Solid Mechanics
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Lecture - 15
Stress-Strain relation

Hello everyone! Welcome to Lecture 15! In this lecture, we will talk about relation between stress and strain.

1 Need for stress-strain relation (start time: 00:25)

Consider a body clamped at one part of the boundary and acted upon by an external load on another part of its boundary as shown in Figure 1. Any general point in the reference configuration (denoted by position vector $\underline{X}$) is displaced by $\underline{u}(\underline{X},t)$ as shown in the figure.

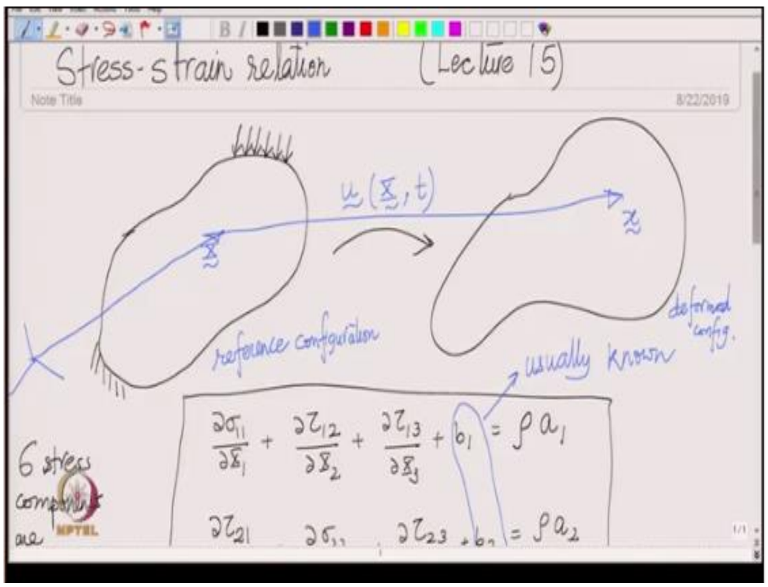

Figure 1: A body deforms due to some external load acting on it. A point $\underline{X}$ gets displaced to $\underline{x}$ due to deformation.

Usually, we are interested in knowing the final deformed shape of the body and the stress generated in it. The knowledge of stress is important because it helps us to check if it is within the prescribed limits so that the body does not fail or fracture. In order to find stress in the body, we can use the following equations of equilibrium derived earlier:

$$\frac{\partial \sigma_{11}}{\partial X_1} + \frac{\partial \tau_{12}}{\partial X_2} + \frac{\partial \tau_{13}}{\partial X_3} + b_1 = \rho a_1$$
$$\frac{\partial \tau_{21}}{\partial X_1} + \frac{\partial \sigma_{22}}{\partial X_2} + \frac{\partial \tau_{23}}{\partial X_3} + b_2 = \rho a_2$$
$$\frac{\partial \tau_{31}}{\partial X_1} + \frac{\partial \tau_{32}}{\partial X_2} + \frac{\partial \sigma_{33}}{\partial X_3} + b_3 = \rho a_3 \tag{1}$$

We have nine unknown components of stress matrix in three equation here. Using symmetry of stress matrix, the number of unknowns reduces to six. The body force components (b_1, b_2, b_3) are not unknowns as they are usually prescribed. For example, if gravitational body force is present, the body force will be ρg in the vertical direction. On the right hand side, we have components of acceleration which can be express in terms of displacement as we show now. The velocity of a particle located at $\underline{X}$ in the reference configuration will be

$$\underline{v}(\underline{X}, t) = \left.\frac{\partial \underline{u}}{\partial t}\right|_{\underline{X}} \tag{2}$$

By keeping $\underline{X}$ fixed here, we are ensuring that the particle does not change while taking the time derivative of displacement function. This is important otherwise the derivative obtained would not correspond to a specific particle. Let us denote the time derivatives with dots above the letter (n^{th} derivative has n dots), e.g.,

$$\frac{\partial \underline{u}}{\partial t} \equiv \dot{\underline{u}}, \quad \frac{\partial^2 \underline{u}}{\partial t^2} \equiv \ddot{\underline{u}} \tag{3}$$

Thus, we can write velocity and acceleration of a particle at $\underline{X}$ as

$$\begin{aligned} \underline{v}(\underline{X}, t) &= \dot{\underline{u}}(\underline{X}, t) \\ \underline{a}(\underline{X}, t) &= \left.\frac{\partial \underline{v}}{\partial t}\right|_{\underline{X}} = \ddot{\underline{u}}(\underline{X}, t) \end{aligned} \tag{4}$$

