Nonlinear Control Design

Prof. Srikant Sukumar

Systems and Control Engineering

Indian Institute of Technology Bombay

Week 5 : Lecture 27 : La Salle's Invariance Principle: Part 5

We will see the proof of the LaSalle invariance principle, ok. And you will see, basically whenever we do these proofs, one of the things that we learnt much later and this is how we catch people doing bad proofs is that all the elements of the, all the assumptions that you made in your proofs should be used. All the assumptions that you made while stating the theorem must always be used in the proof. If they are not, then you know there is some weakness in your proof, ok. Or you have just made too many assumptions, alright. So this is one of the things always keep in mind that we do this.

I will not be giving complete proof of everything and I am referring to Vidya Sagar for a couple of steps. It is your call, not your call actually, it is an exercise. So you will have to go back and check those. You have to read those proofs, that is one of the exercises.

So we will start and we will see how the assumptions get used, ok. So I hope you remember what was the LaSalle invariance principle. We have done this, we have read this a few times. We have seen all the sets that are being constructed and all that. So that is it.

You have a domain, you have an omega compact invariance. You have an E set which is closed and invariant, sorry which is closed and bounded, which is compact not necessarily invariant and then you have finally M which is invariant and compact again. These are the sets, ok, great. So started with omega compact which implies it is closed and bounded and also that it is invariant which means that if I start in omega then my entire trajectory, by the way this is the notation, complete notation for the trajectory. Vidya Sagar uses this notation, so I have also used this now, not, this is, I have been using the shorthand, this is the notation.

Why this notation I am bringing it back is because initial condition is very important, ok. In reality whenever you write this sets it is a function of the initial condition, ok. Limit sets and all the limit sets, limit points these are defined as a function of the initial condition, ok. Remember this. So this is omega of x0.

So this is just the definition of omega being invariant. Further we have bounded trajectories because omega is bounded, correct. I already said that if you start in omega you are remaining in omega and the entire LaSalle invariant states that you have to start in the omega set, great. So let us denote the limit set by omega bar, ok. We are trying to find the limit set now because what do we know? We know that trajectories go to the limit set by the definition of the limit set, ok.

So we are denoting a limit set omega bar as omega bar x0, ok. I know you might have used this notation also to indicate closure of a set in your analysis but this is not closure of omega set, this is a different set, ok. I am just denoting the limit set as this, ok, just a notation. Now by Vidya Sagar's theorem 5-2-30, this is non-empty closed and bounded, ok. This is an exercise.

Why? You can just have to look at Vidya Sagar's proof and copy paste if you want. Hopefully you will understand it also while copy pasting. But this is the, so what I have said is I have asked you to show the proof of that lemma 5-2-30 for autonomous systems. Vidya Sagar's proofs are all for periodic systems. So you have to convert it to autonomous system, the time invariant system, very easy, should be very easy.

You just have to remove the time, ok. But Vidya Sagar's theorem says that the limit set is non-empty closed and bounded if the trajectories that you started with were bounded, ok. If the system has bounded trajectories then limit set is also non-empty closed and bounded, ok, great. Further you also have two other lemmas, ok, which says that again if your trajectories, solution trajectories are bounded which they are, then you will converge to the limit set and the limit set is invariant. You already had non-empty closed bounded.

You further have invariance and of course this result that you will approach the limit set. This notation I don't know if you follow but this is just the distance metric, ok. We define this, right, distance from a set. For example if you have a circle and a point you find the distance between the circle and a point by drawing the, you know, it will be something like this, right. You have a circle and a point then you will find the distance between them by just drawing the normal, ok.

So this would be the distance. Ok. So this is the distance, this is the way of typically defining distance metric, ok. So you can use any metric. So this is basically distance between a point and a set, ok.

And we claim that as t goes to infinity this distance is zero which means that the trajectories, this is again I have used short hand again, this is the trajectories approach the set for large time, ok. So this is again from Vidya Sagar. So these two results are more like enough for us to prove everything. It says that the limit set is non-empty compact and further you will approach the limit set as time goes to infinity and the limit set is invariant. Now the only thing we have to do is relate the limit set to these sets that we have constructed, E and M, ok.

