Nonlinear Control Design

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Week 1: Lecture 2 : Examples of Nonlinear systems

So, what we want to start with is, look at a few examples of non-linear systems. Alright, very simple. All of you, I am sure, have already seen quite a few models. I don't think I have to particularly, you know, tell you that this is non-linear and that is non-linear. You know, if you start to model any single thing including this fan right here, it is a non-linear system. Okay, almost everything that you see around you is a non-linear system.

For the sake of control, of course engineers have been for decades, been using linear approximations and it is pretty fair. I mean, I have seen PID controllers work very well in practice. So I can't say that there is no merit there. But of course, when you want advanced precision, then you have to work with non-linear systems.

And this is one of the things which sort of attracted me to spacecraft, space systems also because there, if you have a really, you know, very very small error in your attitude or your orientation, then you are guaranteed to not send any real signal to the earth. For example, if you have an earth pointing device, then really small errors in your orientation will also, you know, make your antenna worthless for all intents and purposes. So that is one of the nice things, I guess, to study non-linear systems. Alright. So we want to look at examples.

So here the example is little bit more esoteric, just to, you know, sort of drive the point, I guess. Yeah. Alright. So the typical model that we work with, a typical non-linear system model that we are talking about in this course always looks like this. This is also the mathematical sort of representation.

I hope all of you are comfortable with these kinds of notations. Okay. Yeah. If not, I really encourage you to look up some of these notations also.

Yeah. Or you can ask me of while we are discussing. Yeah. But I may not be able to explain every single notation. Okay. So we typically say that our system is of the form x dot is f of x t.

Okay. So this is a non-linear system. This is not a non-linear control system. Okay. Because there is no control here in the expression.

Yeah. So here this is just a non-linear system. The function f is typically a map which is sufficiently regular. What do you think I would imply by sufficient regularity? Any guesses?

Yes, absolutely. So it is the dimension that we are discussing.

I am sorry. Did you say, you said that it is the dimension? It is the dimension of the map. Yes, absolutely. So when you say sufficiently regular, you are expecting it to be sufficiently differentiable in this context. You may be able to do with much less.

Yeah. Again, the differentiability requirement like she rightly said is also depends on the dimension that you are sort of talking about. But a lot of times we just make our lives really simple. We just say that the function is smooth. So it is infinitely differentiable. But again, in reality that may not be the case.

So it is better to work with the least possible assumption. Yeah, we will also look at those. But typically when we say sufficient regularity, this is what we mean. Yeah, we are talking about some kind of a continuity, differentiability, these kind of properties.

Alright. X is what is called, what is it called? States of the system. Thank you. Yeah, X represents the states of the system. Any system is representable in the form of states. Yeah, again the fan, ubiquitous fan because it is right there.

I can say that the angular position, angular velocity, these are the states. May be the states that I am interested in, but of course there is an electrical motor. So there may be you know the voltage. Yeah, which is a state possibly. Or I can think of the current as the state and voltage as the input.

Yeah, so the current is the state and the voltage can be thought of as the control input. Yeah, so depending on how high fidelity a model you want, as that is the number of states you can have. Okay. Yeah, even in civil engineering if you think structural engineering model, structural systems, a little bit flexible structures like you know these suspension bridges and things like that. Yeah, you can have the flexible modes as your states, first mode, second mode, third mode, things like that.

Of course typically it is an infinite dimensional model, but then you sort of discretize it. Yeah, so you can still have you know, you can still put it in a state space form. Yeah, okay. So there is no particular answer to how many states we should consider. It depends on the complexity you are willing to handle in terms of computation and depends on how easy or hard your control design task is.

Yeah, so there is no guideline at all. Yeah, depends absolutely on us. For a system like the fan, the position and velocity are more than enough, more than enough. You will be you know more than comfortably control the system, do computation if you wanted to make it a smart fan. You have smart fans these days, right, whatever.

So you can easily do it with position and velocity, more than enough. But if you are talking a biological system, you are talking electrical system, you can't. So these guidelines are usually, the guidelines on how many states and so on usually come from system properties like observability, controllability. We don't talk about it here. Again I am not sure we may get time or not to talk about non-linear system controllability and observability because that is a huge topic in itself and I am not sure if I will be able to do justice to it.

Because if I do that, I will probably be able, will not be able to do any design. So anyway, so X are the states, t is of course the time. It doesn't necessarily have to be start at initial time 0. So therefore we give some initial time though the close bracket t0 and an open bracket at infinity because there is no infinite time as such. And of course f is a map which takes states and time and gives you something in Rn and X dot is of course telling you how it is evolving.