This description of velocity and acceleration is also called Lagrangian description: whenever we describe velocity, acceleration or any other quantity of a particle in terms of position of the particle in the reference configuration ($\underline{X}$), we call it Lagrangian description. We can now substitute the acceleration components in terms of time derivative of displacement in the equilibrium equation (1), i.e.,

$$\frac{\partial \sigma_{11}}{\partial X_1} + \frac{\partial \tau_{12}}{\partial X_2} + \frac{\partial \tau_{13}}{\partial X_3} + b_1 = \rho a_1 = \rho \ddot{u}_1,$$
$$\frac{\partial \tau_{21}}{\partial X_1} + \frac{\partial \sigma_{22}}{\partial X_2} + \frac{\partial \tau_{23}}{\partial X_3} + b_2 = \rho a_2 = \rho \ddot{u}_2,$$
$$\frac{\partial \tau_{31}}{\partial X_1} + \frac{\partial \tau_{32}}{\partial X_2} + \frac{\partial \sigma_{33}}{\partial X_3} + b_3 = \rho a_3 = \rho \ddot{u}_3 \tag{5}$$

We can observe that it has six unknown stress components and three unknown displacement components. So, we have nine unknowns in total but there are only three equilibrium equations (We have already used the symmetry of the stress tensor to reduce the unknowns for stress from nine to six). Thus, this system cannot be solved without the help of additional equations. These additional equations are given by stress-strain relation. We want to write stress in terms of strain or vice versa. We know that we can express strain in terms of displacement as given below:

$$\underline{\underline{\epsilon}} = \frac{1}{2}\left(\underline{\nabla}\,\underline{u} + \underline{\nabla}\,\underline{u}^T\right) \quad \text{or} \quad \epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) \tag{6}$$

If we can now in some way express stress in terms of strain, then indirectly, we have actually expressed stress in terms of displacement. So, in equilibrium equations, everything will be in terms of the three unknown displacement components. The system of three equations in three variables can then be solved. Thus, stress-strain relation is necessary to get the deformed solution of a body from equilibrium equations. Often, the stress-strain relation is also called constitutive relation.

2 Linear Stress-Strain Relation (start time: 15:44)

Our goal is to express stress in terms of strain, i.e. $\underline{\underline{\sigma}} = \underline{\underline{\sigma}}(\underline{\underline{\epsilon}})$. As stress and strain tensors are both symmetric, they have six independent components each. In general, each of the stress components will depend on each of the strain components.

$$\begin{aligned}
\sigma_{11} &= \sigma_{11}(\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, \epsilon_{12}, \epsilon_{13}, \epsilon_{23}),\\
\sigma_{22} &= \sigma_{22}(\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, \epsilon_{12}, \epsilon_{13}, \epsilon_{23}),\\
\sigma_{33} &= \sigma_{33}(\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, \epsilon_{12}, \epsilon_{13}, \epsilon_{23}),\\
\sigma_{12} &= \sigma_{12}(\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, \epsilon_{12}, \epsilon_{13}, \epsilon_{23}),\\
\sigma_{13} &= \sigma_{13}(\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, \epsilon_{12}, \epsilon_{13}, \epsilon_{23}),\\
\sigma_{23} &= \sigma_{23}(\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, \epsilon_{12}, \epsilon_{13}, \epsilon_{23}).
\end{aligned} \tag{7}$$

The functional form of this dependence can be anything (e.g., linear, exponential, logarithmic etc.) which will vary from material to material. There is no physical law to derive the exact functional form and therefore mechanicians resort to experiments on real material specimen and data fitting technique to obtain these relations. As we can express any arbitrary function as a polynomial function using its Taylor's series expansion, We can do the same for the above stress components about the zero strain state. Let us explore it now.

2.1 Taylor's expansion (start time: 21:19)

The Taylor's expansion of σ_{ij} with respect to strain components will be as follows:

$$\sigma_{ij}(\epsilon_{kl}) = \underbrace{\sigma_{ij}(\epsilon_{kl}=0)}_{\sigma_{ij}^0} + \sum_k \sum_l \frac{\partial \sigma_{ij}}{\partial \epsilon_{kl}}\epsilon_{kl} + \frac{1}{2!}\frac{\partial^2 \sigma_{ij}}{\partial \epsilon_{kl}\partial \epsilon_{mn}}\epsilon_{kl}\epsilon_{mn} + \ldots \tag{8}$$

