Solid Mechanics
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Lecture - 11
Concept of Strain

Hello everyone! Welcome to Lecture 11! In this lecture, we will discuss about the concept of strain. Till now we have been talking about stress and we will now start with strain tensor.

1 Introduction (start time: 0:31)

Let us start with an example which we have come across during our schooldays. We have a horizontal bar, apply a force *F* to it as shown in Figure 1 and measure the change in length of the bar. If the original length is *L* and the final length is *l*, then the strain in the bar is given as

$$Strain = \frac{change\,in\,length}{original\,length}$$

$$\approx \frac{change\,in\,length}{deformed\,length} \text{ (if change in length is very small)} \quad (1)$$

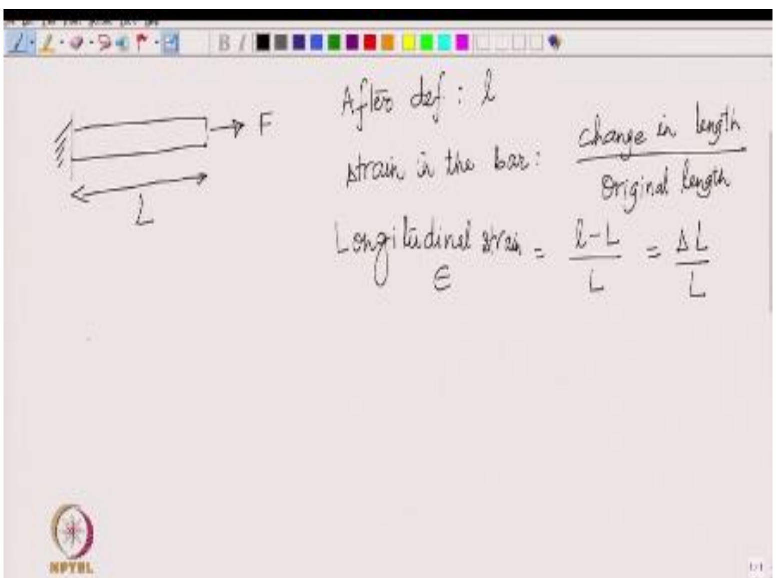

Figure 1: A horizontal bar being pulled by a force applied at its end.

This particular strain is called longitudinal strain (ϵ) and is expressed mathematically as

$$\epsilon = \frac{l-L}{L} = \frac{\Delta L}{L} \quad (2)$$

We have one strain value for the whole bar and hence can also be thought of as the bar's global strain. Let us consider another example in which we have a bar hanging vertically and subjected to gravitational load as shown in Figure 2. Let us find the change in length of small line elements at two locations: one near the point of clamping and the other near the other end of the bar (see Figure 2. We observe that the element near the clamped end undergoes a much higher change in length than the other element. This shows that the change in length is different for different line elements across the length of the bar. We thus define the strain at a particular location to be the local strain. We can conclude that, just like stress, strain also varies in the body.

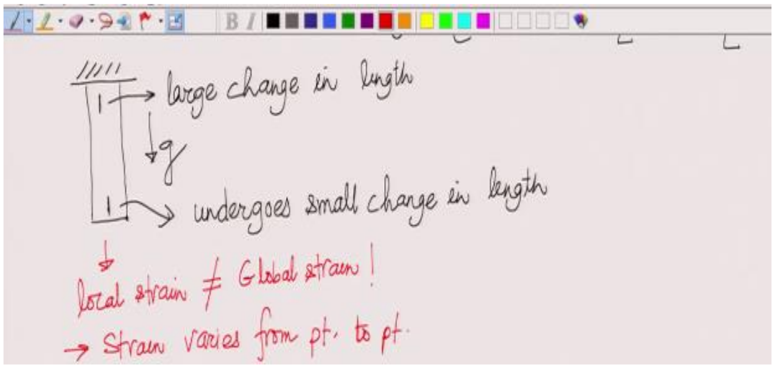

Figure 2: A bar hanging under the effect of gravity

2 Longitudinal strain in a general body (start time: 05:45)

Let us consider an arbitrarily shaped body (which need not look like a bar) subjected to external distributed load as shown in Figure 3. We want to find longitudinal strain at an arbitrary point in the body. In the two examples considered earlier, we had simply considered the change in length along the length of the bar to obtain longitudinal strain. But, here there is no such unique direction as in a bar and the body is, in fact, undergoing change in length or size in all directions. Two such directions are shown in Figure 3 emanating from a point in the body.