So if I integrate this, I can integrate more often than not, I integrate it numerically. You cannot expect to be solving any of these analytically. So you can integrate this to see the evolution of the system. So interesting examples. I spoke to you folks about this infectious diseases model and so on.

So this is one such model, I mean not it. So this is an HIV spread model. So here you have two set of states. So X is the population of uninfected cells per unit volume of blood and Y is the population of the infected cells per unit volume of blood. And the revolution is, and here I is basically the immunoresponse.

So in fact this is a control system. If you think of I as a control, of course there is a natural immunoresponse, but if you, if there is a potential for boosting this immunoresponse, then I is a control. So control is anything that a user can specify or command or in the simplest aero-mechanical systems easy to understand, motors and drives and things like that, propellers, actuators, easy to understand. But of course more complicated in biological systems, social systems, what is the control? So here it is the immunoresponse. So if you can somehow push a drug which will accentuate the immunoresponse, then you have a control.

And then of course you can see this evolution, I mean it is clearly non-linear. I hope all of you understand what is linear and non-linear. How do I say that a system is linear or non-linear? Right. So homogeneity, superposition principle, essentially it has to satisfy the superposition. And what has to satisfy the superposition principle? This right hand side, that is it.

So this right hand side function f, if it is actually satisfying the superposition principle, then it is a linear system, if not it is a non-linear system. And easy to identify of course, so because I have a product of x, y and so on, this is a non-linear system. So both these terms, in fact this is the only term. And interestingly you can see that this appears in the negative and this appears in the positive. So it is sort of saying that the product of the infected and uninfected cells, so these are all of these, typically these infectious disease models, these are all sort of reaction models, chemical reaction models are what they are.

They look like chemical reaction models also. So here you can see that the product of the infected and uninfected cells sort of affects, it adversely affects the uninfected cells and if the product is positive, the infected cells increase. So this is sort of affects the how things go. And of course these are, I mean lot of parameters, I don't, honestly I don't even know what these parameters are. But these parameters are what influence the rate at which things are.

So this is a non-linear model. Again like I said, you will not be hard pressed to find non-linear models, I am just giving you some esoteric models just to give you a feel of where non-linear dynamical systems could be applicable. Alright. This is the actual SIR model. This has been used extensively, extensively in COVID modelling. I mean if you just type COVID model and you pick up, you know just look at the papers, you will find them using the SIR model.

Okay. Why? Because it is the most acceptable model for infectious disease spread, not just COVID. So the only thing that changes with a particular disease are these parameters. The only thing that sort of changes is these parameters. So lot of research went into, you know, what would be the peak number, so that sort of an asymptotic analysis. So the only problem that happened, I felt, was that these SIR type models do not account for new variants coming in.

So when new variants come in, it is like an impulse, almost like an impulse because sure, now the numbers are very small, but it is possible that, you know, say the delta numbers were very small to begin with. But then there was some kind of a critical mass and then there is a huge jump in the numbers. And those are not accounted for with these models, by these models. So as you can see, any model has its weakness. So new variants, there is no way of predicting new variants with SIR models.

What SIR model can do is, once a variant has, you know, sort of entered a population, then it can tell you how fast it is going to progress and what equilibrium it can potentially reach, okay, depending on, you know, how you isolate and things like that. So you have three states here. One is the susceptible, second is the infected and R is the removed or recovered, okay. Makes sense? Susceptible is all the population that is, you know, potentially get it, yeah. So these are all the folks who, well, not using, you know, appropriate protection, right.

Infected is folks who are already infected, right and removed or recovered because typically with most infectious diseases like COVID, if you get, got a particular variant, you can hope that for the next two, three months at least you are safe until you get the new, until the new variant comes in unfortunately. Yeah, I mean, so that was the really sad part. But anyway, so not to say that you cannot get it, but the point is the current variant will be immune to, yeah. It is almost like getting a vaccine shot, yeah, for a while, yeah.

Alright. So then you have this sort of a model, yeah. I would really encourage you to look it up. I mean, you can see so many papers using this SIR model and exploiting it to get, you know, these asymptotic features of, you know, how COVID is going to progress, what is the equilibrium it is going to reach and so on and so forth, okay. Then this is a nice sort of a mechanical system model, yeah and rather complicated mechanical system model. Folks who work with, you know, if some of you have worked with the corporates and like tractor companies like John Deere and things like that, you know that these are rather important problems for them, yeah.