Because we have until now talked about sets omega, E and M but now we have to discuss the connection between omega bar that is the limit set and E and M, ok. That is our job now. Now it should, it is not too complicated. You already, I hope you already are convinced that omega bar is inside omega, right. I hope you, it is clear to you that omega bar is in omega, right.

Because omega bar is, omega is where you are starting. So omega bar has to be inside omega because all trajectories remain inside omega for all time. Therefore, your limit set cannot go outside. So omega bar has to be inside omega. So this is fine, ok, great.

Now we use a result called the monotone convergence theorem. This I discuss in adaptive control but I have not discussed it here but it basically says that if the, if a function is lower bounded and non-increasing, just like our V function. Function is lower bounded by 0 and it is non-increasing, right. Just by our LaSalle invariance assumptions. Function is lower bounded at 0, function is non-increasing.

Then the function has a limit as t goes to infinity. In fact, 0 is not required. It has a limit as t goes to infinity, ok. Basically function is lower bounded, so there is a lower bound. It is non-increasing which means that wherever it starts, it can either stay constant or go down.

Stay constant, go down. Stay constant, go down, ok. If this happens, then the function has a limit as t goes to infinity. This is called the monotone convergence theorem, ok. And that is this result. So we have used a bunch of results known in mathematics and applied control, ok.

And we proceed, alright. So we don't know what the value of the limit is. It is actually not necessarily 0. I should not say it is 0. It is some constant, right. Limit exists, it is a constant, ok.

Great. Now what do we, what are we going to claim? I will already tell you what we are planning to prove. That omega bar is inside, ok. Not just you constructed these sets, right. I know that omega bar is somewhere inside this. But now I am claiming that it is going to be inside E, ok.

How do I claim that? Look at this. You already know this. Yeah. I also put a nice justification. It says since omega is closed, omega bar is the limit set.

Limit set contains limit points and a closed set contains all its limit points. So this is one of the points you have noted down that you have to memorize, ok. So therefore omega bar has to be inside omega. This is also just nice justification, ok. Now suppose I take some point, limit point inside this, some point, limit point P inside this, ok, arbitrary.

Then by the definition of limit point, there is a time sequence such that you have this convergence, right. This is just the definition of limit point. Yeah, that is, there is a time sequence such that if you keep computing x t1, x t2, x t3 and so on in this time sequence, it is going to go to this point P. That is what it means to be a limit point, ok. Now I use the continuity of this function V, right.

How do I use the continuity of the function V? I will write this as V of limit i. Because V is continuous, because x is continuous, I can move the limits outside, ok. This is again a property of continuous function. If the limit, so I can move the limit outside or wherever I want. I can move it here, move it here and so on and so forth, ok.

Result does not change. So all I have done is if the functions involved are not continuous, this is not ok, ok. Remember this. I can get very wrong results by doing that. So I have moved this outside, ok. And what do I know? That from this result here, limit as t goes to infinity, that is V of P, right.

This is actually V of P, right. Just take the limit here and this is just P inside by the definition here and then V of P is this guy and this limit i goes to infinity V x ti, ok. Just by moving it outside, ok, alright. And this is equal to some constant, right. Because of this result, right. This is actually a constant by monotone convergence theorem.

Now notice P was arbitrary, ok. Remember P was arbitrary, ok. You have taken an arbitrary P and I have basically concluded that V of P is a constant, ok. Now what do I say? So what can I claim because my P was arbitrary? I can claim that for all x in this limit set, this is equal to some constant, right. I have just proved it, right. Because I have taken an arbitrary P, I can take any other P, it doesn't matter.

So for all P in this limit set, V of P is actually a constant and the same constant, same constant. So what have I proved? That V is a constant function in omega bar. In the set omega bar, V is a constant function.

V doesn't change, ok. Is this clear? Ok. Because I took an arbitrary P from that and I used the definition and I used the moving limits inside and outside using continuity of V, I proved that V of P is exactly a constant and the same constant. So V doesn't change in omega bar. So if I take a trajectory starting in this guy, consider a trajectory with initial condition in the limit set, ok.

Yes. Ok, ok. Again, let me go back to this definition. It is ok if we don't complete it today. Yeah.

We will look at half of it later on. No problem. So you are saying P is not infinity, right. I hope you understand. P is just a point in this set, right. It is a finite or infinite set, whatever. Just if you are confused about what this P's are, then keep going back to this example.

