Nonlinear Control Design

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Week 10 : Lecture 57 : Feedback Linearization: Part 8

Welcome to Non-linear Control. So, where we were last time was at the Frobenius theorem. We had started discussing the Frobenius theorem. We are going to continue to actually discuss and talk about the actual theorem itself. In order to talk about the Frobenius theorem, we first introduce the notion of a distribution which is basically you take all these generating vector fields and you take a span. So, basically at every point p in the state space or whatever set you are working with, you get a subspace.

At every point p you get a subspace. So, that is what is the distribution itself. Obviously, we do not want the distribution to lose rank or change rank. So, therefore we talk about non-singular distributions which do not alter in rank as you change the point p.

So, we will always work with non-singular distributions. And then we stated and proved this lemma which is saying that if delta is a non-singular distribution then involutivity is exactly identical to the lie bracket being in delta. So, actually I forgot we also defined involutivity which is saying that if there are two vector fields in the distribution. If fg belong to the distribution then the lie bracket also belongs to the distribution. Now, we also discussed this that it will lead us if you want to actually check this in reality you will have to do all these iterated lie brackets.

So, many iterated lie brackets. So, we came up with a single simpler condition which is this lemma 1.3 which says that if you have k generating vector fields for delta then involutivity is identical to just verifying that pair wise they belong to delta. That is it. Pair wise you have to check that fi fj belong to delta.

We also proved this. I am not going to go into the proof of this. We also sort of anyway did this example on feedback linearization. Yeah, this system and anyway we just this is again going back to the previous material. We just took an output which was given to us and then computed the relative degree of the system.

After computing the relative degree of the system we saw that it was 2 and the state space was 3 dimensional. So, obviously I need one more additional coordinate and we saw what we were thinking how to choose it. The simple obvious choice does not work. Like if I choose z1 is x1 the derivative x1 contains the control which we don't like because in the normal form the control appears only in the linear part and not in the nonlinear part. So, we don't like this choice.

So, obviously in order to come up with this good choice we use the definition itself that is you know we want to choose this phi function such that lg phi is equal to 0. And that gives us a bunch of basically gives us just one partial differential equation. Just one partial differential equation which is looking like this. As you can see this is not completely specifying the function but we are just going to guess one possible choice that satisfies this relationship and we guessed it. I would say I guessed it.

If you can come up with a more smarter way of doing this sure. In fact we can go later and see how to maybe I don't know maybe it is possible to do better than this. But as of now no we just have this we are basically guessing this function based on the pd that we have. So, it's not like a completely out in the blue sort of a guess but it's more like you know you have a pd and it's not easy to solve this. So, you will sort of try to do something better.

Yeah. Yeah. Yeah. I mean actually you can do something but I'm not going to and it's going to be very complicated. Yeah. Guessing is way easier here.

Yeah. You just tend to cancel appropriate terms and so on. And we discussed this last time. Alright. This I left for you to sort of look at because this is part of the exercise that was given but I have it looks like I have already done something on it. So, I leave it to you folks to sort of verify this.

Okay. Now we actually move on to the statement of the Frobenius theorem itself. Why do we care about the Frobenius theorem? Because we are trying to answer that original question. When is it possible to feedback linearize the system? Completely feedback linearize system. Until now we've been looking at relative degree something.

Right. Which is less than n. Of course the Frobenius theorem is useful even then but it becomes especially useful to figure out when your relative degree can be made n. Okay. Which means that you essentially have the entire system to look like a linear system.

Right. After suitable state transformation. Okay. So, that is the question we are trying to answer. So, Frobenius theorem is what lets you answer that question. So, that's why we are moving towards the Frobenius theorem.

Alright. So, we make a couple of more, well couple of more, one definition and one theorem. Right. So, the non-singular distribution delta which is generated by these k vector fields is said to be completely integrable on some open set in the state space if there exist n minus k n i highlighters. Okay. If you remember your vector space course, annihilators are essentially what annihilate the elements of the vector space, vector field or vector space itself.

Okay. In this case and remember that the annihilator always has dimension n minus the, yeah, the vector space of annihilators is always n minus the dimension of the vector space itself. That's how it works. So, in this case you have k vector fields. Distribution is non-singular which means delta is of dimension k at every point p. At every point p, delta is going to have dimension k.

Right. Because it's a non-singular distribution. Okay. Therefore, we have n minus k functions. Okay. So, what are these n minus k functions? These are, notice that these are real valued variable.

These are scalar functions. Okay. These are scalar functions. And what, how do we define them? They have to satisfy that L f i h j which is this. Del h j del x f i is exactly equal to 0.

Okay. This is actually like an annihilator. Okay. So, here it is not h itself but you are taking the, it's not, the h is not the annihilator vector. The annihilator vector is the partial of h or d h.

