Nonlinear Control Design

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Week 5 : Lecture 23 : La Salle's Invariance Principle: Part 1

Okay, so one of the reasons for, not one of the reasons, the reason for not including this term in trying to compute delta is of course it would have made the mathematics messier but the most important thing is that if I had included this term in trying to compute delta, my delta would potentially depend on capital T because this term depends on capital T. So if I try to compute delta using this term, it would, delta would almost certainly depend on capital T and that is just not allowed, okay. Because remember, first came delta, okay. There is no T here when we started talking about delta, okay. So we cannot have dependence on T, okay. So that is sort of critical to remember.

That is why we got rid of this term just by, yeah, I mean just by saying that you are always going to go, become smaller in some sense, okay, great. So now we have been able to choose a delta. The important thing to remember is that there are a few elements here. So delta does depend on T0 and R because of V is dependence on T0 and of course R is on the right hand side.

Now you can almost never get rid of the dependence on R but as you can imagine for uniform convergence or uniform attractivity, you will need this dependence to go away. For uniform results, you will need this guy to go away. That is pretty much the key aspect here. Excellent. Now that we have chosen a delta and we figured out that it depends on T0 and R but independent of whatever, independent of epsilon, independent of T which it has to be, now we go on to find our capital T.

So we just, then we use this small idea, okay. What was the idea? We already know that this is upper bounded by gamma R. The term inside this is upper bounded by gamma of R, okay, just because of this guy, okay. That is what we will use, okay. So once you have chosen a delta, you have V T0 X0 which is on the left hand side here is smaller than phi epsilon 1 plus gamma of R goes out and the integral is just capital T, right, integral is just capital T.

So what do I have? I just have V T0 X0 is less than phi epsilon 1 plus capital T times gamma R. Of course, we always already stated by choice of delta, norm of X is less than R whenever norm X is less than delta, okay. I hope that is clear cue, yeah, from here, okay, alright, great. So once I have this kind of an expression, it is pretty straight forward. I need, if you just compute T from here, you just get this expression.

And T has to be greater than this guy, okay. And it is very natural for you to get a expression with T greater than, right, because any T larger than that is allowed. Your conditions are not violated for any time greater than this capital T. So anything beyond this time is okay, okay. And you can see this time depends on epsilon, right, initial time, state, whatever, so on and so forth, yeah, it depends on a lot of things, okay.

Now, if you want to see dependence on delta directly, you can of course use the fact that this is greater than phi norm X0 squared, sorry, phi of norm X0, not norm X0 squared. So this is greater than phi of norm X0, just by positive definiteness. So that will also bring in some kind of delta dependence as you can see, yeah, because phi of norm X0, so norm X0 is less than delta, you will not get an exact result inequality like that, but you can see that there is a dependence on delta also, just by virtue of this guy, just by virtue of this guy. Because the way we have chosen delta is exactly this, yeah, so there will be some kind of connection with delta here, alright, okay, great. Anyway, so there is epsilon and R, alright.

So like I said, in order to prove, I mean the second exercise of course to prove uniform asymptotic stability, yeah, so for that the first exercise is anyway useful because you proved uniform stability, to prove uniform attract, sorry, asymptotic stability you have to prove uniform attractivity. So like I said, this dependence on T0 has to go, okay, the delta dependence on T0 has to go. So therefore you have to think how you get rid of this dependence, okay, what is the additional assumption that you use here, alright, so that is the idea, okay, good. So these are the only two theorems that we prove, okay, but you can see that the other ones will also be very very closely connected to this, yeah. If I wanted to, as you can imagine if I wanted to go to the global version, I don't need, you know, I don't have any ball of radius R, okay, so for all the global versions of the result, the ball of radius R business goes away, okay, so for all the uniformity the T0 dependence goes away, exponential stability is the only thing that I have not actually even approached, okay.