All the derivatives have been evaluated at zero strain state, i.e., $\epsilon_{kl} = 0$. Here, all the indices run from 1 to 3. The first term (denoted as σ_{ij}^0) is called the residual stress and gives the stress in the zero strain state or the reference configuration. The reference configuration is typically chosen such that the stress in the body is zero in this state. Accordingly, we will assume σ_{ij}^0 to be zero for this course. The second term is linear in strain while the third term is quadratic in strain. We had also said earlier said that for this course, we will work with very small strains only. Thus, we can neglect the terms which are quadratic or higher order in strain components as they are much smaller than the linear term. Thus we are left with

$$\sigma_{ij} = \sum_{k=1}^{3} \sum_{l=1}^{3} \frac{\partial \sigma_{ij}}{\partial \epsilon_{kl}} \epsilon_{kl} \tag{9}$$

This is called linear stress-strain relation. We again remark that the derivatives in equation (9) are to be evaluated at the reference configuration. These derivatives have a special significance in solid mechanics: they denote the components of the stiffness tensor. It is a fourth order tensor because it has got four indices all running from 1 to 3 and is denoted as C_{ijkl}:

$$C_{ijkl} = \left. \frac{\partial \sigma_{ij}}{\partial \epsilon_{kl}} \right|_{\epsilon_{kl}=0} \tag{10}$$

Thus, we can write

$$\boxed{\sigma_{ij} = \sum_{k} \sum_{l} C_{ijkl} \; \epsilon_{kl}} \tag{11}$$

2.2 Independent components in tensor *C* (start time: 28:23)

We can observe that C_{ijkl} has $3^4 = 81$ terms since all the four indices go from 1 to 3. These 81 constants are called material constants which vary from material to material. However, it turns out that the 81 constants are not all independent. Let us explore it now.

2.2.1 Minor Symmetry (start time: 30:22)

As σ_{ij} is symmetric, it implies that

$$C_{ijkl} = C_{jikl} \tag{12}$$

For example, $C_{1222} = C_{2122}$, $C_{1323} = C_{3123}$. If this does not happen, then if we work out the multiplication of the RHS in equation (11), σ_{ij} will not come out to be equal to σ_{ji}. Likewise, as ϵ_{kl} is symmetric, one can choose:

$$C_{ijkl} = C_{ijlk} \tag{13}$$

This is because the multiplication in (11) only depends on $(C_{ijkl} + C_{ijlk})/2$ part of C_{ijkl} due to symmetry of strain components as we show below:

$$\begin{aligned}
\sigma_{ij} &= \sum_k \sum_l C_{ijkl}\ \epsilon_{kl} = \sum_k \sum_l \left[\frac{(C_{ijkl} + C_{ijlk})}{2} + \frac{(C_{ijkl} - C_{ijlk})}{2} \right] \epsilon_{kl} \\
&= \sum_k \sum_l \left[\frac{(C_{ijkl} + C_{ijlk})}{2} \right] \epsilon_{kl} + \sum_k \sum_l \frac{C_{ijkl}}{2}\ \epsilon_{kl} - \sum_k \sum_l \frac{C_{ijlk}}{2}\ \epsilon_{kl} \\
&= \sum_k \sum_l \left[\frac{(C_{ijkl} + C_{ijlk})}{2} \right] \epsilon_{kl} + \sum_k \sum_l \frac{C_{ijkl}}{2}\ \epsilon_{kl} - \sum_k \sum_l \frac{C_{ijlk}}{2}\ \epsilon_{lk}
\end{aligned} \tag{14}$$

In the last equality here, we have simply replaced ϵ_{kl} with ϵ_{lk} due to symmetry in strain components. However, this implies that the last two summations become exactly the same and hence cancel each other: just interchange the summation subscripts ($k \rightarrow l$, $l \rightarrow k$) in the last summation. Equations (12) and (13) are together called minor Symmetries of stiffness tensor. By using equation (12), out of the 9 combinations of (i,j), only 6 are independent. Similarly, by using equation (13), out of the 9 combinations of (k,l), only 6 are independent. Thus, we have 6 × 6 = 36 independent components from the total 81 components. We can bring this number further down using another kind of symmetry which is called Major Symmetry.

2.2.2 Major Symmetry (start time: 35:10)

This is another symmetry present in the stiffness tensor which comes from energy considerations. It says

$$C_{ijkl} = C_{klij} \tag{15}$$

Let us prove it. First, let us look at an analogue of a spring.