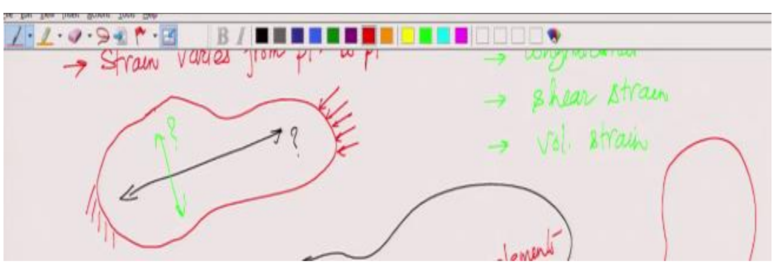

Figure 3: A distributed load applied to an arbitrarily shaped body.

There are, in fact, infinite line elements emanating from this point. As the body deforms, these line elements will also undergo change in shape and length as shown in Figure 4. However, if we assume that the undeformed line elements passing through the point of interest to be infinitesimally small, the deformed curves can be approximated as straight lines too. This assumption later helps us to obtain a simple formula for the change in length of the original line elements. Assuming line elements to be infinitesimally small also allows us to obtain longitudinal strain at a point in the body: if the line elements were of finite length, the longitudinal strain obtained would not correspond to a point but to the small region of the body in which the line element lies.

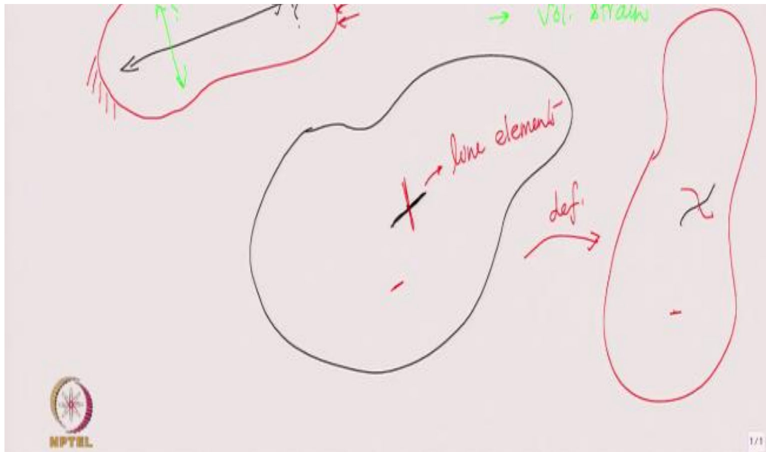

Figure 4: Line elements before and after deformation in a general body

As we have infinite line elements at every point in the body, we can associate infinite strains at that point. So, longitudinal strain depends on two factors: the point at which we are measuring the strain and the line element we choose at that point. Mathematically, we can define a line element at a point by its orientation/direction (denoted by $\underline{n}$). So, the longitudinal strain can be represented as $\epsilon(\underline{x},\underline{n})$. We should recall here that traction was also represented as $\underline{t}(\underline{x},\underline{n})$. The vector $\underline{n}$ there denoted surface normal but in the case of strain, $\underline{n}$ is used to denote the direction of the line element.

