Nonlinear Control Design

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Week 3 : Lecture 16 : Lyapunov stability Theorems- Part 4

Alright, welcome to another class of non-linear control systems, control of non-linear dynamical systems, whatever you want to call it. So last week we had started talking about the central analysis method of non-linear control which is this Lyapunov stability theorems. So we started with some preliminaries that is we talked about function classes, the class K, L, class K R and so on. Then we went on to discuss definiteness and how they are connected to these function classes and also how to extend the notions of definiteness of matrices to definiteness of functions. So basically we understood or we figured that any positive definite or a definite matrix or positive definite matrix is going to lead to a positive definite function. Once we construct a quadratic form out of it.

So we saw some nice examples. Of course we also had these easy conditions to test definiteness and so on. So again definiteness, positive definiteness and so on. So we had these relatively easier conditions.

Then we talked about radial unboundedness. So these were all properties and I think there is also of course once we had radial unboundedness then we also spoke about decrescence for which we did not give any easier characterization. This is because the easier characterization is not easy at all. I mean if you are interested you can look at Vidya Sagar's book to see this easier characterization but it is not very easy. So I would definitely say that it would be very very beneficial for all of you to go back and look at Vidya Sagar's book at least once in a while to see how things are going.

I mean the chapter again I forget the chapter number but it is very easy to find. You can see the Lyapunov stability analysis chapter in Vidya Sagar's book. A lot of the material that is here is being derived from there. It is one of the most comprehensive and very very mathematically precise description of all this. Because now that you have seen this, I have made it of course little bit distilled it and toned it down from the Vidya Sagar language.

So now that you have seen this and you understand this material it will definitely be easier for you to follow what is in Vidya Sagar's book. And then finally we had semi-definiteness properties. So we had four properties, positive definiteness, radial unboundedness, decrescence and semi-definiteness. We already said that positive definiteness was connected to stability, asymptotic stability. So local properties, radial unboundedness is connected to global properties, so global asymptotic stability.

Substantive definiteness is connected to uniformity, so uniform stability properties and finally semi-definiteness is just connected to plane stability. Nothing more than that. So we also started discussing the Lyapunov theorems themselves. I believe that I had mentioned that we will look at the theorems first, understand them, maybe even try to apply them a little bit. Then we will look at the proofs of at least one bit.

We will look at proof of some of it at least. So that we, once you see a proof of sort of one version of the theorem, everything else sort of follows, not difficult to conclude the rest. So we started with this structure of the dynamics and we of course assumed all the nice things that is zero as equilibrium point and f is locally Lipschitz, so that existence of unique solutions is not a problem at all. And then we defined the notion of V derivative or directional derivative. I mentioned very clearly that the function V of x or V of x comma t has no connection to a dynamical system as such.

It's just a function of some variable x and some variable t. Then when you take the derivative or this directional derivative that is when you bring in the dynamics of the system through this term. In fact you will see a lot of times that we use the same structure of V of x to analyze many different systems. So that is why this is very carefully defined. It's a definition.

Whenever I use this notation, this notation implies I am defining something. It is not just an equality. You might think it's an equality because all we did was compute a V dot but it is not. By no stretch of imagination because when I define a function of x and t, I am not saying anything about dependence of anything on time at all. I am just defining functions of x and t.

At max you can take the partial of V with respect to t but there is no notion, there would be no notion of taking partial with respect to x because x is itself an independent variable once I define V x t. But only when I take a derivative and I define it so that you have dependence of x also on So that is why I am defining a function of x and t. So that is why I am defining a function of x and t. So that is why I am defining a function of x and t. So that is why I am defining a function of x and t.

So that is why I am defining a function of x and t. So that is why I am defining a function of x and t. So that is why I am defining a function of x and t. So I did the, I believe the first two statements. We said that we first require to have a candidate Lyapunov function.

