Nonlinear Control Design

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Week 9 : Lecture 50 : Feedback Linearization: Part 2

Relative degree is defined so that it tells you in which derivative of the output you get the input, okay. H is the output function, right. So you keep taking successive derivatives and these are the terms that appear in connected to the control in the successive derivatives, yeah. This is what will happen. You will get H, the derivative will contain LGH multiplying the control. In the second derivative you will have LGLFH multiplying the control and if you keep going on and on, the R minus 1th derivative will have LGLFR minus 2 multiplying the control.

You can see the pattern, right. It is pretty easy, right. LG LGLF LGLF square LGLF cubed LGLFR minus 2, right. So until R minus 2 this is 0, yeah.

You take one more derivative, the term multiplying the control will be LGLFR minus 1. That is not 0. So in the Rth derivative of Y you see that the control appears, okay. So this entire business is to codify in what derivative control appears, yeah. That is what I have written here.

Y is H, Y1 is LFH, YR minus 1. Because of this property, your dynamics in terms of this new variable, this is the state transformation. The state transformation is coming from your output that you chose, from the Y you chose, okay. I choose the Y as H. Its first derivative is LFH because the control term is 0 by virtue of this assumption.

Second derivative so on has 0 control. Similarly all the way to R minus 1th derivative, no control, right. No control because of this assumption. But then when I take the Rth derivative, the control appears because this term is now not 0, okay. Now taking any subsequent derivative is useless as well as feedback linearization is concerned.

And he has a very relevant question, is this the order of the system? No, R is not necessarily N, okay. R is not N. So you can see in this new state transformation, how many variables did I get? New variables? R, right. I just have R, R or R, R plus 1, right. If I take Y, Y1, Y0, Y1, yeah, it is R plus 1 variables, right.

Yeah. So this is invariably going to be less than N, less than or equal to N. And definitely cannot be more than N, yeah, because in the Nth derivative anyway of the control appearing, otherwise you have no control. The control doesn't appear in any derivative then, then means that you have no control, yeah. So the point is that you have only R minus 1 variables that you could design, alright. Now what is the whole purpose here? I took this output Y, just like in the pendulum case I took the output as the X1 state or the Q1 state which is the angle, right.

I took some output, I got this R plus 1 state equations. And in this equation I have the control, right, and this term is nonzero. So I can use this guy, okay. I can use this guy to cancel this. You can see that, right, that I can use this control to cancel this guy and introduce a linear term that makes this system linear, okay.

But then the question is what happens to the rest of this? Are these linear or not? Yeah, are these linear or not? That is not yet clear, okay. But what we can say is this piece becomes linear, okay. At least that much you can say for sure, alright. Alright, let us see. Let us see what happens further.

In order to sort of assess the properties of lead derivatives we have to prove a couple of lemmas, yeah, slightly difficult looking. So please do not get scared. We will introduce one more new notation which is the add bracket that is the add notation, okay, which is basically the notation for successive lead brackets, yeah. So add f kg is basically k brackets. So basically if you think of it simply, add f 0 g is just g itself.

Add f to the power 1 g is f lead bracket with g. Add f to the power 2 g is f lead bracket with f g lead bracket, okay. It is a successive lead bracket notation, yeah. Again why are these important? From a controllability perspective you can move along successive lead brackets also. So it is pretty cool actually.

If you think about the geometric implication of this, it is saying that you can move along f and g and you can move along lead bracket of f g, you can move along successive lead brackets of f g. So these are the directions in which you can move. So again we will look at that later or we may not look at it. So let us not worry about it. But we will use this in the proof of when we can do feedback linearization.

So we need this result which says that if you have these quantities to be 0, all these quantities to be 0 and notice these quantities are 0 for up to k equal to r minus 2, right. This first result is equivalent to the second result is what we have said, okay. And the first one we already have by this, by the relative degree assumption, right. So what we are saying is if the first one holds true, then so does the second one, okay. Again we will use this in the proof.

So bear with me. It is a bit technical. But look at the expressions. Here you had lg, lg lf1 all the way to lg lfk, yeah. And this being 0 is equivalent to we are saying lgh lfgh laddfkgh, okay. So here if you remember the lie brackets is also giving another vector field.

So we are saying you are taking this lie bracket with respect to g, then g and f subsequently and so on is same as saying that these being 0 is same as saying that you take lie bracket with respect to g, add fg and so on and so forth, add fg, add f square g, add f cube g and so on, yeah. So that turns out to be 0. How do we prove this? We prove one key result and that proves all of this. What is that key result? The claim is that lfg is equal to, actually I have written it here, it's not so.

