Nonlinear Control Design

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Week 10 : Lecture 59 : Feedback Linearization: Part 10

For complete feedback linearization we have this new result. This result we will try to prove and understand ok, because it is simply based on using the Frobenius theorem alright and going back to our earlier conditions. The first condition was that the first requirement is that you have a notice that it is a necessary and sufficient condition first of all. Then you are saying that this matrix is full rank and this distribution is involute ok great. How do we prove this? The necessary and sufficiency. If you remember we can go back to what we required for feedback linearization earlier right.

We required that you know you have this sort of thing happening. What is this? Do you remember what this was? If you have an output y equal to lambda x right then you wanted its derivative when you take its successive derivative you want the control to not appear ok. Because we are talking about relative degree degree n so we go all the way to n minus 2 ok. This means that control does not appear in the n minus 1 derivatives ok only in the last one.

And then you also required that LGLF n minus 1 lambda is actually not equal to 0 right. Basically it is just saying that you take when you take the nth derivative the control appears ok. This is what this is saying right. This is how we did it. We said that you had an output function y equal to lambda x yeah or in that case we use hx it does not matter just notation.

And if you keep taking derivatives of lambda control will not should not appear in the first n minus 1th derivative but should appear in the nth derivative. That was this condition ok. Now we are going to basically say that this is going to imply complete integrability ok how? You remember that this set of conditions by lemma 0.1 is identical to this conditions right. All the LGLF can become ADF yeah.

I'm not going to go back and show you the lemma but because we remember this yeah it just becomes ADF 0 ADF 1 ADF 2 all the way to ADF n minus 2 yeah. If these are 0 all the way to ADF n minus 2 g is equal to 0 ok. And similarly using this you can show that ADF n minus 1 g times lambda is not 0 ok. Pretty straight forward to show. We already did this before ok now from this what we have already shown again in lemma 0.

2 is that these are linearly independent vectors ok. How? Ok now I have to swipe back I apologize. Let me go back here and if you remember we proved this using a kind of matrix multiplication right and we had two matrices right. Just look at this yeah. We proved the

linear independence of the DH functions using this product right and we showed that this product is rank R therefore individually each of these must be rank R ok which means these are linearly independent ok which means these are linearly independent as is this.

In this lemma of course our statement said that these are linearly independent but we also proved that these are linearly independent right because if the product is rank R then individually each of them are rank R ok great. So this is what we are using. So we go back and there the R gets replaced by n that's all. So we know that this is now a set of linearly independent vectors ok. So this is the first thing.

We have already proved the first requirement. We are going backwards that we already proved that these are linearly independent done. Now we only are left to prove involutivity of the distribution ok. There we are going to use the Frobenius theorem. What? How do we do that? D is a span of these guys so obviously it's non-singular and n minus one dimensional.

We have already said that right ok. So this condition that you have this guy yeah this condition can be written as this just in matrix form yes this is precisely the integrability condition right. This is the same as this correct. So this is exactly the integrability condition right on G add of G add of n minus 2 G yes. So we are done.

What we have shown what have we shown? We have shown that the distribution is non-singular and we have shown that the distribution is integrable right because in this case you had right you have k equal to in this case this is what k equal to n minus 1 right. This is k equal to n minus 1 right because you have 1 2 n minus 1 vectors in the distribution. So k is equal to n minus 1 therefore how many H functions we are going to get? Only one because n minus n minus 1 n minus k right. So only one H function that is this lambda ok and this is what we had right in our earlier feedback linearization result also that lambda was the output function here this is this lambda alright. So now here if you notice this lambda becomes the output function in the partial linearization I said that it could be the output function but it could also be the extra coordinates that we were talking about ok.

But in any case so you have here this lambda is the output function because we have already proven that D is non-singular and integrable it means that D is involutive right and that is the second statement ok. So one side of the proof is done alright. Now we can also prove the other side similarly ok. So from the first from the two conditions we have that the distribution is non-singular and involutive this is our assumption. Therefore by Frobenius theorem D is completely integrable which means there exists an H such that this happens yeah and what is this? This is nothing but L g L add f L add f g all the way to L add f n minus 2 g yes just by multiplying this out yeah and again I go back to lemma 0.

1 so you see the lemma 0.1 and 0.2 are regularly getting used. So if I go back to lemma 0.1 all these L add f 0 add f 1 add f n minus 2 is basically L g L g L f L g L f n minus 2 ok.

So I got one condition that is required for feedback linearization right. The other one is that L g L f n minus 1 has to be non-zero ok alright. How do I prove that? Ok I take the DH x which is a 1 by n vector and I multiply it with this additional quantity here ok. I take this whole guy but then I multiply with this ok. So what does the product give me? It gives me zeros in the first all of these become zeros right because of this guy but then I get this last piece alright then I get this last piece.