What is a candidate Lyapunov function? It is a C one function of state and time such that V is positive definite. This is the minimum requirement for it to be a candidate Lyapunov function. then you need some of these conditions to be satisfied. Ok, if V. is only negative semi-definite that is it is not definite just that it never takes positive values.

So, V. never takes positive values. Yeah, it takes only non-positive values that is it is negative semi-definite. Then the equilibrium is just stable and on top of this if the V that you started with is also decrescent then you have uniform stability. Ok, so these were the two statements that we did and you remember all the other stability definitions are sort of strengthened versions of this. You start with stability and then you move on to asymptotic versions and uniform versions and things like that.

Exponential versions. The next one again remember this is sacrosanct without this you can't use any Lyapunov stability theorem. Ok, so be careful when you choose a V that is satisfies a condition like this. Ok, alright. So typically an energy function would satisfy this. Yeah, typically energy of a system for Lagrange system Lagrangian system will satisfy something like this or a conservative system the way you know.

Yeah, ok. Next one, local asymptotic stability. All I need is this semi-definiteness is no longer enough. I need negative definiteness. Ok, and this is what I mentioned that definiteness is connected to stability, asymptotic stability. So I am clearly saying local asymptotic stability.

Although we typically don't use this word, we have not been saying this, you just say it is asymptotically stable and the acronym is also AS. Yeah, there is no LAS. Alright, ok. Again the specialization of this would be to start with a decrescent V and then I get uniform asymptotic stability. Ok, so the results are very straight forward.

Once you have the ingredients, the results look very easy. Ok, you have stability, uniform stability. Once you have V dot to be negative definite, you have asymptotic stability. If you start with a V that was decrescent and you have V dot negative definite, you have local uniform asymptotic stability. Again local is not something we state necessarily.

Alright. Then say again, I am going to state all of these before I go to the examples. Alright. Then you have global stability notions. Yeah, now for global stability, I need the negative definiteness of course but I now need V to be radially unbounded. Yeah, a positive definite V is no longer enough.

Remember that the arguments also change. This BR doesn't work anymore. Yeah, V cannot be valid only on a ball around the origin. It has to be valid for all RM. Ok, so therefore you need V to map all states not just in a ball to real number and then C1 positive definite.

Ok. So V is now required to be radially unbounded which means that its arguments have to take all possible states and V dot is negative definite. Then you have global uniform asymptotic stability. Ok. Actually sorry, I also missed saying that it has to be decrescent. Of course if I remove the decrescents, what do I get? What if? Globally asymptotic stability.

You just globally asymptotic stability. So, as soon as decrescents is gone, uniformity is gone.

Ok. Alright. Ok. Then if you remember, I am of course not stating all the intermediate versions because I understand that you understand that if I remove decrescents, I remove uniformity and so on and so forth. You see which word is associated with which word. It's as simple. It's as simple as a word association. Of course when we do examples, it's not as simple.

But for now, the statements are very straight forward. Alright. In fact, it's almost like you can have a cheat sheet in your head.

It's very easy. Ok. Now finally, when we want exponential stability, the conditions are slightly different. You don't use the positive definiteness and all. You don't state them like that. What we say is, if V is decrescent and there exists three class K functions, all of the same order of magnitude such that V t x is lower bounded by phi 1 norm x and upper bounded by phi 2 norm x and further V dot is lower bound, is upper bounded by negative of phi 3 norm x.

Ok. Now, if you notice, this highlighted green thing implies positive definiteness. Alright. And then this highlighted yellow thing implies negative definiteness. Ok. Now here I am already stating decrescents separately.

Ok. However, if you look at sort of the right hand side, it's not exactly decrescents but it is pretty close. Ok. This is how we had stated decrescents. The only difference was there was an absolute value.

Yeah. Here we don't particularly need the absolute value because we have already assumed V to be lower bounded by a class K function which means that it is lower bounded by zero. Because the class K function at x equal to zero will be zero. Right. Which means the left hand side, essentially implying that V is already positive semi-definite by this assumption, at least.