That will require a little bit more, not this kind of analysis that requires more because you have to, like I said you have to use the same order of magnitude ideas and so on and so forth, yeah. So that I am not going into, yeah, you can check out the proof in textbooks like, you know, Khalil and Vidya Sagar and so on, okay, you will have some proof there, okay, if you are interested, alright, great. If there is nothing, no questions here, then I will move on to our next set of lectures which are on LaSalle invariance, yeah, so anyway this is again notes derived by some students in the past, so what we want to do is, we want to discuss the LaSalle invariance principle today. Now what is the motivation? I will simply state it, so motivation for several asymptotically stable systems, V is C1 V positive definite but V dot comes out to be negative semi-definite, okay, this happens a lot, especially for all the dynamic systems specialists, yeah, because typically we are, I count myself more as a control, non-linear control, right, so for me I put some effort into choosing V which is not necessarily the energy of the system, okay, so a large part of my work is actually choosing good functions to work with, yeah, so we typically don't choose the energy of the system as the V, we may start there but then we modify it, okay, but typical dynamical systems guys they want to analyse the system just with the energy, okay, and when you do that in a lot of cases this will happen, that you will have a positive definite and continuous V which is obvious because it is the energy of the system or the Hamiltonian or whatever, I mean you can use the Hamiltonian, you can use the energy, all of these are energy like quantities but V dot will invariably come out to be zero or negative semi-definite, you will never get a negative definite V dot, yeah, this happens a lot and then but you know again like the pendulum case, yeah, you know, I mean I modified the system, of course I played with the system but you know very well that the pendulum, the system itself is asymptotically stable, just because you use the energy of the system and the V dot turned out to be zero does not mean that the system is not asymptotically stable, it is, you can see, yeah, so obviously we need some more mechanism to prove asymptotic stability in such cases, okay, this was the first motivation. The other motivation was, anyway this is the first motivation, the second one is there are systems limit cycle behaviour, yeah, one of the obvious systems is the Van der Waal oscillator, we have already seen that in an assignment I hope, yeah, so you, the trajectories tend to converge to, you know, this closed bounded set, yeah, I mean another example is just this oscillator, here it is a bit more dull than the Van der Waal oscillator, because in the Van der Waal oscillator, I do not remember it is just, whatever it has something like this, I am making it very badly, like this I think, yeah, so trajectories actually converge to this, do this, this is a bit more dull because wherever it starts it just continues in that circle, it does not converge to, it starts converged, yeah, there is nothing to converge to, it just keeps circling like this, this is the standard linear oscillator x1 dot is x2, x2 dot is minus x1.

But then the Van der Waal oscillator which is a little bit more non-linear oscillator tends to do, have behaviour like this and such systems are very very important, I already told you, non-linear oscillators are very critical and are used in lot of bio-rhythm applications, ok, so obviously we want to study such systems. So in, for such systems also we want to have a little bit more general definition of convergence, ok, we want to have more general definition of convergence, not just something as basic as going to a point, we may want something more, in fact there are more modern problems in controls like you have the, say you have the platooning problem, do you know what is platooning? So it is a very modern, you know, I would say transportation theory idea in which basic idea is there are lot of these transportation trucks, yeah, that are carrying a lot of good logistics and it so turns out that if they continue to move in a straight line with uniform distance between each other, that is optimal in the sense of fuel efficiency and so on and so forth, ok. So this is called a platooning problem, however you start, you maintain this straight line formation with uniform distance between these vehicles, very, sounds very straight forward, but of course here the point is there may be traffic on the street and whatever, I mean you may have to change lanes, so once you do a lane change manoeuvre you have to redo the platooning and so on and so forth, right. So these platooning things are automated in some way, so this is actually a formation, a specific example of a formation control problem, ok. Another formation could be you have a bunch of defence vehicles going to, you know, carrying whatever supplies, arms, ammunition, whatever, yeah and you want to guard them, yeah and there is lot of applications now where you have aerial drones which are circling around them, ok and say they are circling around, so you have a bunch of, you know, bunch of drones which are circling this, you know, armoured trucks, yeah and how do you do that? This is also then you are sort of trying to converge to a, you know, circular pattern in some sense, right and this is also convergence to it because obviously you cannot start in the circular pattern and even if you started in the circular pattern and you exactly positioned them, you let them off in this very nice circular pattern around them but the vehicles, armoured vehicles are moving, right and they are doing whatever they are doing depending on road conditions especially in India, right and in border areas, right there may be no roads, yeah.

