Nonlinear Control Design

Prof. Srikant Sukumar

Systems and Control Engineering

Indian Institute of Technology Bombay

Week 5 : Lecture 24 : La Salle's Invariance Principle: Part 2

Very complicated statement, yeah, I understand, okay. I have tried to make a picture, yeah, may or may not help you. So what are the ingredients? These ingredients are pretty simple. We are used to this kind of ingredients. V is positive semi-definite, V dot is negative semi-definite for X in some set. What are the sets? First we have the domain D which is marked in red here.

This domain is like I said, the ball of radius R where everything is happening, yeah, my solutions are not even escaping this. Basically D is this outer set. Now we are saying that there is this omega set which is compact and invariant inside the domain. What is compact and invariant? Means closed and bounded, yeah.

So any disc less than equal to any norm ball, all of these are compact invariant sets because they are closed and bounded. These are examples of compact invariant sets in reals, yeah, okay. So closed and bounded is pretty obvious. So omega is such set, such a set. Also it has to be invariant which means if solutions begin inside the set omega, they don't escape the set omega.

So I think again examples are things like this, okay. So one question, do you think this circle that is the set omega, okay, this is obviously invariant, right, this by the equation. Is it compact? Is it closed and bounded? Boundedness is obvious I hope because x1 square plus x2 square is C. I am asking you. So boundedness obvious I hope because x1 square plus x2 square is C.

So both neither x1 nor x2 can go to infinity. So boundedness is obvious. Compact, do you, sorry, closed, closed, is it closed? Is it closed? Compact means closed plus bounded. So bounded is done. What is, is it closed? Is this a closed set? Is the circle a closed set? How do you define a closed set? Contains all its limit points or supremumism, whatever, fine.

Is it a closed set? Then that is only telling you it is not an open set. Does not say it is a closed set. How do you prove anything is a closed set? How do you prove any set is a closed set? We never prove anything is a closed set. We only know how to prove open sets. Then what do I do now? Its complement should be open.

Is the complement of the circle open? Then R2, in all this big space, complement of the circle is everything meaning the circle out, inside and outside. Is it open? Yes, very simple because I can take any point here, I will make my screen very big and I will draw a circle. That is not going to hit this. I will draw a set is not hitting the circle. Give me another point here.

I cannot make my screen any further. I can draw another open. Complement is open. This is how you prove. You never prove anything is closed.

You prove that the complement is open. The other way if you, that is also very obvious is that if you look at this, this is actually the set F inverse C where F is or V inverse of C where V is this. V inverse of C. V is a continuous function. A single point set C containing only C is a closed set.

Any set with only limited number of finite number of points is a closed set. So V inverse of C is closed by openness, by continuity. That is the other argument. That is a more complicated argument if you are not used to. But it is actually not a complicated argument.

Continuous set, image is single point. It is a closed set. Any single point set is a closed set. So V inverse of C has to be closed. So circle is a closed set.

So it is a compact and invariant set. It is the kind of set we want for LaSalle invariance. So we have a closed and bounded set omega. Now inside it there are more sets. What is the first one? The first one is the set E which is basically all the points in omega where V dot is exactly zero.

So that is probably something like this. The important thing to notice is that omega is an invariant set. Sure. Okay. We have already assumed it.

And inside omega there is E. But E is not necessarily an invariant set. I hope that is obvious. You cannot make an arbitrary set inside an invariant set and say that is also invariant. For example, in this circle I cannot say that this is invariant.

This piece. I cannot say this piece is invariant. Obviously. Because I could be moving around in that circle. I will get out of the set very easily. So just because I have an invariant set does not mean every piece of that invariant set is an invariant set.

Okay. So E is not an invariant set. More often than not it will not be an invariant set. Okay. So what do we do? LaSalle invariants just work with invariant sets. Loves to hence the name LaSalle invariants principle.

Okay. It finds the largest invariant set inside that set E. Okay. And the claim is that if V is greater than equal to 0 and V dot is less than equal to 0 then you will converge to this invariant set M. Okay.