In fact, positive definite by this assumption. So absolute value is not required because V is never going to the negative side at all. So absolute value of V is irrelevant. So what you have here is effectively decrescents. Ok. So we have stated all three requirements which you had for global uniform asymptotic stability.

Right. There is no difference as such. We have stated all the requirements just in this mathematical form rather than writing the words. Ok. What is the difference? The difference is these words.

Ok. So the difference is these words. Same order of magnitude. Ok. And I will get to this soon.

Ok. So this gives me local exponential stability. Why? Because they were only class K functions. The comparison functions were only class K and you can see that I carefully, I was very careful and I said all this is valid for x in some ball of radius r. Alright. So if I want to go to the global version, what do you think I will need? What will happen? Radially unbounded.

Radially unbounded. So I will need, well, yeah, I will need all three to be radially unbounded. I guess. Yeah. Because they are the same order of magnitude. So all three will have to be radially unbounded and of course this will be all of R.

Ok. This will be the only difference. So you can see that I am already saying V is radially unbounded. Although I don't need to. And then you have all three functions.

There exists three functions in class Kr. Yeah. Oh, I see. Such that this happens. Alright. Such that this happens. And if you see, I have also said what is the meaning of functions being of the same order.

It means they are comparable by a constant. This should remind you of the ability to compare the norms. Right. The vector norms are also comparable. This is almost a similar definition. So all three two functions are said to be of the same order of magnitude if they are comparable via constants.

Ok. So and you notice that if f and g can be written like this, then g and f can be written like this. Right. So basically f and g are comparable functions. I mean examples are if you have one function which is x squared, Phi1 is x squared and Phi2 is x4, x to the power 4, then they are not comparable. Because you will never be able to find a constant Gamma1, Gamma2 which will relate that.

Ok. So simply saying, I am just giving a scalar example or you can even take a vector. Norm x squared and norm x4 not same order of magnitude.

Ok. Because they cannot be related by these constants. Ok. Great. So, now we have seen pretty much all the Lyapunov theorems. Ok. It is very quick. I mean once you create the setup, actually it is very quick to state, very easy.

You know how the specialization goes. Right. Start with negative semi-definiteness for V dot, you get stability. Go to negative definiteness for V dot, asymptotic stability and if you go to radial unboundedness for V, you get global asymptotic stability. If you add decrescence on V, you get uniformity in all of these. Ok. And finally for exponential stability, you need, remember we already, it is part of your assignment, first assignment that exponential stability implies uniform asymptotic stability and similarly global exponential stability implies global uniform asymptotic stability.

So, exponential is by definition uniform. Therefore, you need the decrescence conditions also. Right. Whenever you are talking about exponential stability.

So, exponential stability requires existence of these three functions. Ok. Make sense? Excellent. Examples. Let's do examples.

This is where, this is what is our, you know bread and butter. Right. This is the simplest example we can do. The simple harmonic oscillator. The simplest example anybody will start with. What is the simple harmonic oscillator? It is just x1 dot is x2, x2 dot is minus x1.

Alright. For a system like this, it should be obvious to you that, well, or it should be obvious or you must have seen it before, that the phase plane portrait, that is the evolution of this in the phase plane looks like circles. Ok. Why? Because you can think of x1 square plus x2 square and you take derivative of x1 square plus x2 square along this.

Yeah. This evolution makes x1 square plus x2 square constant. Ok. Just you can check it. It's very easy. Anyway we will do it in the Lyapunov function anyway.

Right. So, that's what we choose is a Lyapunov function. It is just half x1 square plus half x2 square. I mean I have just taken this and divided by 2.

Ok. In this case, by the way, this is a conservative system. Conservative system. So, in this case, this is actually the energy of the system. Ok. This is the potential energy plus the kinetic energy.

Ok. And this is energy conserving system. Therefore, you see that it is just moving in circles, concentric circles.

