Nonlinear Control Design

Prof. Srikant Sukumar

Systems and Control Engineering

Indian Institute of Technology Bombay

Week 4 : Lecture 22 : Proofs of Lyapunov Stability Theorems- Part 4

Welcome to yet another class on Control of Non-linear Dynamical Systems, alright. So, we are now, well this is lecture 6.7, we are in I believe the fourth week, yeah and we have already seen all the stability theorems, different variants, we have seen some examples. I believe you already, did you already have your tutorial or is it going to happen this coming Saturday? Okay, alright. So, you will also have your first tutorials where we will look at you know all the well hopefully you will see some examples on how to use the stability theorems, alright. And in the last set of lectures, we saw the proof of the stability theorem, okay.

We of course, we did the proof of one specific theorem, alright. This is the most basic theorem on Lyapunov stability. But you saw that it still gets relatively involved technically, okay. So, we need to make some interesting arguments.

So, we went through multiple steps. So, in the first stage, we, the dominating function that is the class K function was taken to be some simple structure, right. And based on that given an epsilon, we chose a delta, right. And in all cases, the choice of delta always had this kind of a structure, okay, always had this kind of a structure, right. And then we went on to the more general case and here also things were not significantly different, alright.

So, so the way we choose delta was again very similar. Instead of the alpha epsilon 1 square, we just plug in the epsilon 1 into the class K function itself, alright. So, of course, there were some technicalities here and there. I mean, started with an open set and you took inverse of the open set using a continuous function. So, you got another open set.

So, all of those things, some of which I illustrated with pictures like this were also involved, okay. So, it was a game of, I mean, this epsilon delta business is completely a game of finding these open sets, alright. So, that is what we were trying to do because we do not have a structure for V itself, nor do we have the structure for the differential equation, yeah. But you see the power of, I hope you sort of get a feel for the power of the method that this theorem is very qualitative, right. It does not rely on what is your system, what is your Lyapunov function, it works, okay, irrespective of what is your function, what is your Lyapunov function and so on and so forth.

You can get a stability result and this is how the proof will go, okay. Now, of course, I wanted you to complete the uniform stability proof. The homework has already been posted, yeah. I believe one homework is already due, okay. So, anyway, so this is how we proceed.

We will continue to do some homeworks. In between, there will be due homeworks and things like that. So, please go on and continue to do this because the homeworks are relatively small extensions of whatever we are doing in class. So, honestly, if you are sitting in class, the homework should not take you more than 15 minutes, like each problem should not take you more than 10-15 minutes, okay. So, if you are following most things in class, homework should be relatively easy to do, okay.

So, I hope that you are focusing on doing your homeworks. A large part of the grade is, in fact, in the homeworks, okay. Great. So, where do we, what do we want to do now? We stated the stability theorem and we proved the stability theorem, right. Anyway, we stated all the theorems.

So, we proved the stability theorem in the last class. Now, we want to prove the exponential stability, sorry, asymptotic stability theorem. Exponential stability is not a easy target to achieve for non-linear system. So, we are not going to worry too much about it, okay. We want to look at the asymptotic stability theorem.

So, what does it say? I have not written the entire statement. I have just said in addition, what is in addition? In addition to this, that is, you have V which is positive definite in some domain and for all time and it is C1 and further it is negative semi-definite for all X and Br, then the equilibrium is stable. In addition to that, if you also have that V dot is not just semi-definite but in fact negative definite in that same domain for all time greater than initial time, then you have asymptotic stability, okay. So, this is what now we want to prove the next step in some sense. So, because we have already proved stability, yeah, the only thing we need to prove is attractivity.

So, remember that asymptotic stability is just a combination of stability and attractivity. If you add uniformity to stability, uniformity to attractivity, you get uniform asymptotic stability and so on and so forth, right. So, we are already know the terminology pretty well. So, once you know stability and attractivity, we know that we can make all the other definitions also, okay.

Excellent. Alright. So, if that is clear, remember I am only interested in proving attractivity, okay. Now, I am going to state attractivity in a slightly funny way. So, remember when I needed to prove stability, I first defined, I mean in this aside, if you notice I wrote what is stability, just to remind ourselves because this is what I am trying to prove. Similarly, for attractivity, this is where I state attractivity, okay.

It is stated a little bit differently. If you remember, attractivity was stated as there exists a delta which could potentially depend on initial time, such that if you start within the delta ball, as limit t goes to infinity, you go to 0, okay, equilibrium. So, we assume 0. So, we go to 0.