So you have, you typically have a tractor, but then you have some kind of a trailer also attached to it, yeah. So which may carry, you know, some of your produce and things like that or it may be just another piece of equipment that you are carrying. And usually there is a, you know, this sort of a single link between them, yeah. And if you want to control this kind of a system, you have to actually have a, this becomes a rather complicated model in fact, yeah, with a lot of variables. So you can see that you have many variables like the lateral offset, you have the orientation offset which is this guy, yeah.

So the lateral offset is basically the, if this is the path you want to follow, then this is sort of a lateral offset, how far you are from that path. So this is sort of the, seen as the centre of, you know, mass in some sense, yeah. Then you have the orientation offset. What is the orientation offset? You look at the tangent, it is the direction you are expected to move at this point in the curve and the direction you are actually moving and then you look at the orientation offset, yeah. So these are the things you sort of, you can imagine, right, as a control engineer these are things I want to drive to zero in some sense, yeah.

Then you have, yeah, I mean then you have this phi OS and steady state value for phi and things like that, okay. Unfortunately the, this is a different notation and this is a different notation, well, yeah, I mean that is just because I did not write it properly. But these are the same, yeah, okay. So this is the same as this guy.

So this and this are the same. So this is the where phi, where phi, yeah, alright. So the idea is that we want to drive these variables to zero, okay. So typically in control this is a very standard requirement that I always want to drive some variables to zero, okay. So this is usually my aim, yeah. So if, for example, if I am looking at a tracking problem, say a robot tracking problem, like if I want to do a mobile robot tracking problem, yeah, and I want to drive this mobile robot on this path, yeah, but controls guys really like to drive things to zero.

So what will I do? I will take my state as the error between my current, you know, current state and the desired state, yeah, and then create an error and then I will drive the error to zero, yeah. So we really do not like to drive things to some values. We like to drive things to zero, okay, because when you see, you know, all our theorems that come up in the future, you will see that, you know, this is what we look at. We look at all zero equilibrium and things like that, okay.

So that is the idea. So this is the model, very scary looking, honestly speaking, yeah. So this is the lateral offset derivative, yeah. This is the angle offset derivative and this is the phi, where phi offset derivative, okay. These are absolute values where V1, V2, these are all velocities of the, you know, these are the velocities of the trailer, the tractor, yeah.

So this is the very, very complicated looking model. The control is basically the tan delta. So this is, I believe, so this is the control here, yeah. This is what we get to play with and I believe this is the steering, if I am not wrong. I think this is the steering, yeah.

So essentially, the control in this case is just the steering, okay. So this is again a slightly simplified model, I guess. You are saying that you are moving at some uniform velocity and you are just controlling the steering, yeah, alright. I have no idea if the system is controllable or not. So that is a, not an easy question to handle looking at this model. And this sigma is some variable which is plus minus 1 depending on whether it is clockwise or counterclockwise, okay.

What is sigma? I think depending on how you are rotating the, you know, the wheel, you get a sign here for this particular signal, alright. So that is it. It is just because of how the model is being represented. But the control is just 1 and that is this which is the steering, yeah, alright.

Sigma is not a physical variable at all, okay. Sigma is just depending on how you are rotating the steering, clockwise or counterclockwise, it just assumes a sign in the model here, okay. This is just because how the modeling is done, alright. That is it. Then you have like this Lorentz atmospheric model, okay. So this is basically saying how the layers in the atmosphere move around, okay.

So you can see these examples are from very very wide range of areas, okay. X is basically the convective flow, Y is the horizontal temperature distribution and Z is the vertical temperature distribution, okay. So there is a state variable which is representing the flow, okay and correspondingly there is variables which tell you how the temperature distribution in the atmosphere would be, okay. This might be interesting for folks who are looking at how the weather patterns are going to change, okay. So because you are looking at how the temperature distribution in the atmosphere is changing, okay. Here rho is the temperature difference between the top and the bottom slice depending on what you, how big you know, how thick or you know you are looking at how, what is the width of the atmosphere you are actually analyzing, depending on that you will get a different model and beta is width to height ratio of the slice, okay.

So given this you have you know, you have this non-linear. So this particular piece as you can see is pretty much linear, right. This is the convective flow variable. So this is just linear, it depends just on you know Y minus X which is like the temperature distribution and the convective flow variable itself, okay. Then the Y variable here has this non-linearity, right. It is an X times Z and similarly the Z variable, so that is the horizontal temperature distribution also has this non-linearity, alright, okay. Thank you.