The claim is this. Lie bracket, sorry, the lie derivative with respect to fg is the same as lf lg minus lg lf times h, okay. So we are saying that this holds always, okay. So this is all a play of derivatives, yeah. We are just playing a lot with derivatives. It's notationally seems very complicated and maybe difficult to follow for the first time.

But just look at it. It's just derivatives, reordering derivatives, okay. So all I am saying is if I take the lie derivative with respect to this fg bracket, it's the same as taking lf lg minus lg lf. There is always this commutativity type of, you will see in all of lie derivative ideas. Even the linear system, you know, context you have this, the matrix ab minus ba has a very, a lot of value, okay.

So it's almost like that. It's like ab minus ba, right. Lg lf minus lf lg or sorry, lf lg minus lg lf. How do I prove they are equal? I evaluate both of them and show that they are equal, all right. So I don't do anything very complicated. What is lie derivative with respect to the bracket? I expand the bracket, yeah.

We already know this formula, all right. And now I expand the lie derivative which is what? Del h del x multiplied by this guy. That's it because this thing inside this is actually the lie bracket between f and g. I just defined it, right.

Yeah, okay. Now I evaluate both pieces here, okay. Lf lgh. So this is essentially lf of lgh, right. What is lgh? This guy. Similarly, lg lf is lg of this guy, okay.

After that I am again, this is doing successive derivatives like I said, yeah. What is lf of this? Notice that this is a scalar valued function, right, because del h del x is a row vector, gx is a column vector, product is a scalar. Same here, okay. So I am trying to do lie derivative of a scalar. What do I get? Take partial with respect to x, yeah, multiply by fx.

That's it. Only thing is how to take partial with respect to x for this. I just take product rule, take partial of this guy first and then take partial of this guy, okay. There are just two pieces here, yeah. You just have to make sure that you are consistent with the dimensions because now there are matrices involved.

This became a matrix. This is a matrix, yeah. This is a vector. This is a row vector, okay. So that's it. You just have to be consistent with the dimensions.

Otherwise it is just using the product rule, yeah, okay. So all I did was because I have to take another derivative, right. It is like taking second like a Hessian like you have Jacobian Hessian, right. You take first derivative with respect to state then second derivative with respect to state.

It is almost like that. Similarly, I took two, the second derivative now of this guy. Let's hold this guy just using the product rule and then I multiply by gx, alright. Now what? If you look at these guys, this term and this term will cancel. Just look at this.

Del square H del x square gx times fx. Del square H del x square fx gx, okay. This will cancel, yeah. Believe me. There is matrices and vectors involved, yeah.

But this guy will cancel with this guy, okay. Once you have that cancellation, you can see what is left if I subtract the two. Only this much and that's this, yeah. That's it. So in order to prove that these two are the same, I have done nothing but write the two formula, yeah. Very painful looking bookkeeping but that's all it is.

It's bookkeeping and I cancel, okay. So but this is very cool, right. Looks like some new language we are writing, yeah. It's like I mean if somebody doesn't follow this area, what have you even written? Looks like some morse code, yeah. So it's like Lf gi is Lf lg minus Lg lf, yeah.

Looks simple, right. Nice, okay. Good. But remember we were trying to prove that equality, that Lg lf square and all these being zero means that Lg l out of g, l out of square g, those are also zero, yeah. That very equivalent. But what is the nice thing? We can using this simple idea, we can iteratively prove this, okay. The first thing that we have is that Lgh is zero, okay. We also have Lg lfh is zero, right, by the first two, right.

I know that Lg add of l add of gh is what? Is this guy, what I just wrote here. And that's what Lf lg minus Lg lf times h, right. So I expanded Lf lgh minus Lg lfh. What do I know? I know that Lgh is zero, right.

I also know that Lg lfh is zero. Yeah, because that's what I assumed, right. Yeah. So I have already proved that Lfgh is zero, okay.

Done. Okay. So once I have this nice formula, this entire proof goes through very smoothly, okay. Now if I want to do the second level, I just we are not going to show too many levels. I think until second level. What is the add of square g? It is Lffghx, okay.

Again just painful looking but math is not complicated. Yeah. This is actually a lie bracket. So this is Lf lfg minus Lfglfhx.

Again by using the same formula. Yeah. Yeah. Because you can take do this with any two vector fields, right. I have f and I can think of this thing as g now, the new vector field, right. So I can keep doing this with any vector Lf lfg minus Lfglf, okay. I can keep doing this again and again, right.

Now what? Now I can expand this. This is Ladd of g. This is basically the same as Ladd of g, right. Similarly this is Ladd of g and Lfhx. Now notice that Lfh is already 0 by the first step itself, right.