This was attractivity. That was a limit based definition. Now, I am not defining it using the limit symbol, but I am writing the equivalent thing here. How do I say it? Attractivity is stated as there exists a delta which could be potentially depending on initial time, such that if you are given an epsilon, okay, now there is an epsilon also. In the earlier definition, in the limit based definition, there was no epsilon. See, if I am given an epsilon, there exists a time which depends on epsilon and t0, such that your norm of the vector becomes less than epsilon for time larger than t0 plus t, okay.

So, I hope you notice that this is like a convergence. I hope you understand that this is sort of like a convergence. Why? You are, what am I saying? I am saying that, so in all these definitions, by the way, you may not yet be very comfortable converting words to mathematical definitions like this, but in all these definitions, the sequence in which things appear are very important, very important. In the stability definition, there exists, sorry, given an epsilon, there exists delta. First came epsilon, then came delta, alright.

I am not preaching here, but okay, first came epsilon, then came delta. It is not chicken and egg. There is clear certainty here. Well, even in the case of chicken and egg, there is, but okay, we are not going there, okay. But first came epsilon, then came delta, alright.

In this case, first came delta, and then I am saying, given an epsilon, there exists a t, okay. So, the sequence is very important. What am I saying? All I am saying is, all of this mess, all of this is just saying limit as t goes to infinity is zero. It is just this. Whatever is highlighted here is just stating this much, okay.

Why am I saying it in this messy way? You might ask, yeah, you should ask, because that is what I can prove. Because unless I define what is limit, I cannot prove anything. So, this is the definition of the limit. If you say anything, limit is actually convergence, right.

I hope you understand. Limit as t goes to infinity, something going to, actually convergence, okay. In fact, limit as t goes to anything is a convergence result on functions, yeah. Just, it is just, you had sequences which were discrete points. If you move from sequence to functions, you say as t goes to infinity, x of t goes to some value. So, all this goes to, goes to, means convergence, okay.

So, it looks exactly like convergence. If you forget the first part, you forget all this. There exists delta and starting in delta ball and all that, okay. There exists delta such that you start in delta, all this you forget. If I am given an epsilon, I am saying that there exists a time large enough such that beyond that time, my norm x is less than epsilon, okay.

So, whatever epsilon you give me, you can give me say 1, start, suppose you give me epsilon equal to 1, I can give you a time cap T such that my norm of the vector is less than 1 for all time beyond that. If you give me epsilon is half, I can give you another capital T such that the norm of the vector remains below half for all time beyond this t plus capital, t0 plus capital T, okay. This is exactly convergence, okay. Basically, it means that as you keep increasing time, you are moving closer, okay. You are moving closer to the desired point.

In this case, it is 0, okay. This is precisely stating convergence, okay. The only thing is it is keeping it local by saying that there exists delta such that if all this happens and of course if initial conditions start in delta, then all this happens, okay. Otherwise, no. If your initial conditions do not start in the delta ball, nothing is guaranteed. This delta is actually then called in nonlinear systems as basin of attraction.

A lot of you might have heard this term. It is typical to figure this out. Excellent, excellent, okay. Alright, great. Once we have understood this alternate definition for attractivity, let us try to prove this, alright.

Because I would never have been able to prove anything with the limit definition, okay. It is as simple as that. So the assumption we have, it means that V dot is upper bounded by negative of a class K function, okay. I hope this is negative definiteness. Positive definiteness would mean that V dot is lower bounded by a class K function but negative definite means V dot is lower upper bounded by the negative of a class K function.

Just a flipping of signs, that is all. Because V dot negative definite just means minus V dot is positive definite. If you use that logic, you get this, alright. Okay. And we already have this guy that V is positive definite, okay. So inside X is, whenever X is within the R ball, then you have this assumption to be satisfied.

Excellent. Then now the point is how do I prove this? Basically you have to find a delta such that all of this nice thing happens. This convergence happens. Okay, great. How do I prove this? This is saying something about the solution, right. Whenever I write X of t, remember that I am writing the solution itself.

This is a notational simplicity I have assumed. Many books do not use the same notation for solution and the state. But I like to keep life simple. So this is sort of the solution, right. So I want to be able to say something about the solution using the Lyapunov function.

Okay. And that is what I am sort of going to try to do at least. Okay. How do I do it? Look at this.

I am going to write V t0 plus t, X t0 plus t. Okay. Basically I am writing the Lyapunov function value at time t0 plus cap T. Okay. I don't know what cap T is yet. All this will come out of my analysis. I am just saying I added some cap T and I am going to compute this.

How do I compute this? Fundamental theorem of calculus. Integrate, just integrate. Okay.