3 Concept of reference and deformed configurations (start time: 13:45)

When a body gets deformed, every point in the body undergoes some displacement. Let us think of a reference configuration of the body relative to which we would measure displacement of a point (see Figure 5). This reference configuration could be the configuration of the body at time t = 0 or the configuration when there is no load acting on the body or may be any other convenient configuration. The deformed configuration is shown on the right in Figure 5. We also define a coordinate system. The position vector of any point in the reference configuration is denoted by $\underline{X}$. This point (which is also called a material point) displaces to the point $\underline{x}$ in the deformed configuration. In fact, every point in the reference configuration has a corresponding point in the deformed configuration and we can think of a function which relates corresponding points in the two configurations. We denote this function by $\underline{f}$ whose domain is position vector of all points in the reference configuration. The function returns deformed configuration of the body which typically changes with time. The position $\underline{x}$ of a point in the deformed configuration can then be mathematically written as

$$\underline{x} = \underline{f}(\underline{X}, t) \tag{3}$$

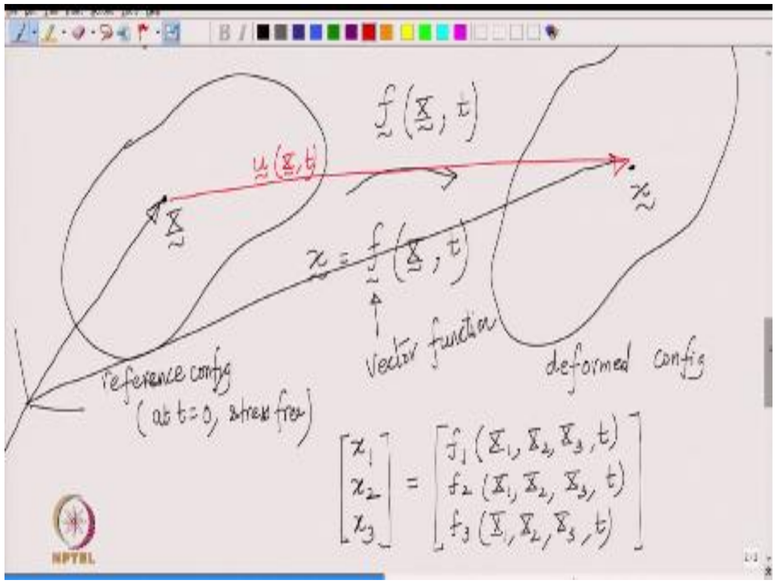

Figure 5: Reference configuration of the body shown on left and the deformed configuration shown on the right along with a global coordinate system.

Note that $\underline{f}$ is a vector function since it has to return the deformed position vector of all points in the reference configuration. In component form, equation (3) can also be written as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} f_1(X_1, X_2, X_3, t) \\ f_2(X_1, X_2, X_3, t) \\ f_3(X_1, X_2, X_3, t) \end{bmatrix} \tag{4}$$

Note that each component of $\underline{f}$ is a function of all components of $\underline{X}$ (X_1,X_2,X_3) and t.

3.1 Displacement vector (start time: 20:08)

The displacement of a point (denoted by $\underline{u}$) is then given by the vector from the reference position to the deformed position as shown by the red line in Figure 5. The displacement $\underline{u}$ will depend on the point we are considering in the reference configuration ($\underline{X}$) and time. Thus, we have

$$\begin{aligned} \underline{u}(\underline{X}, t) &= \underline{x} - \underline{X} \\ &= \underline{f}(\underline{X}, t) - \underline{X} \end{aligned} \tag{5}$$

With these notations introduced, we can now find the expression for longitudinal strain. We look at the reference and deformed configurations of the body again as shown in Figure 6.