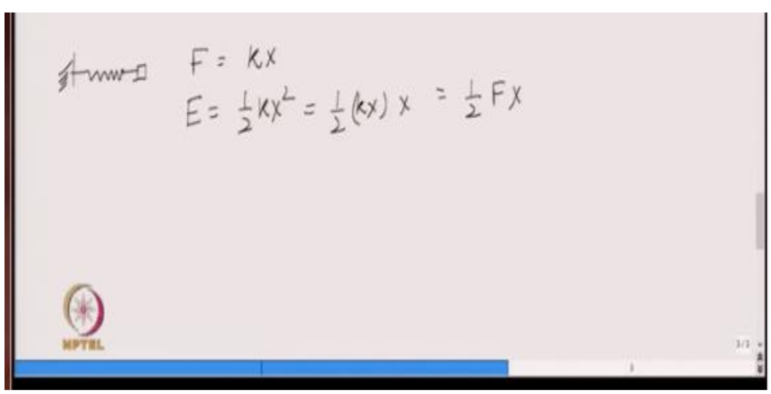

Figure 2: A spring mass system

If the elongation of the spring is x and the spring constant is k, the force F that gets generated in the spring is given by

$$F = kx \tag{16}$$

while the energy stored in the spring is given by

$$E = \frac{1}{2}kx^2 = \frac{1}{2}(kx)x = \frac{1}{2}Fx \tag{17}$$

We also have

$$\frac{\partial E}{\partial x} = kx = F. \tag{18}$$

This is the case of a simple spring. We can think of a three-dimensional body as being made up of multiple springs. The strain components can be thought of as the elongation in an individual spring and the stress components can be thought of as the force that generates in that particular spring. The only difference here is that the force in the spring (σ_{ij}) not only depends on the strain in that spring but also on the strains in other springs. For example, σ_{11} depends on ϵ_{12}, ϵ_{13}, ϵ_{22}, ϵ_{23} and ϵ_{33} too in addition to ϵ_{11}. Thus, a three-dimensional elastic body is just a more complicated/coupled set of springs. Using this analogy, we can also say that the energy is stored when a body gets deformed which comes from the work done on the body by the external load. This stored energy is called strain energy. We usually work in terms of strain energy density which is strain energy stored per unit volume. To get the total energy stored in a body, we can integrate the strain energy density over the entire volume of the body. Comparing with spring energy, we can write the strain energy density E as

$$E(\text{strain energy/volume}) = \frac{1}{2}\sum_i \sum_j \sigma_{ij}\epsilon_{ij} \tag{19}$$

This is analogous to equation (17) for a single spring. As i and j go from 1 to 3, there are a total of 9 springs. Let us now substitute for σ_{ij} using linear stress-strain relation given in equation (11):

$$E = \frac{1}{2}\sum_i \sum_j \left(\sum_k \sum_l C_{ijkl}\epsilon_{kl}\right)\epsilon_{ij} = \frac{1}{2}\sum_{i,j,k,l} C_{ijkl}\,\epsilon_{kl}\epsilon_{ij} \tag{20}$$

This is the energy stored per unit volume in the body. Now, we can also write the stress component σ_{mn} (corresponding to equation (18)):

$$\sigma_{mn} = \frac{\partial E}{\partial \epsilon_{mn}} = \frac{1}{2}\sum_{i,j,k,l}\left(C_{ijkl}\frac{\partial \epsilon_{kl}}{\partial \epsilon_{mn}}\epsilon_{ij} + C_{ijkl}\epsilon_{kl}\frac{\partial \epsilon_{ij}}{\partial \epsilon_{mn}}\right) \quad \text{(using equation (20))} \tag{21}$$

We have used the fact that C_{ijkl} is a constant and does not change with strain. Assuming that the strain components are independent of each other, the partial derivative of one component with respect to another $\left(\frac{\partial \epsilon_{kl}}{\partial \epsilon_{mn}}\right)$ will give 1 if they are same and 0 otherwise. To express this mathematically, we need to multiply two Kronecker Delta functions, e.g.,

$$\frac{\partial \epsilon_{kl}}{\partial \epsilon_{mn}} = \delta_{km}\delta_{ln}, \quad \frac{\partial \epsilon_{ij}}{\partial \epsilon_{mn}} = \delta_{im}\delta_{jn} \tag{22}$$

A more rigorous approach is to also account for symmetry of strain components in the above derivative which we have skipped for simplicity. We thus have

$$\begin{aligned}\sigma_{mn} &= \frac{1}{2}\sum_{i,j,k,l}\Big(C_{ijkl}\delta_{km}\delta_{ln}\epsilon_{ij} + C_{ijkl}\epsilon_{kl}\delta_{im}\delta_{jn}\partial\epsilon_{mn}\Big)\\ &= \frac{1}{2}\Big(\sum_{i,j} C_{ijmn}\epsilon_{ij} + \sum_{k,l} C_{mnkl}\epsilon_{kl}\Big)\end{aligned} \tag{23}$$