Okay. Because h is a scalar field or this real valued function. Yeah. But when I take its partial with respect to x, I now get a row vector.

Right. It's one by n dimension. Okay. So, this is the annihilating vector. Vector. Okay. So, and further you want that these d h, d h is of course this guy itself.

We have already defined this notation. d h is this. These d h are supposed to be linearly independent.

Okay. Okay. So, if this, if these conditions are satisfied, then the vector field or the, sorry, or the distribution is said to be completely integrable. Okay. This is just a definition. So, just keep this in mind. We don't want to go into too much more detail than this.

Yeah. So, distribution generated by k vector fields, I want the existence of n minus k functions such that their d h multiplied by f i is zero for all i and all j. Yeah. i is ranging from one to k, j from one to n minus k. Okay. And of course these d h have to be linearly independent.

As soon as I said this d h has to be linearly independent, you should be reminded of this lemma 0.2. Yeah. If you remember, I mean, I will scroll back.

What was lemma 0.2? Yeah. We started saying that some row vectors are linearly independent. Okay. So, exactly like this. As soon as I say that these d h are linearly independent, you should be reminded of that.

Yeah. So, you are already, it looks like there is a connection between what we are saying here or what we want here and what we were looking for there. Okay. Alright. Great. Then we talk about, I mean, in order to actually verify this result, the Frobenius theorem, we need the inverse function theorem.

But the inverse function theorem is such an inverse function and implicit function theorem. These two theorems are such powerful results that all of you who are doing anything in systems and control just read and understand this theorem, these two theorems, the implicit function theorem and the inverse function theorem. Okay. Implicit function theorem is an out byproduct of the inverse function theorem.

It is very straightforward. What does the inverse function theorem say? It just says that if you have a continuously differentiable function from an open set in R n to R n and if the total derivative, that is dt of p is invertible. Okay. At some point p. So, dt of p is invertible. Then the function is not just invertible at p but in a neighborhood of p.

Okay. So, so all these result, both the implicit and inverse function theorem results. Remember, I connected, if you remember when I talked about the diffeomorphism. Right. And I said that it is essentially like the equivalent of a similarity transformation for non-linear systems.

Right. So, when I talked about the diffeomorphism, how did I verify something is a diffeomorphism? I took its Jacobian and the Jacobian has to be invertible. Right. This is where it all that comes from. Because if the Jacobian is invertible, that is the dt is invertible at a particular point, then there exists a neighborhood around p in which the entire function is invertible.

Okay. So, it is almost like what we do with linearization in, you know, in control also we do this. Right. We take a non-linear system and we linearize around some equilibrium point. And then we look at the linearization, the ad matrices and if they have some nice property like if A is a Hurwitz matrix, we say that the system is locally exponentially stable or locally asymptotically stable.

So, this is almost like that. The derivative of a map or the Jacobian of a map actually gives you something about the invertibility of the map. Okay. And this is why these results are rather powerful and applicable in so many different places.

Yeah. So, that is what this is saying. It is just saying that if you have a continuously differentiable function from one open set in Rn to another and the derivative or the Jacobian is invertible at some point p, you just have to check at one point, then there exists a neighborhood around p in which the entire function is invertible. Yeah. Not talking about Jacobian, Jacobian is already invertible.

Yeah. The function itself is invertible in this entire set. Okay. So, that is the inverse function theorem. And not only invertible, it is in fact, the inverse is also continuously differentiable.

Okay. So, similarly if you started with a smooth map, then the inverse will also be smooth.

This is exactly what we use to define a diffeomorphism. Okay. So, whenever you have to check that a map is a diffeomorphism, by the way, the map T can be, is a dt can be invertible only if it is going from the same dimension to the same dimension. Right. So, therefore, we are always talking about Rn to Rn.

Same in diffeomorphism. Right. It is a state transformation. So, the number of states here on the left has to be same as the number of states on the right. Right. How do you check it is a diffeomorphism? Just compute the Jacobian of the map and the Jacobian is invertible, you are good to go.

It is a diffeomorphism. Okay. And these are admissible transformations, state transformations like similarity transformations.

Yeah. Okay. Excellent. Great. Now, we are ready to state the Frobenius theorem. Okay. Before I state the theorem itself, I will tell you that I am not going to do the proof extensively.

The proof is written here. Yeah. If we actually end up having time, I might just do a session separately just for the proof. Yeah. Proof is quite involved. And, but I will encourage you to read it in.

I will just give you a short sketch. We don't expect you to do the proof itself. So, I am not going to actually cover it. But, later on if there is extra time, I will just do a separate session on just the proof.