Alright. Alright. So, I take my V as exactly the energy of the system. Notice V is C1. In fact, it is radially unbounded. I hope you are convinced that this V is radially unbounded. Yeah. Goes to infinity.

I mean first of all, it is strictly positive whenever norm x is non-zero. Alright. And it goes to infinity as norm x goes to infinity in any direction. It does not matter.

Alright. Therefore, V is positive definiteness. In fact, V is radially unbounded. Ok. So, and anyway this is a linear system. So, V is radially unbounded.

Alright. So, you can see that, that this V is radially unbounded. We can just focus here. Now, if I take the derivative V dot, there is no time argument. Right. So, uniformity is free. Yeah. Just like I said, there is no time argument in the system, no time argument in V, uniformity is free.

We don't even talk about uniform stability motions. Ok. Right. So, partial of V with respect to x times the evolution. So, what is partial of V with respect to x times this? It is just the way you take derivative. Ok. After all this definition, all you have to do is take derivatives.

It is just x1 x1 dot plus x2 x2 dot. Just taking derivatives and plugging in from the dynamics. If I plug in x1 dot from the dynamics, it is x2. If I plug in x2 dot from the dynamics, it is minus x1.

Right. So, essentially it is 0. The sum is just 0. Ok. By Leah. And so, what have we done? What have we shown? We have shown that V dot is always 0.

Exactly 0. Which means it is only negative semi-definite. Right. It is not negative definite. Yeah. Because even for non-zero values of the state, V dot will always be 0.

Ok. Make sense? Yes. Alright. Good. Ok. Fine. So, by Lyapunov stability theorem, I started with the V which was positive definite, radially unbounded in fact. And V dot turned out to be only negative semi-definite.

Therefore, my equilibrium that is the origin is stable. Ok. This is all I have. Yeah. And this is a fact. There is nothing more.

You can't get anything more for this system. Because the phase plane portraits all look like this. Yeah. So, if you start at some point here, you will just follow this circle. If you start at another point, you will follow this circle. If you start at another point, you will follow this circle.

Wherever you start, you will just start tracing a circle of that radius in the phase plane. Ok. It is as simple as that. It is one of the simplest systems to illustrate Lyapunov theorems. Next one is a complicated one.

See and you start seeing how things get very complicated very soon. That's probably the aim of this example.

Yeah. I just played with this system a little bit. Yeah. I just made it time varying. Alright. So, x1 dot remains x2. Yeah. And x2 dot is minus x1 divided by 1 plus t.

I just made it time varying a little. Now I want to see if I can do anything. So, what do I do? I sort of choose my Lyapunov function in a slightly more smart way because otherwise I think I will not be able to proceed at all. So, I choose it as half x1 square. The first term remains the same.

And by the way, many people ask me how do you choose Lyapunov functions and such.

There is no way. It is an art. Ok. You either start with the energy of the system and then try to modify the term. It doesn't have to be energy of the system or it is motivated by some literature.

Ok. It is not a guaranteed process that this is what will work and this is how I can get a Lyapunov function.

Ok. No. You cannot do that. Alright. Great. So, I play with the terms. I take the same term as the first case. But then in the second term, I add this guy. Because I want to do some cancellation because of this guy.

Because I want to do this time cancellation. Right. So, notice already, well before going there, I should say something. What about V? Is it positive definite? Yes.

Yes. Because it is greater than half x1 square plus half x2 square. Right. So, it is positive definite. In fact, radially unbounded. So, V is positive definite.

In fact, radially unbounded. Is V decrescent? Is V decrescent? No. You remember, we did this example. Right. Whatever class K function you give me that needs to be, that needs to upper bound this guy, I will just dominate it by bumping up time. Because the class K function will have no argument of time. So, once x is fixed, this is fixed, this is fixed, the class K function is fixed.

I will just bump up time arbitrarily. And I will beat any class K function that you give me.