So they are doing whatever they are doing, so but these guys have to keep, you know, converging again to another formation, right. So every time the formation requirement changes, centre changes and so on and so forth, so obviously they have to reconverge. So this is also a sort of limit cycle behaviour you are looking at. If you want to study it in that sense, many people don't, most people don't study it in that sense, they think of it as a coordination problem but most coordination problems can lead you, formation problems can lead you to limit cycle type behaviours, ok. This is very important set of results actually, Lassare invariance, alright.

Excellent, after this mighty introduction, we will look at the systems that we are interested. These results work for non-linear autonomous systems only, the ones I am stating, yeah. There are extensions, it is a topic of great research of trying to do this for non-autonomous systems also, yeah and there are results there and you can look at it but it is still an active topic of research, right. It is not like, it is not textbook material yet, yeah and most of what we do here is textbook material, ok. So we start with, as usual a nice continuous Lipschitz, locally Lipschitz continuous vector field with some initial conditions, we succinctly denote the solutions as X of t, right, we have been doing that, ok and we make some definitions, ok.

So remember, we start with autonomous systems, yeah, that is the main thing to remember, alright. First we define what is an invariant set. What is an invariant set? A set omega is said to be invariant, if you start in this set, you remain in this set for all time, ok, that is what is an invariant set. If you start in the set, you remain in the set, ok, great. So I mean in this case, it is pretty obvious, I hope it is obvious that for system like this, what is the invariant set? It is a circle, yeah, of whatever radius you started with.

If you started with a circle of radius square root of C, you remain in that circle of radius square root of C, yeah. Obviously, we are not discussing cases where there are disturbances and errors in sensing and all that stuff, yeah, this is the precise case, yeah, this is the ideal case if you mind, ok, great. So, omega is this, ok, it is a set of all points in the circle, this is how you write, yeah, omega is exactly this, yeah. Notice it is just the circle, not inside the circle, not the disc, it is just the circle, ok. So this is what is a typical limit cycle, yeah, if you are inside a disc that is not a limit cycle, that is just stability, ok, that is what we do.

If you are within, if you are given an epsilon, then you start within delta, you remain within epsilon that is not, nothing to do with limit cycle or anything, it is, you are in a disc, yeah, here you have a circle, that is a limit cycle behavior, ok, remember that, ok. So, limit point, what is a limit point? A point P is said to be a limit point of this function x of t, again whenever I write x of t, it is the solution of this equation. So a point is said to be a limit point of x of t, if there exists a time sequence, a sequence of time such that as tn goes to infinity, such that tn goes to infinity as n goes to infinity and x of tn goes to this point, whatever point we are denoting as a limit point, ok. So basically there is a time sequence such that if you keep writing terms x t1, x t2, x t3, x t4, x tk, but the important thing is this tk has to go to infinity as k goes to infinity. Remember, whenever I say sequence, it is an infinite, it is infinite size, not finite, ok.

So as k goes to infinity, tk has to go to infinity. The basic idea is it is limit point, I mean we want to look at behavior asymptotically. Therefore, as k goes to infinity, tk has to go to infinity and if in that case x tk converges to some point P, then it is a limit point, ok. A very simple example I have constructed just to illustrate how things are different. If you look at this series or sequence, sorry not series, yeah, it is half 1, half 1, half 1 and so on and so forth.

So if I take the time sequence and I have denoted this as 0, 1, 2, t0, t1, t2 and so on, if I take the time sequence t0, t2, t4, ok, where, what is the limit point? Correct. If I take t1, t3, t5, t7, 1, ok. So remember, any one function or a sequence can have multiple limit points, not just one. This and this is what constitutes a limit set, ok. This is what constitutes a limit set.

You can have multiple limit points and the set of all those limit points is the limit set, ok. Just like half, the set containing half and 1 is the limit set in this case. And if this limit set is a cycle, ok, and then the question is how do you define a cycle? You can define it mathematically. I do not want to get into the mathematical definition, but think like a circle. It is somehow closed in some sense, closed, compact, those are the requirements.

So if the limit set is a cycle, then it is a limit cycle. Then you have a limit cycle behavior. Then Van der Pol oscillator, ok. That weird looking set is a limit set. It is a limit cycle because it has, it is a cycle here, it has a cyclical thing and there is a periodicity here.

Basically it has periodicity, ok, great. I hope those points are clear. Once those points are clear to us, we can state the most general form of the LaSalle's invariance principle. So are these three definitions relatively clear to you? Invariant set, limit point, limit set and limit cycle is just an extension of limit set, ok, ok, great. Let omega subset of D subset of Rn be compact, that is closed and bounded.