We already have Lfh is 0. We have already assumed Lfh is 0. Yeah. Now what we are left with is Ladd of g Lfhx, this guy.

Yeah. No. Did I get that right? No, no, no. Sorry, sorry, sorry. I did not get that right. I apologize, I apologize.

We know that Ladd of g hx is 0. We have already proved that. Yeah. I am wondering why do I need the next step because I already have Lfh also to be 0, right. So this term is actually 0. It should be done anyway.

Yeah. I don't think I need this additional step. This additional step is not needed. See because Ladd of g hx is already 0 by this guy and Lfh is already 0 from here. What did I miss? What did I miss? Oh, thank you.

Lgh is 0. Thank you folks. Thank you very much. See one tends to make mistakes like this. So all I have proved is Ladd of g hx is 0.

So this guy goes away. This does not go. Of course Lfh is not 0. Lfh is already 0.

Lfh is already 0. Lfh is already 0. Lfh is already 0. Lfh is already 0. Lfh is already 0. Lfh is already 0.

Lfh is already 0. Lfh is already 0. Of course Lfh is not 0.

LgLfh is 0. Okay. Okay. Thank you. Thank you. Thank you. Alright. So all I am left with is this guy. Okay. What do I do with this guy? I expand this again.

Ladd of g I expand again. Ladd of g is what? LfLg minus LgLf. Right. By the same formula. Yeah. So LfLg minus LgLf multiplied by Lfhx.

Now I am back to where I want things. This is LgLf squared h. This is 0. By this whole thing.

Yeah. So LgLf squared h is 0. Similarly LgLfh is also 0. Again by this guy. Now it is good.

Now we are good. Now this is 0. Okay. So I hope you are able to follow this fun math. Yeah. Just practice it a little bit. What I would recommend is you try to do this.

Write out this proof by hand on your own. Yeah. Write out a few more terms. Yeah. Go to addf cube g. Yeah. Once you do addf cube g I think you will learn because it is painful enough to write this much. Yeah. So but what have we shown essentially now that we have until now we have used that Lgh is 0, LgLfh is 0 and LgLf squared h is 0.

Right. We used it here. So we have used these three are 0 and what have we obtained? We have obtained that Laddf0gh is 0 because Laddf0 is just Laddfg0h is just I did not say this is actually just Lgh because that is what we defined this notation. So that is already 0.

Okay. We will also prove that Laddfg is 0. We also prove that Laddf square g is 0. So we use three equalities from the first one to prove three equalities in the second one. So this should be evident to you that if I go further I will be able to prove that Laddfkgh LfLh is 0 if and only if LgLfk plus Lh is 0.

Okay. This is just pattern. We are using just the pattern. You are just matching the indices. Yeah. Until now we used what? Like I said Lgh0 LgLf is 0 and LgLf square is 0. We used three things.

So 2 plus 1 3 and we proved what Lgh is 0 again. We proved Laddfh is 0 and we proved Laddf2h is 0. Right. So that is essentially what I am generalizing here.

Okay. That is what I am generalizing here. We have proved Laddf here k is 2 until now. Until now what we have proved we have shown with k equal to 2.

Right. And L equal to 0 I guess. Yeah. Yeah. We used k equal to 2 and L equal to 0. And so we have used LgLf square h is equal to 0 until this point we have used. Okay. Does that make sense? Yeah. Okay.

Again what I will strongly recommend is please go back and write out these terms yourself in a notebook. If you write it out it will be clear to you. If you just look at it, it will not be clear to you.

But I am not doing anything complicated. This is the only thing that we needed to prove. Yeah. Once we proved this we keep using this successively again and again to do all this entire proof go through. Okay. But anyway this is enough for the proof. For this lemma this is enough. So basically we can so basically using this idea we can actually prove that Lgh is 0, Laddf h, Laddf gh is 0 and all the way to Laddf k gh equal to 0.

Okay. We can basically go on like this is what I am saying. You can do the same thing again and again. Yeah. You can take another derivative take addf cube g addf to the power 4g.

You can do this all the way. Alright. That is essentially what I am saying. Yeah. Alright. Alright. Fine. Then we have another lemma which is sort of based on this lemma which is essentially saying that if I have a relative degree r for the system. Okay.

By the way in these nodes we have been rather specific. It says the relative degree r at some state x0. Yeah. Usually it is better if you have relative degree r in a set. Yeah. Because otherwise doing feedback linearization only at one point is not going to help you in terms of control.

