Nonlinear Control Design

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So now I am going to talk about some theory. So I am going to define what my, what would I call some safe set or forward invariant set. Yeah. And this is defined as C equal to say, suppose I define my set using some, yeah. Notice that I have defined it the other way round. Right.

Here if you see my, no I think this is fine. Right. I have also defined it using 25 minus x1 square greater than equal to 0, 9 minus x2 square greater than equal to 0. Same thing.

H can be an amalgamation of 25 minus x1 square, 9 minus x2 square. Same thing. That is the set C. The boundary like I said is basically where hx is exactly equal to 0 and then I have C naught, I would say the interior of C is the set of x in Rn is basically greater than equal to 0. Okay.

Great. Now, so typically if you see the way we have been defining this is that we have sort of, I mean yeah that is fine I mean in particular cases. So the reciprocal barrier function, that is what we have just looked at, is basically something like say Bx is equal to you know something like I guess 1 over hx. Right. This is how we have been using it.

Right. It is like 1 over 25 minus x1 square or 1 over 9 minus x2 square. Okay. This is how we have been doing.

Right. And what we, the condition we sort of imposed was that B dot is less than equal to 0. Okay. This is the sort of condition we were imposing. However, we do not need this. It is too restrictive.

Yeah. It can be shown that this is again going back to the same idea that everything inside is also being made invariant in some sense. Okay. So we do not particularly need this. Alright. So if you go back to our example also, again if you go back here, if you even in this example what did we do? We said that V dot is less than equal to 0.

Same thing that we do with our typical Lyapunov analysis. Right. Not exactly the barrier function but the Lyapunov analysis. Right. So what does this mean? This means that this set is a square.

Right. So if I start inside this square. Right. If I started close to the boundary. Okay. What will it do? It will make some kind of a set.

Right. This V is now complicated by the way. I cannot really make a shape out of it. But this V is complicated. What it will do is it will make a set which is something like this say for example. We will try to get close to a continuous version of a square basically.

If you started here somewhere. Right. But if you started smaller it will make a smaller square. Right. Because that is how the invariant set for the V dot less than equal to 0 works.

Okay. We already saw this. If you start in the inside ellipse you will remain inside the inside ellipse only. Right. You do not have to. Actually let me think about it.

In this case will it be true? Actually it is not very clear if that will be true in this case. Does it mean that the inside set actually depends on how the shape is? Actually I am not sure. Honestly speaking. Yeah. Because I have to see the shape.

And this will be a good exercise to see what is the shape of the level sets. That is V equal to constant. What is the shape that you get here? It is not very obvious to me what the shape will be.

Yeah. Yeah. I do not think it is. Because this is looking like what this is. Is this. Can I write this as 25 minus 25 minus x1 square divided by.

Yeah. I can write this as 25 minus x1 square divided by this. So this becomes 25 divided by 25 minus x1 square minus 1. And similarly for the other one. This becomes 25 over 25 minus x1 square minus 1. This becomes 9 over 9 minus x2 square minus 1.

Right. So it is actually some kind of an ellipse only. Right. Some funny kind of a.

Is it still. No. It is not an ellipse. Right. Yeah. It is not very clear what the shape will be.

Yeah. But the sort of point that I am trying to make is as you keep reducing your initial conditions you will get smaller and smaller sets. That is the hope I guess. Right. You will start getting smaller and smaller sets. And the smaller set will become invariant.

You will not end up exploring the larger set at all because of this condition. Okay. And this is a little bit restrictive again. And this happened. Why? Because we took V dot less than equal to 0.

This kind of a condition is what we used. Right. So that is apparently not required. This is what some of the research has been. Apparently all you need is this is what is the relaxation is that B dot is less than equal to gamma over B where I guess gamma is some positive constant.

Okay. Gamma is some positive constant. I hope you sort of understand what this equation is doing. Okay. What is it doing? It is saying that if you are very far from the boundary.

Okay. If you are very far from the boundary what happens to B? Typically I mean if you look at our barrier function B is some finite value. Okay. It will give you some you know finite sort of decay. Hopefully it will give you a finite decay I guess.

Let us see. If I look at this barrier 1 over h x type of case. So in our case we are looking at what? Yeah I am trying to sort of make sense of how this will look for us. Let us see.

Suppose let us look at our example. Okay. So in our case B x was 1 over 25 minus x 1 squared. Okay. So unfortunately this may not satisfy this condition which is why I am not very sure. Okay let us see. What is B dot of x? This is 2 x 1 x 1 dot divided by 25 minus x 1 squared whole squared.