Okay. Alright. It is very, actually this is Vivek's note. He has done a very good job because usually it is not easy to find the proof of Frobenius theorem in Euclidean space. Yeah. Typically, whenever, if you remember I told you that the reference was Alessandro Estalphi's book. So, most of the notes and everything out there, whenever we talk of Frobenius theorem, they are working on manifolds.

Okay. Not Rn. And then there is lot more notation and you know, careful bookkeeping there. Yeah. So, here everything is proved in Rn, which is rather nice.

So, the proof is very nice. Please go through it. Okay. If you get a chance. So, let's look at the statement. The statement is actually just two pieces. One, it says that, if you, of course you start with a non-singular distribution of dimension K.

Okay. Always. I mean, that is what we have been doing. Then, in, what this entire thing says is two things. One, involutivity and complete integrability are equivalent.

Okay. There is two pieces here. Although it has long statement here. Involutivity is equivalent to complete integrability. That's the one statement. And the other statement is that, other piece is that, if you have involutivity, then your, there exists some transformation such that your delta looks like a span of unit vectors.

Okay. I don't know if you can appreciate it yet. So, I started with what? I started with, yeah, that's what I started with. The delta is what? It's a span of this guy, right? Some K vector fields evaluated at a particular point. What are we saying? We are saying that, something rather powerful. We are saying that, if delta is involutive in the state space, then it's equivalent to saying that this delta is in fact the span of identity vector fields or the unit vector fields.

What is the unit vector field? E i is this, E 1 is this. So, obviously it's not affected by point P or anything. Everywhere it's the same. So, it's pretty powerful.

It says that there exists some smooth change of coordinates. Okay. Such that all these vector fields that you have, F 1 to F 2 to F k looks exactly like E 1, E 2, E 3 and E k.

Okay. Alright. I hope you can appreciate the power of the result. Okay. And if you remember, what are these vector fields? They are just right hand sides of differential equation. Right? If you, that's how you have been, if you think about the control vector field, it is something like x dot is u i summation of u i F i. Okay. That's how we have been looking at.

Right? And it basically gives you what directions of movement at every point. At every point, a distribution test tells you that what are the directions where you can move using this particular control. Okay. So if I now say that this span F 1 to F k, which is some complex non-linear function, became just span of E 1 to E k.

Now E 1 to E k is just, you know what it is. It is the k dimensional R k. It is a subspace R k in R n. Right? So it essentially you immediately can say that at every point in the state space, I can move in a k dimensional subspace of that state space. Yeah? That's pretty powerful. Okay. Also, if you replace F 1 by E 1, F 2 by E 2, F 3 by E 3, what is the dynamics come out to? Somehow looks like some integrators are happening.

Right? x 1 dot is x 2, x 2 dot is x 3. No? Does it look like it is happening? Maybe not exactly.

Okay. That maybe not. That no, no, no, no. It's not. The integrator is not happening very well. Let's not worry about that. But the point is, the key point is, you are reducing this rather complicated non-linear, you know at every point P you get a different subspace, different shape of the subspace to a very nice hyperplane. Okay? So this is the power of the Frobenius theorem. One, if you have in-volatility of this delta, then you get this very nice hyperplane at every point.

Yeah? Which is the R k dimensional subspace in R n. Okay? And two, in-volatility and complete integrability are identical. Yeah? If you have a non-singular delta. Yeah? And that's pretty cool because in-volatility was some kind of an algebraic computation, you know whatever.

I mean you were just doing lie brackets. Yeah? And you just do, it's easy to verify. Yeah? And by doing that you actually get that you have complete integrability. And when we actually say it's completely integrable, you actually got these functions h 1 to h n minus k. Yeah? That's what it means right? Complete integrability means I actually get these functions h 1 to h n minus k which the derivatives of which act as annihilators.

Okay? Alright? So these are what will become a new coordinates. Right? So alright. Proof. I'll just sketch it for you. Proof goes in three steps. In the first step, in step one what is being done is just this proof. That if you start with in-volatility, any such f 1 to f k will become e 1 to e k.

Okay? That is what this step one is doing. Okay? It does it in a rather nice cool way. Okay? But I mean it doesn't do it okay. It doesn't do it in completeness but it more or less does that. Okay? That you are, if you see these f 1 to f k, you started with f 1 to f k, these are and then you ended up with something like e 1 to e k but with something more here.

Yeah? With something more here. Not e 1 to e k itself but up to the kth element this is consistent. Right? It is just the unit vectors. First unit vector, second unit vector and so on. But there is some terms here. Okay? That's what this does in the first step.

Yeah? It, the other thing it also does is it shows that this new vector fields commute. What are the new vector fields? These guys. Okay? And when we say vector fields commute, we mean that the lie brackets are linear. Okay? That's how you define commuting vector fields.