In the case of reals, compact and closed and bounded are equivalent and invariant. What is D in this case? D is the domain, just like your ball of radius R, right. D is that ball of radius R type of thing. This is the domain in which you are working. So you want the existence of an omega which is compact and invariant inside this domain.

Let V mapping B to R be a C1 function, ok, such that Vx is greater than equal to 0. Yeah, look at the interesting things that are already happening. We are denoting it as V, again scalar valued C1 function, looks like a Lyapunov candidate at least, but it is not necessarily because you only require semi-definiteness. Positive definiteness not required, so it is not a Lyapunov candidate, ok, does not have to be. And further, you want that V dot is negative semi-definite in this compact invariant set omega, ok.

Then you define E as the set of points in omega such that V dot is exactly 0. Yeah, it is evident that V dot is 0 somewhere, otherwise no need of saying V is only semi-definite, right. So there are points other than the origin inside omega such that V dot is exactly 0, ok. And so E essentially captures those points where V dot is exactly 0, yeah. Of course, origin is also there, but there are potentially points beyond the origin because V is semi-definite.

V dot is, sorry, yeah, V dot is semi-definite. And let M be the largest invariant set inside this E, ok. So lot of technical terms coming up now, ok. We will try to clarify this, ok.

So we have used many sets, yeah. So we define E as the set of points where V dot is 0, we define M as the largest invariant set inside E. Then we can claim that if your initial conditions start in this omega, then as T goes to infinity, your solutions will lie in this largest invariant set M, ok. And M is obviously a positive limit set, ok. Not necessarily a cycle, ok, but it is a positive limit set. So the immediate obvious thing is the LaSalle invariance principle does not require Lyapunov candidates.

Let us look at the obvious things, then we look at the more non-obvious things, ok. Does not require Lyapunov candidates because it is starting with V only greater than equal to 0. The other thing, it gives a, actually seems to give a more general result than Lyapunov theorems, right. Because it talks about convergence to positive limit set, yeah. It basically at the end of a LaSalle invariance analysis, you could potentially say that my solutions go to say if the limit set contains 10 points or 10 equilibria, yeah.

Then you can actually say that the solutions can go to any one of these 10 equilibria, ok. So that is stronger than, not stronger but more general than saying it goes to this one point, ok. It is a more, so obviously if it is a limit cycle, then LaSalle invariance actually gives you a way of saying that you converge to this limit cycle, ok. There are hardly any other results which will talk about how you go to limit set. I mean there is also the Poincare's theorem but there are very few results which tell you that you will actually converge to a limit cycle or a limit set because or converge to a set for that matter because until now we are only talking about converging to a point, yeah.

Even we have not even said anything about convergence to a set, ok. In fact, if you guys notice this notation is in itself not very obvious. If M is a finite set of points, suppose M is a finite set of points, then this is ok. You are just saying that as N, time goes to infinity, your states go to one of these points.

So you can analyze any all the points. But if M is, suppose M is a continuous set like a circle, then how do you even, I mean this is not a very simple notion to understand, ok. When I use this terminology, this terminology is difficult. All your, in a sense what you are saying, actually I would put it this way. If any of you knows this notation, ok, this would be the more precise notation that we are used to following. Do you know, sorry, do you understand this norm of X t subscript M? What is an M norm? You mean like L0, L1, L2, M norm? No, no, it is not that.

Although I have used similar subscript notation, I agree, but it is not that. This notation is, or maybe I know there is an overloaded operator here, but this notation is also used for, and if it is not a number and a symbol or a letter, it is definitely used for norm with respect to set or distance from a set. Ok, this is a distance from a set. This is defined with an infimum.

This will be defined as inf over all X0 in M norm of X. Ok, it is basically telling you shortest distance from the set. Yeah, so this is actually a set norm because we are, until now we are used to measuring. So all of this is nice and easy to do in Rn. So yeah, we can do it. In more complicated topological spaces, well I will say it is their problem.

Whoever is working with that, they have to figure out all this. So this is actually a more special norm. It tells you distance not from a point like origin or something, it tells you distance from a set. So that is rather interesting. Thank you.