This largest invariant set M. You will. Okay. Obviously there must be a reason why such a claim is being made and we will look at the proof which I promise you is complicated. But we will look at the proof. These are the only few proofs like this we do.

That is the end. After that no more proof. Yeah. After that it is results and examples and design and so on. So only three proofs actually.

So alright. So we will look at this proof because non-linear folks will feel I do not give that proof in a non-linear control course. No? Okay. Alright.

Great. So these are the sets. Pretty straight forward. If you start inside omega notice I am not allowed to start anywhere in the domain. I am starting inside the omega set because the point is Lasal wants to restrict all the trajectories inside that invariant set. So you start in that invariant set so obviously you are not roaming about in this random domain. You are within this invariant set and then you construct the E and the M.

So you are saying that I move from the larger invariant set to the smaller invariant set. Okay. If V is greater than equal to 0 and V dot is less than equal to 0. So without the use of V upon of candidates even I have a pretty solid thing going.

Okay. Pretty strong result is what I would say. So before going forward what I would like to do is directly go to this example once. Let us see.

This okay. Let us look at this system. Okay. Now this is not done very properly here. So I will write a few things. What is this? This is the pendulum. Right? This is the standard non-linear pendulum not the modified one that I gave.

This is the standard non-linear pendulum. And if you look at the analysis it is pretty straightforward. K is the, K is your whatever I mean. This is just a normalization. Nothing more.

K contains the length and all that. C is the damping factor on the joint. And so if you look at it this is the again the kinetic energy because it contains the theta dot and this is the potential energy. We actually saw this. And if you actually compute V dot, so V is actually positive definite in an open ball. But I am going to say it is only greater than equal to zero.

I am going to invoke the LaSalle invariance not the specialized results, not this. So I know it is greater than equal to zero. I hope you believe me it is semi-definite for all x1, x2. So I am going to say for x1 in, I am trying to see what the domain should be.

So x2 in R is fine. But in x1 what do I want to say? If I say zero to 2 pi I have a problem because I am missing this guy. Then what do I say? Minus pi to pi will contain this guy but it will not contain the minus pi. No, minus pi to pi will not contain the top point. How do I make sure I have both of them? I will have to take say minus 2 pi to 2 pi. This will make sure I have the bottom one also and the top one also.

Zero and pi both, both equilibrium. Because I want to make limit set. So I want to sort of have some kind of a limiting behavior.

So let me see. Let me see what happens. Okay. Right, right. Okay, okay. And x2 can be in R. This is conservatively the domain.

This whatever I have stated now is the D set. Because this is not, can this be the omega set also? I guess this can be the omega set also. But the problem is it is, it would have been okay but it is not a bounded set. So it cannot be the omega set. It can be the D set. But if I want something bounded then I have to choose some bound here.

Alright, alright. How we typically choose the bound, let me be honest, how do we create the omega? There is a certain trick to it. Is we know that by the fact that V dot is going to be less than equal to zero. We know that V is non-increasing.

V of x t is less than equal to V x zero. Okay. And suppose I call, say this is some constant C.

Okay. Okay. So this is some constant C. Alright. Now remember also, well I have only semi-definiteness, positive semi-definiteness. But suppose V is by positive definiteness I can of course do a few things. But let me see, let me see. So what I will typically do is I will say that my omega is the set of x1, x2, I am going to make this bigger, is this set of x1, x2 in the domain such that V of, yeah fine, I will just say x in the domain such that V x, okay, such that V x is less than equal to C.

So let me use the x zero. Okay. Actually I should not use this. I know this is getting a bit complicated. But just try to follow my argument. It is very important when applying LaSalle invariance to be able to define these sets very carefully.

Okay. That is why I am putting a little bit of effort. It is obvious that this domain is fine. I hope you are okay. Yeah. Because my only aim with the domain was to be able to include both the equilibrium. Because I want to use LaSalle invariance so there has to be multiple limit points.