Nothing complicated. Just the fundamental theorem. Starting value plus integration t0 to t0 plus cap T of V dot. Okay. Just wrote the fundamental theorem.

Okay. Now I start using all my cool assumptions. Okay. So by the way, on the left hand side I already have by positive definiteness this guy. Yes, by this. From here I have this. Right. So the left hand side already has some class K function of the state which is good for me because I know that it is monotonic.

State increases, class K function increases. So it is monotonically behaved. Very nice. Okay. On the right hand side also I would like something like that.

Let's see what happens. Okay. Now this is what I have and now I know from the first assumption, the negative definiteness assumption that this can be written as less than equal to this guy. Why? Because V dot is less than equal to minus gamma norm x. So I have just substituted this here. And because there is a less than equal to, the equality became a less than equal to.

Alright. Okay. Now I will immediately say what I want. This is a method of proof I like to, this is how I like to write proofs. Okay. This is a good way to write proofs because it helps you see where you want to go. Otherwise if I keep telling you one step of the proof after the other you will be lost as to why are we even doing all these steps.

Okay. So I just want this. Now what is epsilon 1? Anybody else? It is already down here. What is epsilon 1? Exactly. You already know what is epsilon 1.

It is just minimum of epsilon and R. Yeah. You always use epsilon 1 and not epsilon. Okay. So what happens if this inequality happens? I know that this inequality, this and this together implies this. I have just written them. And that means that norm of x t0 plus cap T has to be less than epsilon or less than equal to epsilon.

Okay. And this is what I need. Right. And by the way I notice that I chose capital T arbitrary. So for any capital T beyond this also this will work.

Nothing is going to change. Okay. I hope that is evident to you. Okay. I mean because the time I chose is pretty much arbitrary. In fact I didn't have to write it as t0 plus T or anything. I could have just written t1 greater than t0.

But just to keep our notation simple we will return it in this. Alright. Okay. So this is what I want. Now obviously how do I get it? I have some expression.

Great. So obviously what will I work on? I will work on this. Any guesses how do you think we will proceed? So this is, see because everything else is clear. Right. I have to just work on this inequality. Everything else is pretty much set. How do you think I will proceed? First what all other things you have to find? You mean in this definition.

First I want a delta. Right. Okay. I hope it is evident to you that delta is connected to this term. Size of this term purely governed by delta.

Okay. So if I get something on this term I get something on delta. Okay. Great. And then what about capital T? I also have to find capital T because epsilon is a given quantity. Where do I get capital T from? Where is capital T in this expression? From the second term.

So somehow from this term. Okay. So one thing should be sort of evident to you. This quantity is always going to be dropping in value. In fact definitely dropping, not staying constant. Staying constant is not a possibility because you see this is a class K function inside. So this is going to give some positive contribution. So this is if you integrate, integration is just so you are integrating positive, so you are summing up positive areas.

Okay. This is scalar quantity. So you can even think of it as areas under a graph. So you are integrating positive quantity. So it should be evident to you that this value is dropping. So this should give you some hope that if you give me any epsilon which is much smaller than delta. Typically you can assume epsilon will be much smaller than delta.

Anyway you can give me any epsilon. It has got nothing to do with, no connection with delta unlike stability. In stability delta is less than epsilon. But here it is not that. Epsilon can be less than delta, more than delta. But the point is for any epsilon you should be able to find a capital T and the small epsilon is what is important for us.

And large epsilon is irrelevant. You are trying to get to zero. So your epsilon needs to be small. Okay. So this should give you some faith that whatever delta related value I start at, I will always be going down. So therefore there is hope that you know you will drop to some class K function of epsilon.

Okay. So there is hope that I am going downwards. Okay. So this should give you some hope. Okay. So good we understand that our delta term has some connection to this guy and our T term has some connection to this guy.

Okay. Excellent. Okay. So anyway I have stated this again. But it should be obvious that for any T bar greater than T, this inequality has to hold. Okay. Why? If T bar is larger than T, again more positive contribution.

So anyway it is going smaller only. It is not going to go larger or anything. So if you take any time larger than the T you started with, you will get smaller values here. So this inequality anyway getting maintained.

We are not worried. In fact strict inequality. Okay. Great. Alright. So like I said this is what we need. I rewrite it in this form. So now I have to figure out how to choose a delta.

How do I choose a delta? I remove the effect of the T somehow. Okay. So what do we say? We know that we are going to be within this R ball. Okay. Anyway by stability we have already proved this. Remember. By stability proof itself with all these assumptions we have already proved that we are not going to exit the R ball.