Now what do I know? I know that this is full rank right this is a full rank matrix and this is of course not zero rank right this is rank 1 at least because it is just one coordinate right. So if I take partial and it is zero it is not a function at all yeah it is a constant it is not a coordinate anymore. So this is at least rank 1 well at least at most rank 1 because it is a row vector so what do I have? I have the product of a rank 1 matrix and a rank n matrix ok. This is a rank n matrix this is a rank 1 matrix so the product can be cannot be less than rank 1 right because the rank of the product is actually the minimum of the rank of each of the entities. So the minimum is 1 so this has to be rank 1 cannot be rank 0 if this is rank 1 this is not 0 right ok.

So same tricks if you remember the tricks that I am using are the same use lemma point 1 lemma point 2 take products of matrices show that product has some rank ok that is how we are doing things ok. So because the individually each of them this is rank 1 this is rank n so obviously the product cannot be less than rank 1 yeah so therefore this has to be non-zero. So we have just proved that this guy is non-zero. So we are done right this is what is the conditions we require for feedback linearization in fact h again becomes the output the desired output ok. So and this is what we can use to come up with the output ok.

So we did the DC motor example right if you let's go back this was the DC motor yeah what did we do we verified this point 1 point 2 and so on and did we try to yeah we also found a feedback linearization right we found a feedback linearization and in order to do the feedback linearization we actually already had an output correct we already had this output which is this yeah in this case at least the model that I had used I don't know what's model what is the model coming up I don't know but at least the model that I used had relative degree what what does the relative degree in this case 2 right. Relative degree was 2 not 3 so this was not completely linearizable ok so the relative degree was only 2 that is why if you remember I had chosen this extra state right I had chosen this additional state right which was the phi to make the diffeomorphism alright. So let's keep this in mind what was the dynamics this and this was the output ok. So somehow my feeling is that the dynamics that we have now is something different ok alright that's fine let's let's let's go and see what happens let's not worry alright. So what is the DC motor dynamics that you have here this is what I mean there is a proper with network and electrical voltages in L and R and so on yeah so we have all the nice the proper equation the electrical dynamics on the stator side then you have the electrical dynamics on the rotor side then you have the mechanical dynamics of the rotor ok.