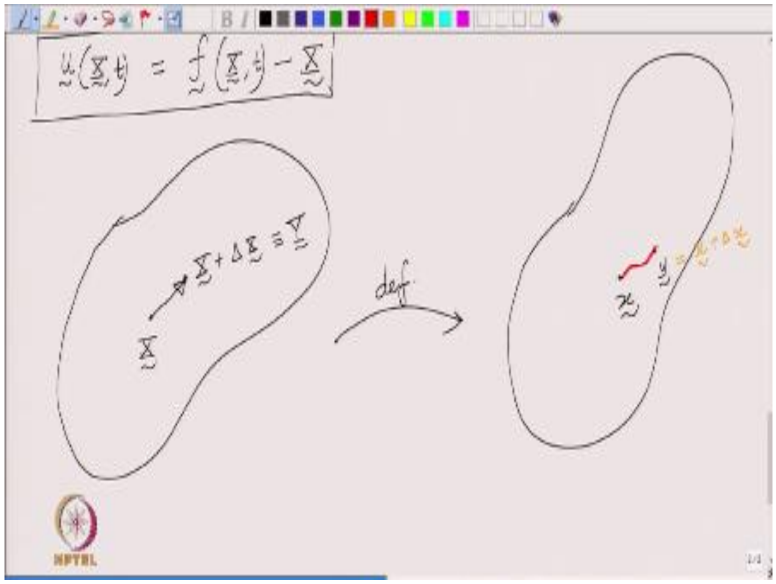

Figure 6: Deformation of a small line element when a body gets deformed.

4 Relating line elements in reference and deformed configurations (start time: 22:06)

We consider two points $\underline{X}$ and $\underline{Y}$ ($\underline{X}$ +Δ$\underline{X}$) very close to each other, in the reference configuration. We are interested in the line element that joins these two points. As mentioned earlier, the material present on this line forms the material line element. After deformation, point $\underline{x}$ corresponds to $\underline{X}$ while $\underline{y}$ corresponds to $\underline{Y}$. In general, the deformed line element need not be a straight line but when the initial line segment is small enough, the deformed curve will be very similar to a straight line. In further analysis, we'll let Δ$\underline{X}$ approach zero and so this assumption will be valid in our analysis. While doing that, we would also ensure that we find the strain at point $\underline{X}$ itself. We then write the point $\underline{y}$ as $\underline{x}$ + Δ$\underline{x}$. The unit vector direction for the initial line element ($\underline{n}$) will be given as:

$$\begin{aligned} \underline{n} &= \frac{\underline{Y}-\underline{X}}{||\underline{Y}-\underline{X}||} \\ &= \frac{\Delta\underline{X}}{||\Delta\underline{X}||} \end{aligned} \tag{6}$$

Finally, using equation (2), the longitudinal strain at point $\underline{X}$ for the line element in the direction $\underline{n}$ will be given as

$$\epsilon(\underline{X},\underline{n}) = \lim_{||\Delta\underline{X}||\to 0} \frac{||\Delta\underline{x}|| - ||\Delta\underline{X}||}{||\Delta\underline{X}||} \tag{7}$$

The vector Δ$\underline{x}$ can be rewritten using equation (3) as

$$
\begin{aligned}
\Delta \underline{x} &= \underline{y} - \underline{x} \\
&= \underline{f}(\underline{Y}) - \underline{f}(\underline{X}) \\
&= \underline{f}(\underline{X} + \Delta \underline{X}) - \underline{f}(\underline{X})
\end{aligned} \tag{8}
$$

To get all expressions in terms of the quantities at $\underline{X}$, we can use Taylor's expansion to expand $\underline{f}(\underline{X}+\Delta\underline{X})$. As $\Delta\underline{X}$ has got three components: $\Delta\underline{X}_1 \Delta\underline{X}_2, \Delta\underline{X}_3$, the Taylor's expansion will involve derivatives with respect to all components, i.e.,

$$
\Delta \underline{x} = \underbrace{\underline{f}(\underline{X}) + \frac{\partial \underline{f}}{\partial X_1}\Delta X_1 + \frac{\partial \underline{f}}{\partial X_2}\Delta X_2 + \frac{\partial \underline{f}}{\partial X_3}\Delta X_3 + \dots}_{\text{Taylor's expansion}} - \underline{f}(\underline{X}) \tag{9}
$$

We can also write the above in a compact form as

$$
\Delta \underline{x} = \begin{bmatrix} \frac{\partial \underline{f}}{\partial X_1} & \frac{\partial \underline{f}}{\partial X_2} & \frac{\partial \underline{f}}{\partial X_3} \end{bmatrix} \begin{bmatrix} \Delta X_1 \\ \Delta X_2 \\ \Delta X_3 \end{bmatrix} + \frac{\partial^2 \underline{f}}{\partial X_1 \partial X_1}\frac{\Delta X_1^2}{2} + \frac{\partial^2 \underline{f}}{\partial X_1 \partial X_2}\Delta X_1 \Delta X_2 + \dots. \tag{10}
$$