I hope you remember that. We proved it as part of the proof. Okay. Right. So we know that norm x is going to be less than R. This is a class K function.

So therefore this has to be less than gamma R, gamma of R. Right. Just by monotonicity. So I can pull this guy out. Okay.

Anyway that is what we are saying. Anyway we are not, we will use this later on. Don't worry about it. We are not using it immediately. Anyway. So let us go to the choosing, this was just an aside, a fact that we are going to use very soon. Let us go to how to choose the delta. I know that V is upper bounded, sorry lower bounded, sorry, yeah, V is lower bounded in this way because of positive definiteness.

And V is in fact upper bounded just by this. So actually I can completely neglect this term as far as my delta choice goes.

Okay. And what do I know? Not what do I know. What do I want? I want that this happens. Okay. Because obviously this implies norm x is less than R. We obviously want to maintain that. We don't want to violate that in this proof either.

Okay. So if, because if phi norm of x is less than phi R, then norm x is less than R. So we are not interested in violating that condition in this proof. So what do I do? I choose my delta such that the supremum of this guy is less than phi R. Okay. Because this is the, this is the term that gives me delta, yeah, evident.

So from here, from just this inequality, I choose a delta. That's it. There is no magic here. And you already understand that such a delta exists. Right. Because we have already done this argument many times. That is, we know that, let's see, if I want to say existence of delta, yeah, I know that V t0, 0 is 0.

And I know that phi R is strictly positive. And I know that V t0 x is C1 in X. It is continuous, more than continuous. Okay. What does it mean? So V starts at 0 here, yeah, and I want it to take maximum value phi R.

Yeah. So I want to find the x such that for all x in this domain, it has to take maximum value phi R. Okay. So that's pretty obvious, right. I mean in the sense that this is already, yeah, this is, I have this much space to play with.

So I can always find the bound on x. Right. I can always find a bound on x such that this happens. Right. Just by continuity. You have already proven this argument.

This is my intermediate value theorem. Okay. So there has to exist such a domain on x. Okay. You can use many other arguments also, by the way. It is not impossible. If you, the other argument for again those of you who have seen and are interested in analysis type evidence, I will tell you that your set you are interested in, so if you look at the function, this guy V t0 x.

Okay. Which is a C1 function. Right. And you want the image to be here. So if you take the inverse of this set under this function.

Right. This is also a closed set. Okay. This is a closed set in x. Okay. And so you can from that closed set you can obtain whatever your x bound is. Why? Why? Again analysis freaks.

Why? If it's a closed set, so you know that I hope you understand that my image is 0 to phi r. Right.

Closed set. Because I said less than equal to. And lower bound is 0. Well known. Zero, phi r. This is where I want to lie. It's a closed set. I know that the inverse under continuous map is a closed set. Why do I say that I can find a delta now? How do I, what would be the delta then? Not what would be the delta, but how do I say I can find a delta? Just because I have a closed set in x now.

This is a closed set in the state space. State space is some Rn. So it could be some closed sphere, closed square, basically square with boundary, sphere with boundary, hyper sphere with boundary, ellipsoid with boundary. That would be the sort of sets you are looking at. Why do I say that then I can find this delta? Okay. Again, you probably folks still don't, are not completely comfortable with the analysis ideas.

But the simple argument is a closed set always contains its supremum. Okay. This is it. Yeah. This is all you have to say. Okay. Which means that you can always find the boundary, which means that at the boundary is a delta.

The boundary point is the delta point. That is the delta. That is which will give you norm x less than delta. Okay. As simple as that. Yeah. Again, if you did not follow that, no problem. But the basic idea is that just by intermediate value theorem, we have always done this even before. Because Vt, Vt0x takes value 0 at 0 and you wanted to take, you wanted to take maximum value phi r, there has to exist some bounded x such that for all values inside that bounded x.

Okay. This is not that obvious. It may seem to you that I am just restating all the obvious things. But I can create some funny functions.

I will create it for you. So easy. If I make something stupid like this. So here if I want to claim anything for norm x less than 6. In this case it is scalar. If I want to claim anything for norm x less than 6, can I make such a claim? No.

Right. Because at x equal to phi it explodes. Then it may come back and do nice things. But at x equal to phi it explodes. Okay. The problem is what? Why? Why is this not behaving nicely and I am saying that my functions will behave nicely? Continuity. This is not continuous at x equal to phi. Just by basic continuity, nothing more.

You don't need C1 or anything like that. Just by basic continuity, this existence of such a delta is guaranteed. So that you can say that for all values of state within the delta ball, I am guaranteed to be within phi r. Okay. Alright.