All points in this circle are the P's. All points in this set, this circle is the omega bar, ok. So if you have any confusion in this proof, always go back to this. This is the omega bar set, ok. So all P is just some point here, some point here, ok. Now what did I do? I started with some point on that circle for example, alright.

I started with some point on that circle and then I used the definition of the limit point which said that I will converge to that point as time goes to infinity, right. As time goes to infinity, I converge to that point, ok. In this case, I will always remain here, yeah, in this particular case. But in general, I will converge to this point.

That is the definition of the limit point, yeah. It is the definition of this point P, ok. So I have just written the definition. I have not done anything more than write the definition of the limit point here. Now you forget this side, ok.

I know that you can forget this side, do not worry about it. I know that limit i goes to infinity Vx Ti is actually equal to this, yeah. I can move the limits around because of continuity. And once I move it around, you see that as i goes to infinity, Ti goes to infinity. So x of this quantity goes to P. This guy is just P, just from here, just by this definition, right.

x of Ti goes to P as Ti goes to infinity, yes. So this thing whatever I have written is actually V of P. And I have proven what? That V of P, so I have just proven that V of P is actually equal to this guy, correct. But this guy by monotone convergence theorem is just a constant, yeah, by two different results, yeah. This right hand side became V of P for some point in the limit set. And the left hand side became a constant and the same constant, it is not a changing constant because there exists a limit, ok.

And limit cannot be multiple points, one point. As T goes to infinity, this function has to go to this point C. It cannot go to something else, ok. Therefore V of P for all P is actually equal to that same constant, ok. Again for those who understand analysis well enough, for them this is basically somehow saying that subsequent, subsequence convergence and sequence convergence becomes the same in this case, ok. Because this is like saying I have many many limit points but I am, but all of them map to the same limit point in once you put it inside the V function.

All of them map to the same point which is the C, ok. So all this entire set omega bar maps to this same point C, ok. In fact this is like a, sort of like a level set argument, ok. We have not talked about definition of level sets, in fact Vidya Sagar uses that notation also. But that is the idea, ok.

There are this set omega, omega bar could be anything, ok. But the function V maps it to one single point C, ok. All these points. Yeah, subsequence is irrelevant. See notice this, the subsequence is only used to construct this set, ok. What will your typical trajectories look like? I am telling you, it could be like this, could be this, could be this, could be this, ok.

So this is some P1, this is P2, this is P3, this is P4, ok. I have not drawn a very nice picture I can tell you, this is not a unique solution, does not look nice and it is not exactly like this.

But the idea is the subsequences are used only to find these individual points, ok. But one, after that the work of subsequence is done, exactly. You got this entire set and because of monotone convergence theorem, these can only map to one single point and not multiple points.

And this is the magic of this result, ok. So, notice it is ok, we will stop here, but go through this, go through Vidya Sagar's proof also. Yeah, once you see it again, you will see how cool this is. In fact, very powerful that you took one result on the left hand side and then another set of results on the right hand side to conclude that the entire limit set will map to one single point, ok.

And that is pretty cool. It is not very evident in this example, ok. The problem is this is the transient set, limit set, everything because once I start here, I will only be on this circle, ok. And in this case, talking about this point and this point, there is no differentiation. You do not understand, you cannot say that I will converge to this point. Not easy to say that I will converge to this point or this point or this point.

This is where you will need the subsequence idea, ok. So, this is a little bit complicated. It is easier to see in the Van der Pol oscillator type examples, but when you have these continuous limit sets, things are not that easy, ok. Things are not that easy to visualize is what I am saying. That is why we have to do this analysis type proof which is beyond visualization.

You do not visualize anything. It is very algebraic, yeah. Because if I had to visualize this and do this proof, it would be very complicated, ok. That is why I gave you this example. You have the subsequence, it goes here, it goes here, it goes here. This helps you construct the omega bar, ok.

So this is what gives you this sequence that goes here. But what our result on the left hand side is saying is that all of these map to the same point C, ok. And what we will do? We will use this, I am not doing it now, but we will use this to say that V dot is zero on omega bar, right. Because V is constant in omega bar, therefore V dot is zero in omega bar which means omega bar lies inside E, because E is exactly the set where V dot is zero.

Omega bar has to be inside E, ok. So this is what we will do, ok. Alright, we will stop here. Thank you. Thank you.