Two higher order terms are written to understand the pattern. We can see that both these terms have derivatives multiplied to expressions that are proportional to the square of magnitude of $\Delta\underline{X}$ ($||\Delta\underline{X}||^2 = \Delta X_1^2 + \Delta X_2^2 + \Delta X_3^2$). So, we can see that these terms will be O($||\Delta\underline{X}||$)^2. So, all higher order terms can be clubbed together as O($||\Delta\underline{X}||^2$) terms. Here 'O()' (big-O) notation means that the terms are of the same order, i.e.,

$$
lim_{||\Delta \underline{X}|| \to 0} \frac{O(||\Delta \underline{X}||)}{||\Delta \underline{X}||} = \text{a finite number} \tag{11}
$$

Finally, we have the following in component form:

$$
[\Delta \underline{x}] = \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial X_1} & \frac{\partial f_1}{\partial X_2} & \frac{\partial f_1}{\partial X_3} \\ \frac{\partial f_2}{\partial X_1} & \frac{\partial f_2}{\partial X_2} & \frac{\partial f_2}{\partial X_3} \\ \frac{\partial f_3}{\partial X_1} & \frac{\partial f_3}{\partial X_2} & \frac{\partial f_3}{\partial X_3} \end{bmatrix} \begin{bmatrix} \Delta X_1 \\ \Delta X_2 \\ \Delta X_3 \end{bmatrix} + O(||\Delta \underline{X}||^2) \tag{12}
$$

4.1 Deformation gradient tensor (start time: 36:32)

Let us begin with the definition of the gradient of a general vector quantity $\underline{f}$, i.e.,

$$\begin{aligned} \underline{\underline{F}} &= \underline{\nabla}\, \underline{f} = \frac{\partial \underline{f}}{\partial X_1} \otimes \underline{e}_1 + \frac{\partial \underline{f}}{\partial X_2} \otimes \underline{e}_2 + \frac{\partial \underline{f}}{\partial X_3} \otimes \underline{e}_3 \\ &= \sum_j \frac{\partial \underline{f}}{\partial X_j} \otimes \underline{e}_j = \sum_i \sum_j \frac{\partial f_i}{\partial X_j} \underline{e}_i \otimes \underline{e}_j. \end{aligned} \tag{13}$$

It turns out to be a second order tensor. Its matrix form in ($\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$) coordinate system is [$\underline{\underline{F}}$] whose components are given by

$$F_{ij} = \frac{\partial f_i}{\partial X_j} \tag{14}$$

It is this matrix which is appearing in equation (12). In first year maths course, we have come across the gradient of a scalar quantity which yields a vector and is defined as

$$\underline{\nabla}\, f = \frac{\partial f}{\partial X_1} \underline{e}_1 + \frac{\partial f}{\partial X_2} \underline{e}_2 + \frac{\partial f}{\partial X_3} \underline{e}_3 \tag{15}$$

If we represent this vector in the coordinate system ($\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$), we get

$$\left[\underline{\nabla}\, f\right]_{(\underline{e}_1, \underline{e}_2, \underline{e}_3)} = \left[\frac{\partial f}{\partial X_1} \quad \frac{\partial f}{\partial X_2} \quad \frac{\partial f}{\partial X_3}\right]^T \tag{16}$$

This is the column from of the gradient that we have seen in first year math course. Let us get back to equation (12). We can recall from the first lecture that the tensor form of a matrix-column multiplication is just the second order tensor corresponding to the matrix multiplied with the vector corresponding to the column. Thus, we can also write equation (12) in tensor form as follows:

$$\Delta \underline{x} = \underline{\underline{F}}\, \Delta \underline{X} + O(||\Delta \underline{X}||^2) \tag{17}$$

The vector function $\underline{f}$ is also called the deformation map and $\underline{\underline{F}}$ is called the deformation gradient tensor. This is a very important equation since it relates material line elements $\Delta \underline{X}$ at a point $\underline{X}$ in the reference configuration to the corresponding deformed line elements $\Delta \underline{x}$ at $\underline{x}$ in the deformed configuration. It is the deformation gradient tensor at $\underline{X}$ which relates the two.

