to be expanded using Taylor's series. For the Taylor's expansion, we will consider $\sigma_{\theta\theta}\underline{e}_\theta$ and $\tau_{r\theta}\underline{e}_r$ as the functions to be expanded. We thus obtain

$$\begin{aligned} \underline{t}^{+\theta}\left(r, \theta + \frac{\Delta\theta}{2}, z\right) &= \left[\sigma_{\theta\theta}\underline{e}_\theta + \frac{\partial}{\partial\theta}(\sigma_{\theta\theta}\underline{e}_\theta)\frac{\Delta\theta}{2} + ...\right] + \left[\tau_{r\theta}\underline{e}_r + \frac{\partial}{\partial\theta}(\tau_{r\theta}\underline{e}_r)\frac{\Delta\theta}{2} + ...\right] + \left[\tau_{z\theta}\underline{e}_z + \frac{\partial\tau_{z\theta}}{\partial\theta}\frac{\Delta\theta}{2}\underline{e}_z + ...\right], \\ \underline{t}^{-\theta}\left(r, \theta + \frac{\Delta\theta}{2}, z\right) &= \left[-\sigma_{\theta\theta}\underline{e}_\theta + \frac{\partial}{\partial\theta}(\sigma_{\theta\theta}\underline{e}_\theta)\frac{\Delta\theta}{2} + ...\right] + \left[-\tau_{r\theta}\underline{e}_r + \frac{\partial}{\partial\theta}(\tau_{r\theta}\underline{e}_r)\frac{\Delta\theta}{2} + ...\right] + \left[-\tau_{z\theta}\underline{e}_z + \frac{\partial\tau_{z\theta}}{\partial\theta}\frac{\Delta\theta}{2}\underline{e}_z + ...\right] \end{aligned} \tag{19}$$

Upon adding the two contributions, we finally obtain

$$\underline{F}^{+\theta} + \underline{F}^{-\theta} = \left[\frac{\partial}{\partial\theta}(\sigma_{\theta\theta}\underline{e}_\theta)\Delta\theta + \frac{\partial}{\partial\theta}(\tau_{r\theta}\underline{e}_r)\Delta\theta + \frac{\partial\tau_{z\theta}}{\partial\theta}\Delta\theta\underline{e}_z\right]\Delta r\Delta z. \tag{20}$$

We can see from this equation that the product rule will give us extra terms in the form of the derivatives of basis vectors with respect to θ. We finally obtain

$$\begin{aligned} \underline{F}^{+\theta} + \underline{F}^{-\theta} &= \frac{1}{r}\left[\frac{\partial\sigma_{\theta\theta}}{\partial\theta}\underline{e}_\theta + \sigma_{\theta\theta}(-\underline{e}_r) + \frac{\partial\tau_{r\theta}}{\partial\theta}\underline{e}_r + \tau_{r\theta}\underline{e}_\theta + \frac{\partial\tau_{z\theta}}{\partial\theta}\underline{e}_z\right] r\Delta r\Delta\theta\Delta z \\ &= \frac{1}{r}\left[\frac{\partial\sigma_{\theta\theta}}{\partial\theta}\underline{e}_\theta + \sigma_{\theta\theta}(-\underline{e}_r) + \frac{\partial\tau_{r\theta}}{\partial\theta}\underline{e}_r + \tau_{r\theta}\underline{e}_\theta + \frac{\partial\tau_{z\theta}}{\partial\theta}\underline{e}_z\right]\Delta V. \end{aligned} \tag{21}$$