As, *i* and *j* are just used for summation and are dummy variables, we can change them to *k* and *j* respectively which yields

$$\sigma_{mn} = \frac{1}{2}\sum_{k,l}(C_{klmn} + C_{mnkl})\epsilon_{kl} = \sum_{k,l}\tilde{C}_{mnkl}\,\epsilon_{kl} \tag{24}$$

where $\tilde{C}_{mnkl}$ is defined as

$$\tilde{C}_{mnkl} = \frac{1}{2}(C_{klmn} + C_{mnkl}) \tag{25}$$

This $\tilde{C}_{mnkl}$ has got major symmetry as shown below:

$$\tilde{C}_{mnkl} = \frac{1}{2}(C_{klmn} + C_{mnkl}) = \frac{1}{2}(C_{mnkl} + C_{klmn}) = \tilde{C}_{klmn} \tag{26}$$

From this analysis, we can conclude that although equation (11) has a general *C* tensor, we can always work with an equivalent $\tilde{C}$ which has both major and minor symmetries. Due to major symmetry, the number of independent constants in C_{ijkl} will further reduce. We can think of a 6×6 matrix with the rows representing the values of C_{ijkl} corresponding to independent combinations of (*i,j*) and the columns representing the values of C_{ijkl} corresponding to independent combinations of (*k,l*):

$$[C_{ijkl}] = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{1112}\\ C_{2211} & C_{2222} & C_{2233} & C_{2223} & C_{2213} & C_{2212}\\ C_{3311} & C_{3322} & C_{3333} & C_{3323} & C_{3313} & C_{3312}\\ C_{2311} & C_{2322} & C_{2333} & C_{2323} & C_{2313} & C_{2312}\\ C_{1311} & C_{1322} & C_{1333} & C_{1323} & C_{1313} & C_{1312}\\ C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1213} & C_{1212}\end{bmatrix} \tag{27}$$

From major symmetry, we know that we can flip (*i,j*) and (*k,l*) which implies that the above matrix is symmetric. Thus, out of the 36 components here, only 21 are independent. We can summarize the reduction in number of independent components as follows:

81(34) ---------Minor Symmetry→ 36 ---------Major Symmetry→ 21.

Thus, a general linear elastic material will have maximum 21 independent material constants.

3 Voigt Notation (start time: 54:08)

We know that the stress matrix is written as

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \tag{28}$$

Due to the symmetry of stress matrix, it has got only six independent components. So, we can also think of a 6×1 vector formed by the six independent components. The sequence in which they are written is shown in Figure 3.

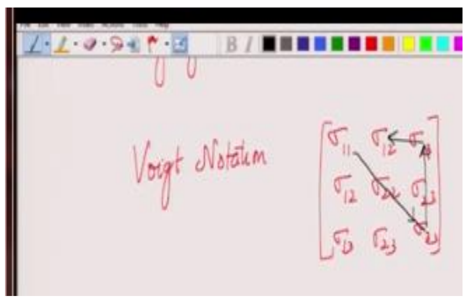

Figure 3: The arrow directions shows the sequence in which the six independent components of stress matrix are written in Voigt notation

We first go along the diagonal starting from (1,1) component. Then we go up along the third column and then left along the first row. Thus, we obtain the following vector:

$$[\underline{\sigma}] = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} \tag{29}$$

This vector is also called the stress vector. We can follow the same steps to get the strain sector from the strain matrix as

$$[\underline{\epsilon}] = \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \\ 2\epsilon_{12} \end{bmatrix} = \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{bmatrix} \tag{30}$$

For a linear stress-strain relation, there exists a matrix which relates the stress vector with the strain vector.

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} & 6 \times 6 \text{ matrix} & \\ & & \\ & \text{(formed by} & \\ & C_{ijkl}) & \\ & & \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{bmatrix} \tag{31}$$

This 6×6 matrix is called stiffness matrix. It is the same matrix as in (27) which is symmetric and contains 21 independent constants. We can also write the strain vector ($\underline{\epsilon}$) in terms of stress vector ($\underline{\sigma}$) using another 6 × 6 matrix:

$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{bmatrix} = \begin{bmatrix} & & \\ & 6 \times 6 \text{ matrix} & \\ & & \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} \tag{32}$$

This 6 × 6 matrix is called compliance matrix which turns out to be the inverse of stiffness matrix. Although, the most general material has 21 independent material constants, there are materials with additional symmetries. For example, isotropic materials have only two independent constants. We will discuss about such materials in the next lecture.