5 Formula for longitudinal strain at a point in the body (start time: 44:52)

Once we have the expression for $\Delta \underline{x}$, we can express the square of final length (l^2) of a line element as

$$\begin{aligned} l^2 &= \Delta \underline{x} \cdot \Delta \underline{x} \\ &= \left[\underline{\underline{F}} \Delta \underline{X} + O(||\Delta \underline{X}||^2)\right] \cdot \left[\underline{\underline{F}} \Delta \underline{X} + O(||\Delta \underline{X}||^2)\right] \\ &= (\underline{\underline{F}} \Delta \underline{X}) \cdot (\underline{\underline{F}} \Delta \underline{X}) + O(||\Delta \underline{X}||^3) \end{aligned} \tag{18}$$

Note that the higher order term here is written as $O(||\Delta \underline{X}||^3)$: this is because $\underline{\underline{F}}\Delta \underline{X}$ is an $O(||\Delta \underline{X}||)$ term and when it is dotted with an $O(||\Delta \underline{X}||^2)$ term, we get an $O(||\Delta \underline{X}||^3)$ term. Similarly, the square of reference length (L^2) of a line element can be written as

$$L^2 = \Delta \underline{X} \cdot \Delta \underline{X} \tag{19}$$

The square of stretch λ^2 in the line element then becomes

$$\lambda^2 = \frac{l^2}{L^2} = \frac{(\underline{\underline{F}}\Delta \underline{X}) \cdot (\underline{\underline{F}}\Delta \underline{X})}{\Delta \underline{X} \cdot \Delta \underline{X}} + \frac{O(||\Delta \underline{X}||^3)}{\Delta \underline{X} \cdot \Delta \underline{X}} \tag{20}$$

We can write $\Delta \underline{X} \cdot \Delta \underline{X}$ as $||\Delta \underline{X}||^2$. Then, for the first term, we distribute the square term in the denominator over the two terms which are dotted in the numerator. Also, the $O(||\Delta \underline{X}||^3)$ term when divided by $||\Delta \underline{X}||^2$ gives an $O(||\Delta \underline{X}||)$ term. This finally yields

$$\lambda^2 = \left(\underline{\underline{F}}\frac{\Delta \underline{X}}{||\Delta \underline{X}||}\right) \cdot \left(\underline{\underline{F}}\frac{\Delta \underline{X}}{||\Delta \underline{X}||}\right) + O(||\Delta \underline{X}||) \tag{21}$$

Using equation (6), we can write the above expression as

$$\lambda^2(\underline{X},\underline{n}) = \underline{\underline{F}}\,\underline{n} \cdot \underline{\underline{F}}\,\underline{n} + O(||\Delta \underline{X}||) \tag{22}$$

Our final aim is to express the above formula in terms of displacement. So, we need to somehow write $\underline{\underline{F}}$ in terms of the displacement. We begin with

$$\underline{f} = \underline{x} = \underline{X} + \underline{u} \Rightarrow f_i = X_i + u_i \tag{23}$$

To find $\underline{\underline{F}}$, we need the derivatives of $\underline{f}$, i.e.,

$$\frac{\partial f_i}{\partial X_j} = \frac{\partial X_i}{\partial X_j} + \frac{\partial u_i}{\partial X_j} = \delta_{ij} + \frac{\partial u_i}{\partial X_j} \tag{24}$$

This is because X_i's are independent coordinates in the reference configuration: the derivative of a component with respect to the same component will give 1 otherwise it will be 0. The tensor form of this equation is

$$\underline{\underline{F}} = \nabla \underline{f} = \underline{\underline{I}} + \nabla \underline{u} \tag{25}$$