5 Total force due to traction (start time: 40:32)

We have found the force due to traction on all the faces of the cylindrical element. We can add equations (1), (15) and (21) to obtain the total force on the cylindrical element due to traction to be

$$\underline{F}^t = \Bigg[\underbrace{\frac{\partial \sigma_{zz}}{\partial z}\underline{e}_z + \frac{\partial \tau_{\theta z}}{\partial z}\underline{e}_\theta + \frac{\partial \tau_{rz}}{\partial z}\underline{e}_r}_{z \text{ plane}}$$
$$+ \underbrace{\left(\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr}}{r}\right)\underline{e}_r + \left(\frac{\partial \tau_{\theta r}}{\partial r} + \frac{\tau_{\theta r}}{r}\right)\underline{e}_\theta + \left(\frac{\partial \tau_{zr}}{\partial r} + \frac{\tau_{zr}}{r}\right)\underline{e}_z}_{r \text{ plane}}$$
$$+ \underbrace{\frac{1}{r}\left(\frac{\partial \tau_{r\theta}}{\partial \theta} - \sigma_{\theta\theta}\right)\underline{e}_r + \frac{1}{r}\left(\tau_{r\theta} + \frac{\partial \sigma_{\theta\theta}}{\partial \theta}\right)\underline{e}_\theta + \frac{\partial \tau_{z\theta}}{\partial \theta}\underline{e}_z}_{\theta \text{ plane}} \Bigg] \Delta V + o(\Delta V) \tag{22}$$

6 Body force (start time: 44:40)

For the body force, we just need to find the body force at the center of the cylindrical element and multiply it with the volume of the element. We further decompose the body force into components along $\underline{e}_r$, $\underline{e}_\theta$ and $\underline{e}_z$ to obtain

$$\underline{F}^b = (b_r\underline{e}_r + b_\theta\underline{e}_\theta + b_z\underline{e}_z)\Delta V + o(\Delta V) \tag{23}$$

7 Rate of Change of Linear Momentum (start time: 45:55)

Just like the body force term, the rate of change of linear momentum term ($\frac{d}{dt}\vec{P}$) will also be very similar to that of the Cartesian coordinate system. Only the acceleration vector has to be decomposed along the cylindrical coordinate basis vectors, i.e.,

$$\frac{d}{dt}(\overrightarrow{P}) = \rho(a_r\underline{e}_r + a_\theta\underline{e}_\theta + a_z\underline{e}_z)\Delta V + o(\Delta V) \tag{24}$$

8 Final Balance (start time: 46:49)

We have got all the terms for Linear Momentum Balance. We can now plug in equations (22), (23) and (24) in the following balance law

$$\underline{F}^t + \underline{F}^b = \frac{d}{dt}(\overrightarrow{P}) \tag{25}$$

We also divide both the sides by ΔV and take the limit $\Delta V \to 0$. By doing this, the $o(\Delta V)$ terms will drop out. We finally separate the terms in the final balance equation along different basis vectors to get the component form of the balance equation, i.e.,

Equation in *r*-direction:

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + b_r = \rho a_r \tag{26}$$

Equation in *ϑ*-direction:

$$\frac{\partial \tau_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + 2\frac{\tau_{r\theta}}{r} + b_\theta = \rho a_\theta \tag{27}$$

Equation in *z*-direction:

$$\frac{\partial \tau_{zr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{z\theta}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\tau_{zr}}{r} + b_z = \rho a_z \tag{28}$$

The first three terms in each of the equations are similar to what we get in Cartesian coordinate system. There is one extra term in each equation. The extra term for the θ-direction contains a factor of 2 because the two terms $\frac{\tau_{r\theta}}{r}$ and $\frac{\tau_{\theta r}}{r}$ can be combined as one using the symmetry of stress matrix. These equations can be remembered easily. First of all, we need to list down all the stress components in the direction we are considering. For example, for the *r*-direction, we need σ_{rr}, $\tau_{r\theta}$ and τ_{rz}. Then, we need to take their partial derivatives with respect to their second indices. Whenever we do a partial derivative with respect to θ, we need to also divide by *r*. However, the extra terms need to be remembered explicitly.

We can conclude from this derivation that the equations for cylindrical coordinate system are quite different when compared with the equations in Cartesian coordinate system. So, it was really important to work out this derivation explicitly. In the next lecture, we will see how the strain matrix can be represented in cylindrical coordinate system.