It is like in the matrix case also right? You a b minus b a equal to zero. That's actually the same. Exactly this. For the matrix case, yeah? If you say linear system case, x dot is a x, x dot is b x. This is the lie bracket.

It is easy to compute. If I say f 1 is a x, f 2 is b x, lie bracket is exactly a b minus b a. Okay? Not difficult. You can think about it. In fact just a thought experiment. So f i, f j commute. Okay? So earlier when you started with f 1 to f k it was not evident.

But once you do this transformation it is evident that the vector fields commute. Okay? Alright. What do we do in the second step? Hmm. Second step is the complicated step.

Actually, sorry, I apologize. Here all you show is that the vector fields commute. That's all. You don't show that it becomes equivalent to e 1 to e k. That is done at the, in the second step. If you look at just the last line of the second step you can see f 1 to f k are mapped to the constant distribution e 1 to e k.

Okay? And what the second step does is it actually comes up with the transformation. The state transformation. You need a state transformation right? To go from the f 1 to f k vector fields to the e 1 to e k vector fields. That's what the second step does. It's rather complicated because it uses flows and things like that. I have actually not introduced this terminology too much.

Again, if time permits we will do a separate session on this but not right now. Yeah? All I am telling you is what each step is proving. Alright. The final step. Okay? Once you know that your f 1 to f k's are smoothly mapped to e 1 to e k's.

Okay? You try to come up with the h functions for the e 1 to e k system. Okay? What did we do? We went from the f 1 to f k vector fields to the e 1 to e k vector fields. Alright? And now in order to prove complete integrability what do I need? I need to come up with the h.

h 1 to h n minus k. That will annihilate this guy. This is much easier to do. Right? Because even annihilating e 1 to e k is much easier. Right? What do I say? All I do is I pick this annihilating functions as x j plus k for j starting from 1 to n minus k.

Okay? I hope this is not getting already too complicated. Okay? Just look at this. x 1, x 2, x k, x k plus 1. I am going to write this as x k plus n minus k. Yeah? You see this vector? Yeah? Let's go back to what we are doing in feedback linearization. Let's forget this annihilator business. What were we doing in feedback linearization? Whatever was the relative degree of the system, we were choosing the rest of the coordinates. Right? If the relative degree was r, we were using n minus r coordinates on our own so that the entire thing becomes a diffeomorphism.

Right? And in the normal form of course l g phi is zero. Right? It's almost like that. Almost exactly like that. Yeah? If you look at this, this almost looks like l g phi. Doesn't it? Yeah? Because the g is the flipped version.

The g is this guy, the vector field and the phi is this guy. Right? Exactly looks like that. Even if I go back here, l f h. Exactly looks like l g phi. This is how I was choosing the new coordinates.

Right? So that the control doesn't appear. That's all this. This looks exactly similar to that. You make something zero. Some l g phi to be zero. It's exactly like that. In this case, the g is this vector field and phi is this function.

Okay? So that's all we are doing. If you notice, I am choosing the g in a smart way so that this partial multiplied by E i is zero. Now i ranges from what one to k. Right? So E i has one in what position? In the ith position.

Right? E i has one in the ith position and zeros everywhere else. And this i is maximum k. Cannot be more than k. Now if I choose my, this guy is j plus k. Okay? Then where will it have its, if I take a partial with respect to x, this row vector, where will it have one? Where will it have one? In the j plus kth position. Right? It will have one in j plus kth position and zero everywhere else.

Right? So definitely, I will, so this guy will definitely have one in the j plus kth position. This will have one in the ith position. And i is less than or equal to k.

Therefore, they don't have, this is a dot product. Right? So they don't have one in the same position. Can never have. That's how I chose it. Right? This E i will have one say here. E two will have one in the second position. Okay? This will have one in the j plus two ith position.

Okay? So therefore they are not going to be, have one in the same position. So the dot product has to be zero. Okay? That's it. I am just making the smart choice. Okay? Why do I do it in the E i, in the E one, E two, E k frame coordinates? Because it's easier. It's just unit vectors. Yeah? Much easier to construct this.

Okay? Now once I have constructed this annihilator in the E coordinates, E i coordinates, I can take it back to the original coordinates. That is the F one to F k system. Yeah? Just by doing a transformation.

That transformation comes from here. Yeah? That transformation comes from here. Which is why this thing is a bit complicated. And it's pretty easy to show that once I make this backward transformation, this will also have the same property. That del H j del Q F i Q will be zero. Because I moved back and forth with the same coordinate change.

Right? I got a very nice coordinate change. Okay? And I constructed the annihilator in the new coordinates.

Then I went back. I took that annihilator back in the original coordinates. That's it. Okay? Yeah. The process is simple. Notation is complicated. Alright? And so what have I done? I started with involutivity. Okay? And I proved that I have integrability. Okay?