Correct. Thank you. Right. Okay. No. No minus. It is a plus. That is fine. Okay. That is fine. But this is not going to be very obvious how we are doing it.

No no no no. This is not very evident. But the rationale for doing this is at least how it stated here is that the inequality allows for B dot to grow when solutions are far from the boundary. Okay. Yeah. So basically what this will do is that if x is far from delta c in C naught then obviously your B is positive right because of whatever how you have constructed it.

B is going to be positive. So there is going to be some positive quantity on the right hand side. B is positive right because I took 1 over h x and h is positive in the interior. Right. h is positive in the interior. In fact it will be more and more positive in the interior.

Right. As you go further in the interior it is more and more positive. Okay. That is the definition of the set.

Right. So obviously h is positive. 1 over h is also positive. Okay. That is the idea. So if you are in the interior if you start in the interior your B is going to be positive and so this is right hand side is going to be positive.

Okay. So what am I saying? That the derivative is less than some positive number. Okay. So basically B can increase. Okay. Because all I am saying is that derivative is less than a positive number.

So it can also be positive. So this allows for B to increase. What does the increase of B mean? Remember B becomes infinity at the boundary for the reciprocal type construction. Right. This becomes infinity at the boundary.

So B increasing means you are going closer to the boundary. Okay. Makes sense? So B can increase implies x moves towards delta C. This is allowed by this construction.

Okay. So this is allowed only in this concern. If B dot was less than equal to 0 this is not allowed. Because it is always going down or not reducing or it is never increasing. B dot less than equal to 0 means B is never increasing. Here there is a possibility that B can increase. However what happens when you go close to the boundary? If x near delta C implies B is large.

Right. That is how we have defined it. You remember? 1 over 25 minus x1 squared. If x1 reaches 25 the denominator is exploding. Sorry the denominator is going to 0.

So B is becoming large. Okay. Alright. If B becomes very large what happens to this guy? Almost 0. Almost 0. So therefore B dot implies B dot approximately less than equal to 0.

So B either doesn't increase or starts to decrease. Okay. So what is it saying? It is saying basically only this boundary H of x is in effect. The inside boundaries are not you know like in this case I could have remained inside. Right. You know I could have remained inside this and not even gotten here.

That is not allowed here. Yeah. Here the right hand side is positive for some time. You are allowing some growth inside the set. Okay. But as you start getting closer to the boundary of the set this growth stops.

Okay. This growth stops. Okay. Which means implies x cannot escape C. Yeah. And this can this is not difficult to show. These folks have actually shown it.

So basically how they are saying it is that so this is the B dot. Sorry. So what is B dot? B dot is equal to minus 1 over H x square times H dot and if this is less than equal to gamma over B this is gamma H. Right. So this condition means that H dot plus gamma H cubed greater than equal to 0. Yeah this is what you will get.

So what did I do to get this? I think it is evident. Right. B is defined as 1 over H x.

Right. So B dot is this guy. Correct. I think I did it correctly. Verify. B dot is minus 1 over H x H dot H x square H dot and the right hand side is gamma over B and that is this guy. Yeah. This is gamma times H.

Right. So that gives me this kind of a differential inequality. Yeah. This can be solved. Yeah using your comparison lemma type results and they have actually solved it. I am going to just write it. H x T x 0 is greater than equal to 1 over square root 2 gamma T plus 1 over H square at x 0.

Okay. Okay. So you get this. Basically just by solving this with initial conditions on x and all that. Not difficult.

Right. You can just solve this equation. This is a differential inequality. You can just solve the equality and then by the comparison lemma make it greater than equal to.

Done. So you will get this solution. Yeah. Yeah please verify this. I am also not sure. But it is not difficult to see that this is already giving me H greater than equal to 0.

Right. Right hand side is positive. Yeah. Right hand side is positive. Not a negative quantity. Right. So this is you are always getting some positive number on the right hand side.

Which means what? You are inside C. Right. For all time. Right. So this implies what? C is forward invariant. Okay. So this kind of barrier function already did something better than what we did. By making B dot is less than equal to gamma over B.

Of course you have to be careful how to choose this B and all and how to make it. And so those things are not obvious here. Yeah. Unfortunately I am not sure we will have time to discuss those.

But this is how you define a barrier function. Okay. This is the right way.

You want this to happen. Okay. Not less than equal to 0. Yeah. That is just too much to ask. If you can enforce this via the control design.

Excellent. It would be the best way to do that this happens. Okay. Now one final notion. Yeah. That I sort of want to talk about is basically the notion of which is slightly more than this.

It is zeroing. Yeah. Just a second. Yeah. Which. This.