So, no, not decrescent. Only this much. So, then I go on and take the derivative. Right. I have three terms, x1, x1 dot. Yeah. 1 plus t, x2, x2 dot. But then by the chain rule, I have to take derivative of this guy also.

So, I have x2 squared divided by 2. Ok. So, then I have x1, x2 here. Plugging in x1 dot, I get x1, x2. Plugging in x2 dot, this 1 plus t cancels out, I have minus x1, x2.

Right. So, this term and this term cancels out. And then I am left with x2 squared by 2. Ok. Something pretty bad happened. Right. Because v dot turned out to be x2 squared by 2 which is greater than equal to 0.

Yeah. Doesn't mean anything. Because it may just be the case that I chose a bad v. Does not mean that the system is unstable. This is not enough to say that the system is bad or unstable or whatever.

Remember. Yeah. It may just be the case that I chose a bad v, I cannot choose a good v. So, that's my problem. Alright. Great. So, that's what I have said.

I have not conclude on stability yet. Ok. I thought about it a bit, I could not find any good v honestly speaking.

Yeah. Which would let me conclude anything. But maybe you guys can try. I don't know. Yeah. You can try what you can get. But as far as I could see, the system is rather is unstable. And why I conclude that is that if I just look at the dynamics of the system.

See it's very difficult, it's not easy to solve the system. Can I solve the system actually? No. It will be rather hard. Yeah. Because these two are coupled. If they were not coupled, this would have been ok. But it's not going to be very easy to solve the system.

I mean I may be able to use some time varying linear system tricks to solve it. But it's not obvious how to solve this. Ok. Because of this guy and the fact that this is a coupled system.

Ok. So, what did I do? I thought about it. I tried to see the phase portrait. Yeah. But for large values of time. Ok. Let's look at what happens for very small values of time.

Very small values of time, this is almost equal to 1. Right. Almost equal to 1. So looks like a harmonic oscillator.

Yeah. Cycle. So for very small values of time, it looks like this. This circle. Ok. But as time increases. Right. So, first of all drawing a phase portrait for a time varying system is also unintuitive. Because the time argument is not visible here. I can't use a time argument here unless I draw a third axis and make something very complicated.

Yeah. But, but you see it's not possible to do a very good phase portrait based analysis for time varying systems also. Ok. So, you see with such a small change, adding a time dependence, things can go rather messy.

Ok. But why I conclude that it is possibly not stable? What happens for large time? This guy dominates. You can forget 1. This term is going to 0 almost.

Ok. So, x2 dot is 0. x2 dot doesn't change. So, wherever I start, I stay at that same level in the vertical axis. But x1 dot keeps increasing. Right. Because, because if I started far away, that's why you see the size of my arrows. This is what is in it. If I start close to the origin, small horizontal velocities.

If I start further away, larger horizontal velocities. Further away, very large horizontal velocities. Similarly, if I start here, small negative velocities, larger negative velocities, very large negative velocities. Ok. So, I can see that there exists initial conditions which are never coming to the origin or doing anything nice.

Just think about it. If I make a ball, what is stability? You give me an epsilon ball, I have to give you a delta ball. Can you do that? No. Right. Because that everything is going away. Some initial conditions will push you this way, some initial conditions will push you this way. And, and large time is where we are thinking of things happening.

Right. I mean we don't care about transient, I mean stability, sorry, this asymptotic stability has no concern with transients. Ok. So that is much newer results in non-linear control where you start talking about transients. Yeah. Also one of the complaints of linear system folks that you don't care about transients.

Ok. So non-linear systems all the analysis rotates around or converges to asymptotic results. Yeah. So large time basically. So large time, in large time I can see that things will not work well.

Yeah. All my trajectories will start to explode in some sense. So I guess in some sense this is not, this was not so wrong. Ok. But you see it took me a lot of intuition and effort to even get a result like this. Yeah. This very tiny system. Ok. Thank you.