The tensor $\nabla \underline{u}$ is called the displacement gradient tensor. Replacing deformation gradient in terms of displacement gradient in equation (22), we get

$$\lambda^2(\underline{X},\underline{n}) = \left(\underline{\underline{I}} + \nabla \underline{u}\right)\underline{n} \cdot \left(\underline{\underline{I}} + \nabla \underline{u}\right)\underline{n} + O(||\Delta \underline{X}||) \tag{26}$$

We then use the following identity whose proof is provided in the next section:

$$\underline{\underline{A}}\,\underline{a} \cdot \underline{b} = \underline{a} \cdot \underline{\underline{A}}^T \underline{b} \tag{27}$$

This means that we can bring a second order tensor to the other side of the dot product but we have to transpose it. In equation (26), we then take the tensor $\underline{\underline{I}} + \underline{\nabla}\,\underline{u}$ to the other side of the dot product which yields

$$\begin{aligned}
\lambda^2(\underline{X}, \underline{n}) &= \underline{n} \cdot \left[\left(\underline{\underline{I}} + \underline{\nabla}\,\underline{u} \right)^T \left(\underline{\underline{I}} + \underline{\nabla}\,\underline{u} \right) \underline{n} \right] + O(||\Delta \underline{X}||) \\
&= \underline{n} \cdot \left[\underline{\underline{I}} + \underline{\nabla}\,\underline{u}^T + \underline{\nabla}\,\underline{u} + \underline{\nabla}\,\underline{u}^T\ \underline{\nabla}\,\underline{u} \right] \underline{n} + O(||\Delta \underline{X}||)
\end{aligned} \tag{28}$$

Here we used $\underline{\underline{I}}^T = \underline{\underline{I}}$ since it is a symmetric tensor. We will continue the strain formulation from here in the next lecture.

6 Proof of the identity (27) (not present in the video lecture)

6.1 Method I

$\underline{\underline{A}}\,\underline{a}$ is a vector and the dot product of two vectors in a coordinate system is done by transposing the column of the first vector. So, we get:

$$\begin{aligned}
\underline{\underline{A}}\,\underline{a} \cdot \underline{b} &= \left([\underline{\underline{A}}]\ [\underline{a}] \right)^T [\underline{b}] \\
&= \left(\left[\quad \underline{a}^T \quad \right] \left[\quad \underline{\underline{A}}^T \quad \right] \right) \left[\underline{b} \right]
\end{aligned} \tag{29}$$

Using associativity of matrix multiplication, we can write it as

$$\begin{aligned}
\underline{\underline{A}}\,\underline{a} \cdot \underline{b} &= \left[\quad \underline{a}^T \quad \right] \left(\left[\quad \underline{\underline{A}}^T \quad \right] \left[\underline{b} \right] \right) \\
&= \left[\quad \underline{a}^T \quad \right] \left[\underline{\underline{A}}^T \underline{b} \right] \\
&= \underline{a} \cdot \underline{\underline{A}}^T \underline{b} = R.H.S. \text{ (proved)}
\end{aligned} \tag{30}$$

6.2 Method II

Let $\underline{c} = \underline{\underline{A}}\,\underline{a}$. We can write this in component form as

$$c_i = \sum_j A_{ij} a_j \tag{31}$$

Hence

$$\begin{aligned}
\underline{\underline{A}}\,\underline{a} \cdot \underline{b} &= \underline{c} \cdot \underline{b} \\
&= \sum_i c_i b_i \\
&= \sum_i \sum_j A_{ij} a_j b_i \\
&= \sum_i \sum_j A^T_{ji} a_j b_i \quad (\because A_{ij} = A^T_{ji}) \\
&= \sum_j a_j \underbrace{\left(\sum_i A^T_{ji} b_i \right)}_{j^{th} \text{ component of } \underline{\underline{A}}^T \underline{b}}
\end{aligned}$$

$$= \underline{a} \cdot \underline{\underline{A}}^T \underline{b} = R.H.S. \text{ (proved)} \tag{32}$$