So this is the mechanical part and the other two are the electrical parts you have inductances the resistances the back EMF the rotor inertia angular speed friction and torque ok so this is the torque developed at the rotor shaft alright so there is yeah I am just going to skip I am just going to skip to this part ok yeah. So this is the dynamics of the system yeah fx plus gxu ok now yeah yeah there was an error here there was an additional zero this should be only three dimensional right cannot be four dimension because it's a three dimensional state space so I just remove this but does this match the dynamics we had for the DC motor is what I am wondering yeah this was it so because I took the dynamics from Khalil I really don't know if so let me paste it here now does this look similar to this guy x1 dot is minus x1 plus u yeah similar alright x2 dot contains some x2 term and a constant and an x1 x3 similar x3 contains x1 x2 and an x3 this is not similar ok there is some additional term here I don't know what it is from I think in that model clearly looks like the friction term is missing ok in the earlier model in this model you see the x3 dynamics has this missing yeah so it looks like there is some and that x3 is coming from this guy yeah that's the friction model it looks to me like or am I missing something yeah x3 is omega yeah yeah yeah looks like it's missing the I mean the model I believe that I took from Khalil is missing the friction ok maybe whatever very good DC motor alright anyway so we have this model alright now the question that we want to ask is can we fully feedback linearize the system how do we verify this we are given two conditions right first is that G you have to verify this G add of G and add of squared G right why because the three dimensional system right and we want add of N minus 1 G until add of N minus 1 G we want this to be full rank now let's do some computations ok what is G is this guy ok what is add of G I have already written it add of G is this ok how do you compute add of G I want you to compute I want you to verify how do you compute add of G otherwise you folks will get completely lost ok you have given G 1 L S 0 0 and you have given F well actually I am not going to rewrite so add of G is what is the same as lie bracket of F G excellent excellent alright and this is what formula first you do the second one DGF minus DFG always remember the first term is Jacobian with respect to the second argument Karu I will do this in a smart way first of all DG if you look at DG what is DG what is DG Jacobian of this yes 0 it's a constant right constant thing don't have derivatives right so this guy is gone already ok no first term all you are left with is the second term second term ok now you have to play it smart I would not compute all of it what would I do I am doing DF times G so some matrix multiplied by G so what do I know the G has last two elements 0 ok so do I have to compute the entire matrix what will I compute first column right first column of DF is enough because it will be multiplied by the first element of G so that's what I am doing just computing the first column what is the first column of DF no I mean what is the expression how do you compute first column of DF ok what is DF expand DF for me in terms of actual partial derivatives first column is what now absolutely del F1 del X1 del F2 del X1 del F3 del X1 right that's the first column right that's how I am doing this so all I am doing is taking partial of each of these with respect to X1 right so that's what I did look at this minus RS by LS how did I get a square I have no idea I don't know god help me I don't know I got this square no square there is no square here right did I miss something am I right what about del F by del X1 of this guy just this right minus K LR what why am I missing everything acha I have already multiplied oh see you guys don't tell me when I am right this is correct I have already multiplied by G no so obviously 1 by LS is multiplied so RS by LS 1 by LS right this is correct yes partial with respect to X1 is just minus RS over LS multiplied by 1 by LS RS LS square done partial with respect to X1 just this 1 just this multiplied by LS is this guy third term partial with respect to X1 is this multiplied by LS again this excellent excellent excellent now done what about add F square G what about F this is F of FG now now it's not nice anymore because FG is now this yeah FG is now this guy let's see did I do this apparently I did not do this okay let's let's put some effort and do it and what else what else is ok this 47 mutual capital G of tell me I have F G ok alright folks now we are going to work now what is add F squared G ok this is equal to F comma F G ok this is what we have to verify linear independence as of now what have we obtained we have obtained G which is which has something in the first element alright then we have obtained this guy which has also something positive in the or whatever non-zero in the first element but then you have some funny things happening here ok you know alright but we are going to compute this, this is what D F G F minus D F F G ok. Now what first row compute D F G F G is written here take Jacobian and tell me what is the first row cannot take so much time for first row thank you very much just be confident I will shout at you later alright ok second row correct got it right third row I cannot hear anything ok alright then you have this F nice whatever it is F I am not even going to write it I will do the multiplication later on D F minus R over S sorry R over L S 0 0 ok second R R L R and then third is minus K L S L R times x 1 third one yup yup K L S J x 2 but do you prefer this or the Frobenius theorem proof this one? I think you can write something right in that one it is K L S by J times x 1 and F by J times what minus R S by L S squared minus K L R x 3 K by J x 2 good anything nice is coming out of here or what? Nothing is going to be nice here is it that is the whole idea right ok let us just look at this guy what is this guy the first row is 0 so I have a 0 here what about the second row it is just minus K over L R multiplied by F by J x 3 plus K L S by J x 1 x 2 wow nice oh this is just not nice we will try to avoid such cases minus R R by L R x 2 plus V R by L R minus K L S by L R x 1 x 3 minus this whole guy ok I am not even going to try to expand this ok what do we have to do we have to now say that this is full rank so what he is saying is that it is full rank if x 2 or x 3 are non-zero ok alright then alright then let us see we already have G which is giving me 1 here if either x 3 or x 2 is non-zero this is going to give me another independent vector alright this one I really don't know I mean I don't see how this can be simplified any further yeah I ideally don't know I mean I assume that this will also give me something like unless I miss something this is correct though this is correct because I have already written it before also yeah this is not turning out to be simple but I guess if x 3 and x 2 are non-zero then this should be fine too yeah I really have to check or you guys can check I don't know yeah I wouldn't usually make up an example like this but that's fine yeah in fact what I was wondering was in our case we did not have the friction term so how did that impact the linearizability is what I was wondering because all that is getting rid of is some of these f terms here which is here and here in some very few places to be honest yeah this is I will say to be completed offline yeah you can do it on a leisure time yeah or you can even you can even check it numerically yeah you just feed it and then just plot it and if it doesn't hit zeros you're fine okay rather than actually doing the whole thing doesn't make sense to be honest okay let's go to the second condition what was the second condition you want the distribution to be actually we need involutivity right we need involutivity okay we want involutivity of the in this case g and add of g okay only two vectors the add of square g is not required yeah because we are looking at n minus 2 here n is 3 so we are looking at the distribution which is g and add of g and this you want to be involutive that should be relatively easy but actually what is there to prove because we have only two okay alright sure sure we have g and add of g and what we want to claim is that g comma fg belongs to the distribution okay this is the involutivity right because you the distribution is just consisting of yeah but this should be easier to find yeah why because this is what dg times fg minus d fg times g what do I know now what is dg 0 right so dg is 0 so this term is 0 okay so and I know that g only has element in the first there is only the first term so I only have to compute the first column of this guy okay I believe we've already done dfg is this guy dfg was this right what is the first column 0 so so obviously 0 belongs to delta I mean if you have any vector space 0 as an element of the vector space you are confused yeah or if you don't like to look at it like that you can think of I take g 1 times g minus 1 times g still has 0 any vector space contains 0 okay so this is the 0 so trivially true okay so so implies delta is involutives okay which means that delta is completely integrable so what does that mean it means that if you take some function beta and you take its gradient or whatever you can call it d beta multiplied by the elements of the distribution that is g and add fg that is equal to 0 this is the condition right and so this gives again what a partial differential equation right because this grad B is what partially with respect to x 1 x 2 x 3 multiplied by these okay alright so let's actually try to evaluate this what does it what does it give me I mean I don't know why he keeps changing notation anyway you have d beta multiplied by terms of the distribution which is g and I will say add fg and this is equal to 0 yeah by complete integrability alright so we use this to actually compute you