Right. Because you are not going to operate only at one point. So you typically want to have some at least partial feedback linearization in a set. Okay. Some set at least.

So if you operate in that set you can apply this feedback so that the system looks linear. Alright. Alright. So what are we saying? We are saying that if you have a relative degree r system then these row vectors are linearly independent. What are these vectors? D H is simply partial H with respect to x. Okay. This is just the small d notation because this is H is a scalar valued function.

When I had the matrix G sorry when I had vector fields G and F then I used the capital D notation for the Jacobian. Okay. So again these vectors should look familiar to you. This D H, D L F H and so on and so forth because we have been taking partials of we have been taking these partials.

Right. This comes up in the first derivative of H. Right. This will come up in the second derivative and so on and so forth.

Yeah. You keep going on and on. This will show up in all the derivatives. That is essentially what these vectors are. Okay. It will show up in successive derivatives of the output. Yeah. What we are seeing is that these successive derivatives until the r minus oneth again r being the relative degree these are linearly independent.

Yeah. If you say that your system is relative degree r. Okay. How? Okay. How? How do we claim that? We claim that by doing a matrix multiplication.

Okay. I will just go to the sort of the end. We use this. We want to use this. Yeah. That is rank of product of matrices is equal to the minimum of the two ranks of each of the matrices. This is what we want to use to prove that these are linearly independent vectors.

So what we do is we construct a matrix out of this.

Right. Because these are all row vectors. So I construct this matrix. Right. Out of this.

Right.

So this matrix. Yeah. And I multiply with another matrix. Okay. And I look at the rank of the product. Okay. So let's see. What we are seeing is this product is at this inner product is actually equal to this.

This is not difficult. Just look at this. This dlf ih is basically this guy. Yeah. It is just whatever is this d, d is just replace d by the del del x. Right. So I am just doing del del x of this guy.

And that is sort of multiplied by this. Inner product in our case is just vector multiplication or matrix multiplication.

Right. So if you multiply these two, this is just the lie derivative. Right. Yeah. This is how we have defined lie derivative. Okay. Why? Because this is a scalar valued function. Right. Any lie derivative, lie derivative of a scalar value function gives you a scalar valued function.

Okay. So this is a scalar valued function. I took partial of that function and I multiplied it with a vector field.

That's the lie derivative. Right. With respect to this vector field. So that is what it is. L add f g multiplied by lf. Yeah. The only difference is there are some indices here. Here there is i add f i and here there is j.

That's all. Just to account for the fact that you could have lf square or you could have add f 3 and things like that. That's it. Yeah. But other than that, this notation is the equivalent notation of taking the lie derivative with respect to a vector field. Okay. If I put i and j equal to 0, what will I get? Just to see if you are following. If i and j are 0 in this expression, what will be the left hand side and right hand side? What will this term be? What is the derivative for j equal to 0? If j equal to 0, what is this? What is lf 0 h? Is h.

Lf 0 h is just h. Okay. When I put the power as 0, it is just h. No derivative. A 0th derivative.

Yeah. So lf 0 h is just h and d of h is just del h del x. Okay. Similarly, if i is 0, what happens? What is adf 0 g? g. Nothing happens. Yeah. So that will be basically inner product between del h del x and g. What is that giving you? What is the inner product between del h del x and g or product del h del x times gx? What is that? lgh.

Okay. That's what this is. You can do it for different parts. That's what this expression is. Nothing too complicated. It is just saying that again unfortunately notationally so complicated that you have to wrap your mind around it. That's why I am saying please go and write it out.

Yeah. Please make sure you write this one out by hand in a piece of paper. If you do that, you will follow it. Alright. Alright. So what do we know? Now by relative degree argument, we know that this guy l adf k g lf l h is going to be non zero for k plus l equal to r minus 1 and it is going to be equal to zero for k plus l less than r minus 1.

Here we have used the previous lemma. The lemma 0.1. Yeah. Because the lemma 0.1 says that if you have this happening, you have this happening.

Yeah. And what is this? This is what I have written just now. Just saying that it is using this what I wrote here. Yeah. It is just using this. Yeah. Basically just by using that successive lg lf lg lf square is zero, I will be able to prove that as many l adf g h are also zero. Okay. That is essentially what.

Therefore, we know that lg lf lg lf square is zero until r minus 2, r minus 2th power. And so therefore, I will get similarly zero here.

Yeah. Until r minus 2. And then for r minus 1, I will get non zero. Okay. That is exactly what I have written.

Okay. Until r minus 2, it is zero. Less than r minus 1 means r minus 2 because this is integers. Yeah. Yeah. And this is non zero at r minus 1. Alright. Thank you.