Why? Why are you saying it is negative? Yeah. It is positive. No. No. It is the positive square root.

No. No. No. Don't worry about that. It is always the positive square root. You can that you can verify in the solutions.

Yeah. Yeah. Okay. I mean it is a see the basic ideas is the continuous the solution is continuous right. Yeah. So either you are always positive or always negative.

Okay. And at initial time if you are positive there is no point in taking the negative square root.

No. The solutions are continuous. So you will always work with one side only. Yeah. Yeah. Yeah.

Yeah. Okay. Following. Zeroing barrier functions. Okay. This is what is a little bit more advanced. Right.

Not much. Okay. So sort of the final notion. I will sort of take a few more minutes and we will be done. Yeah. Here even in the previous case if you see your function still became infinity on the boundaries. So you can imagine whatever you do whatever these tricks you do to make sure that the inside side does not become invariant and all that.

Great. That works. But your control might still go unbounded if you start close to the boundary. Because that is a feature of how you define it. Just like here right. Control contain this guy. You start close to the boundary same d even there because that is how you define your barrier function 1 over hx.

Okay. So this is one unseemly thing that you want to avoid. You do not want the control to become explored as you get start close to the boundary. So that is where the zeroing barrier functions sort of come in.

They define in a slightly different way. Let's see. I will talk about it. First we define the notion of a extended class k function. You already know what is a class k function because it's 0 at 0 and strictly increasing. But the arguments are always taken to be argument of a class k function is also taken to be positive.

Whenever you define a class k function alpha you say that it goes from r plus to r or whatever. Now we do not do that. Now it's allowed to go from r to r. Okay. So this is basically the notion of an extended class k.

So alpha 0 is 0 and alpha strictly increasing. Basically negative arguments are allowed. Okay. Negative arguments are allowed. That is the idea. It is not necessary that the argument must be positive because if you remember we always take this class k function as norm of x and norm of x square and which is basically argument is the norm.

So the arguments are positive. Here the arguments do not have to be positive. Okay. So basically negative argument allowed. That's the whole point because once you start at 0 at 0 of course it's going to be positive for values greater than 0 of the argument.

But values less than 0. No. Right. So that's the important thing. So this is why it's sort of this is the sort of extension of if you remember Lyapunov functions are always positive definite and how was positive definite is defined by comparing it with a class k function. Right.

Now we compare this barrier functions with a class extended class k function. Okay. This gives us flexibility on both sides. Right. There it was the function was always positive. So the Lyapunov candidate was of course always positive.

Here the function can also be negative. So therefore this function can also go negative. That's the whole point here. All right. Okay. So suppose you have again a definition.

So 0 in Z b f is the zeroing barrier function. How do we define it? Yeah. So this is h belongs to C 1 is Z b f for set C. Yeah. We have defined the set C, the safe set. Right. If there exists extended class k function alpha and set D with C being a subset of D being a subset of R n such that for all x in D L f h x is greater than equal to minus alpha h x.

Okay. Yeah. So remember dynamics we were looking only at a dynamical system. There is no control. Therefore only L f h no L g and all that. So L f h is what just h dot right. It's just h dot in the along the trajectories of the system. So this is the direction there. So what am I saying? I am just saying that there is a larger domain of course the C has to be part of a larger domain and in that larger domain this h dot must be greater than equal to minus alpha h.

Right. Now the one of the simplest so now notice that h itself the way we have defined C notice yeah we have defined C h can is not stopped from being negative.

Right. In this definition it doesn't say h cannot be negative or anything like that. Right. h can be negative. Right. It is all we all we are saying is that whenever it is greater than equal to 0 it is the set C.

Right. Otherwise outside the set C h is potentially negative. Right. Because whenever it is positive it is in the set C. Right. So whenever you are outside just like here in these examples I mean I guess you can see this example itself. Right. Then when h is positive x1 is less than phi plus minus phi x1 is between plus minus phi if h is negative it's outside.

Okay. So h negative is allowed. No problem. Yeah. Okay. Great. Now what do you want? We want h dot to be equal to greater than equal to minus alpha h. So this is basically a this is a or a simplification actually this is a generalization that way. This is a generalization of what? You take alpha this extended class k function as h itself alpha of h is just h.

Yeah. Because it is it will be 0 at 0 and then increasing on the negative side it will go down. No problem. Straight line. Basically it is a straight line. Yeah. See if alpha function is taken as the unit function itself then this is the generalization of this guy.