Yeah. If I just take the bottom one and use minus pi to pi, I do not even have the top guy. I am not so intrigued by it. Yeah. And LaSalle invariance does not do anything special.

And x2 can be anything in R. Now my problem is that this domain is not bounded. Therefore, I cannot use that as an omega. But I know that V is less than equal to C for all time.

Because from my V x zero. Okay. From my V x zero. So if I define my set as this guy, then this is an invariant set. I hope that is evident to you. Yeah. Because if I start anywhere in omega, V x will be less than equal to C. And if V x is less than equal to C, then it will remain less than equal to C for all time.

Therefore, I remain in the set omega. Because set omega is defined using this. Okay. So if I take any point in this set, V will be less than equal to C. And as I propagate it through the dynamical equations, it will remain less than equal to C, which means I am still within this set.

Okay. And by the way, such an omega set exists, very simple, because for positive definite functions, you have class K function dominated and so on and so forth. Very easy to construct. In this exact specific case, I just, this is, I just have to solve this guy.

Okay. That is very easy. I will give you a conservative estimate. I just want x2 from here, from here, I just want x2 to be what? Can anybody tell me? If I need to satisfy this, what should be the bound on x2? What is the largest value x2 can take? Under root 2C. Absolutely. Why? Because forget the contribution of this term. This is whatever, this is non-negative.

It will be maximum 2K, minimum whatever, 0. Minimum 0, maximum 2K. So take the minimum value 0 and then x2 has to be less than under root 2C. Okay. Because x2 square by 2 is less than equal to C. Done. So I got my, and by the way, this is an absolute value, because there was a square, so when I took the square root, it is an absolute value, which means x2 has to be between minus root 2C and plus root 2C.

Done. My omega set is very straight forward in this case then. Omega is exactly equal to minus 2 pi comma 2 pi cross minus root 2C root 2C.

Yeah. Very easy. I just used the same logic. So I have constructed my omega. Now the, whatever, the directional derivative is too simple. You already have done this before. If I take the derivative of this, I will get x2 x2 dot and K sin x1 x1 dot. Basically this term and that term will cancel out and I will be left with V dot is just minus C x2 square.

Okay. So V dot is negative semi-definite. Right. Why is it negative semi-definite? It can be zero whenever, you know, only x2 is zero, x1 is arbitrary. So basically whenever all states do not appear in any function, cannot be definite. It is only negative semi-definite.

Okay. So it satisfies the two requirements of the LaSalle invariance. V is positive semi-definite, V dot is negative semi-definite. Excellent. I have constructed the invariant set omega in which I am starting.

That is also done. Now I have to construct the set E. What is the set E? It is the set of states in omega such that V dot is exactly zero. What is that set? In this case, where is V dot exactly zero? When x2 equal to zero. So this is actually all the points of the form alpha comma zero. And alpha of course, I have to be careful because I am still in the invariant set.

So alpha belongs to minus 2 pi. Yeah, because I am still in the invariant set omega. Obviously alpha still belongs to minus 2 pi 2 pi. Now I want to find the largest invariant set M inside me.

This is the complicated part. Okay. And I do not think I have enough time to do that now. We will do this in the next class. Okay. But we will continue with this example in the next class. Of course, I will recap a little bit. But I hope you have seen how we start by constructing an omega which of course has to be closed and bounded, can contain multiple equilibria.

That is what we have. And we already have shown V is positive semidefinite, V dot is negative semidefinite. Okay. Remember the V chosen was very simple. Energy of the system which like I said dynamical systems folks absolutely love.

They do not like to hunt for more complicated Lyapunov functions. You know backstepping type Lyapunov functions will be a completely no-no for them because the typical question would be your Lyapunov function adds position and velocity. What does it even mean in the real world? How can you add position and velocity? So there are many points there. So anyway remember that we have more or less constructed lot of the ingredients.

The only quantity left to compute is M. Okay. Starting invariant set done, ending invariant set is the only thing we need to compute. Okay. Alright. We will do that in the next class. Okay. Great. Thank you. Thank you.