So what does this give you? So what do you want to do? You want to basically maintain h greater than equal to 0. Yeah. I hope that's obvious. Now if I solve this equation what do I get? Let's look at this equation because solving the this one general one is difficult. Let's try to solve the simpler one. What is the solution? h of t is h is it? Tell me everything.

e to the power of minus gamma t let's say initial time is 0 and okay I am done. h naught thank you very much this is important h naught. Okay. Now suppose if starting in c or I should be more precise if x starting in c then what do I know? I know that h naught x or how do I say h 0 h naught which is defined as h at x 0 is definitely greater than 0 I will say in t here of c. So h naught is positive right because that's how I defined h.

So what does this mean? Implies h t is always greater than 0 and h t goes to 0 as t goes to infinity. Okay. Alright. So what's the good thing in this case? In this case what's the nice thing? Nice thing is that again I mean it should be obvious to you that x of t is also in c.

Yeah. Also in the in the interior for all time forget infinity. Infinity is not in real numbers. So x of t is going to remain in the set c. So it is definitely this method also made it invariant. Okay. But it has a nice structure right I mean it is nicer structure there is no reciprocal happening here.

There it was h dot is gamma over h here it is gamma times h. Okay. So also nice. Right. So again a similar feature that you will see is that here as you go close to the boundary h is going to go to 0.

Going to go towards 0. Therefore by this law as you go close to boundary you will stop moving. Gamma h is close to 0. Stop moving. Stop further increasing or anything. Yeah.

If you are far from the boundary h could have large values. It could have large but you are inside the set. Suppose you are of course you are inside the set.

Let's assume you are inside the set. Yeah. h is positive. Okay. Could have large positive values as you go further in and in the set. Yeah. So h is decaying which means it is being pushed towards the boundary. Earlier we were looking at everything in the perspective of 1 over 2.

Therefore we were thinking other way round. Now is the other way round. 0 is h equal to 0 is the boundary. So as you are in the interior h is large therefore minus gamma h is large and you can potentially get pushed towards the boundary.

Right. Because h dot equal to minus gamma h is basically going to push you towards the boundary. It is going to reduce h.

Right. That's fine. You are allowed to explore that region. And as you go close to the boundary h is going to become 0. You don't further move. You have stopped. Okay. So that's the idea that this there is no reciprocal nature here. So when you do a control design with this we are of course out of time.

We will not be able to do a control design and all that. But typically constructing these is not easy. That's the thing. Typically how this is done is folks usually solve quadratic program. They don't actually what they do is they don't actually they only construct the h and they put a quadratic these are all quadratic program conditions.

Right. If you see this is a QP type conditions. These are actually QP type conditions. If you don't see also I am telling you this QP type conditions. Yeah. This is like the barrier condition and corresponding there will be a control condition which is going to make states go to 0. So there will be a control condition barrier condition and the two are solved simultaneously using a quadratic program using some say I want to minimize control or something like that.

Okay. And this so they don't actually solve this by hand analytically. So it's not typically easy to solve all of these by hand. But the reciprocal ones you can see you can do by hand.

I would really strongly suggest that for the same sort of example you try to construct these zeroing barrier functions. Okay. Also. All right. So why is it called zeroing barrier function because at the boundary we didn't do anything. Earlier we took a reciprocal right one over hx and things like that.

Here we took the h itself as the barrier function. Right. And we were just just ensuring that the h is such that h dot is greater than equal to minus alpha h. Okay. Which means that it has a nice zeroing property.

Yeah. The barrier function becomes zero at the boundary because the barrier function and this function are kept the same functions to be honest. Yeah. So the barrier. So that's why it's called a zeroing barrier function.

On the other hand when we took the reciprocal ones at the boundary they explode. Here they become zero. Right. So it's like you stop moving at the boundary. Therefore you will not need large force to push it back typically.

Okay. That's the idea. So I would really recommend that for some simple example like I tried you try to come up with a zeroing barrier function. Okay. It's essentially this h of x function that h dot is greater than or equal to minus alpha h.

Can one come up with that? And so that if you are looking at the control problem there is no difference per se.

Right. It's all these conditions this condition becomes. Yeah. Exactly the same thing just with the control. Right. And this condition becomes.

Okay. This is just in the clf condition. Right. This is the conditions. That's all. It's just that you have a control term here to play with. This helps you in the design. In the control case you have a control term now and you can play with the control so that with whatever b you have or whatever h you have with using this control you can get this.

That's the whole idea. Yeah. I would strongly recommend you try because that's what we did right. We did the control problem. I will post this article and you can take a look.

Yeah. There are rather nice articles. The entire field of bipedal robot walking is now relies on this. Okay. This is the Aaron Aims's group and these guys are pretty much experts at it. Okay. Alright. Okay. Thank you. Thank you.